Math 152 Week-in-Review

Exam 3 Review

Comparison

1. Determine if the series converges or diverges. FULLY explain your reasoning.

(a) $\sum_{n=1}^{\infty} \frac{2+3\cos n}{n^3+4n^2}$ $-1 \le \cos n \le 1$

$$\sum_{n=1}^{\infty} \frac{2+3\cos n}{n^3+4n^2} \le \sum_{n=1}^{\infty} \frac{2+3(1)}{n^3} = \sum_{n=1}^{\infty} \frac{5}{n^3}$$

Converges p-series $p=3>1$

$\sum_{n=1}^{\infty} \frac{2+3\cos n}{n^3+4n^2}$ Converges by comparison Test

Comparison

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{5n^2-2} \geq \sum_{n=2}^{\infty} \frac{n}{5n^2-2} \geq \sum_{n=2}^{\infty} \frac{n}{5n^2} = \sum_{n=2}^{\infty} \frac{1}{5n}$ Diverges p-series $p = 1 \not> 1$

$\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{5n^2-2}$ Diverges by Comparison Test

Comparison $-1 \leq \sin(n) \leq 1$

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{5+\sin n}{n-4\sqrt{n}} \geq \sum_{n=3}^{\infty} \frac{5-1}{n}$ Diverge. p-series $p=1 \ngtr 1$

$\displaystyle\sum_{n=3}^{\infty} \frac{5+\sin n}{n-4\sqrt{n}}$ Diverges by Comparison Test

Alternating A.S.T.

(d) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n}$ Alternating!

$b_n = \frac{1}{n}$

(1) $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

(2) Decreasing? ✓

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n}$ Converges by A.S.T.

2. Determine if the series converges absolutely, converges conditionally, or diverges. FULLY explain your reasoning.

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(3+\ln n)^3}$

Absolute: $\sum_{n=1}^{\infty} \frac{1}{n(3+\ln n)^3}$

$f(x) = \frac{1}{x(3+\ln x)^3}$

(1) Cont. ✓
(2) Pos. ✓
(3) Dec. ✓

$\int_1^{\infty} \frac{1}{x(3+\ln x)^3}\,dx$

$u = 3+\ln x$
$du = \frac{1}{x}\,dx$

$\int u^{-3}\,du$

$\frac{u^{-2}}{-2}$

$\frac{1}{-2(3+\ln x)^2}\Big|_1^{\infty} = \frac{1}{\infty} - \frac{1}{-2(3+\ln 1)^2} = \frac{1}{18}$

Integral Converges

$\sum_{n=1}^{\infty} \frac{1}{n(3+\ln n)^3}$ Converges by Integral Test

$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n(3+\ln n)^3} \text{ is absolutely convergent}}$

b.) $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$ Absolute: $\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$ Converges p-series $p = \frac{3}{2} > 1$

is absolutely convergent

c.) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$

Absolute: $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$

Diverges by Test for Divergence

Conditional: $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$ Alternating

$b_n = \frac{n}{2n+1}$ (1) $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$

Diverges by Test for Divergence

$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$ Diverges d.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$

Absolute: $\sum_{n=1}^{\infty} \frac{1}{5n+1} \leq \sum_{n=1}^{\infty} \frac{1}{5n}$ Diverges p-series $p=1 \not> 1$

$$\lim_{n\to\infty} \frac{1}{5n+1} \cdot \frac{5n}{1} = 1 = L$$

$0 < L < \infty$ ✓

$\sum_{n=1}^{\infty} \frac{1}{5n+1}$ Diverges by L.C.T.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$ Alternating

(1) $\lim_{n\to\infty} \frac{1}{5n+1} = 0$ ✓

(2) Decreasing? ✓

$b_n = \frac{1}{5n+1}$

Converges by A.S.T.

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1} \text{ is conditionally convergent}}$$

e.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n n!}{(2n+3)!}$

$$\lim_{n\to\infty} \left| \frac{10^{n+1} \quad (n+1)!}{(2(n+1)+3)!} \cdot \frac{(2n+3)!}{10^n \quad n!} \right|$$

$$\lim_{n\to\infty} \left| \frac{10^n \cdot 10}{10^n} \cdot \frac{(n+1)\cdot n!}{n!} \cdot \frac{(2n+3)!}{(2n+5)(2n+4)(2n+3)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{10\cdot(n+1)}{(2n+5)(2n+4)} \right| = 0 < 1$$

$\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n n!}{(2n+3)!} \text{ is absolutely convergent}}$ by Ratio Test

3. Find the radius and interval of convergence for $\sum_{n=2}^{\infty} \frac{(x+3)^n}{5^n\sqrt{n-1}}$. FULLY explain your reasoning.

Center : $a = -3$

Goal: $|x-a| < R$

$$\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{5^{n+1}\sqrt{(n+1)-1}}\cdot\frac{5^n\sqrt{n-1}}{(x+3)^n}\right| = \lim_{n\to\infty}\left|\frac{(x+3)^n(x+3)}{(x+3)^n}\cdot\frac{5^n}{5^n\cdot 5}\cdot\frac{\sqrt{n-1}}{\sqrt{n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x+3}{5}\cdot\frac{\sqrt{n-1}}{\sqrt{n}}\right| = \left|\frac{x+3}{5}\right| < 1$$

$|x+3| < 5$

$$\boxed{R = 5 \qquad I: [-8, 2)}$$

$-8 \xleftarrow{-5} -3 \xrightarrow{+5} 2$

Let $x=-8$

$$\sum_{n=2}^{\infty}\frac{(-8+3)^n}{5^n\sqrt{n-1}} = \sum_{n=2}^{\infty}\frac{(-5)^n}{5^n\sqrt{n-1}} = \sum_{n=2}^{\infty}\frac{(-1)^n\cancel{5^n}}{\cancel{5^n}\sqrt{n-1}} = \sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n-1}}$$ Alternating!

$b_n = \frac{1}{\sqrt{n-1}}$

(1) $\lim_{n\to\infty}\frac{1}{\sqrt{n-1}} = 0$ ✓

(2) Decreasing? ✓

Series converges by A.S.T.

Let $x=2$

$$\sum_{n=2}^{\infty}\frac{(2+3)^n}{5^n\sqrt{n-1}} = \sum_{n=0}^{\infty}\frac{\cancel{5^n}}{\cancel{5^n}\sqrt{n-1}} = \sum_{n=2}^{\infty}\frac{1}{\sqrt{n-1}} \geq \sum_{n=2}^{\infty}\frac{1}{n^{1/2}}$$ Diverges p-series $p = 1/2 \ngtr 1$

Series diverges by comparison Test

4. Find the radius and interval of convergence for $\sum_{n=0}^{\infty} \frac{(2x-3)^{n+1} n!}{100^n}$. FULLY explain your reasoning.

Center: $a = \frac{3}{2}$ Goal: $|x-a| < R$

$$\lim_{n\to\infty} \left| \frac{(2x-3)^{(n+1)+1}(n+1)!}{100^{n+1}} \cdot \frac{100^n}{(2x-3)^{n+1} n!} \right|$$

$$\lim_{n\to\infty} \left| \frac{(2x-3)^{n+1}(2x-3)}{(2x-3)^{n+1}} \cdot \frac{(n+1)\cdot n!}{n!} \cdot \frac{100^n}{100^n \cdot 100} \right|$$

$$\lim_{n\to\infty} \left| \frac{(2x-3)(n+1)}{100} \right| = \infty < 1$$

"Never" less than one

"Never" converges

$\boxed{R = 0}$ $\boxed{I: \left\{\frac{3}{2}\right\}}$

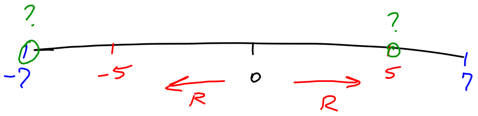
~~5. If $\sum_{n=0}^{\infty} c_n(x+2)^n$ converges at $x = 5$, on what interval are we guaranteed convergence?~~

New Question:

Suppose we know:

$\sum_{n=0}^{\infty} c_n x^n$ Converges at $x = -5$

Diverges at $x = 7$



$\sum_{n=0}^{\infty} c_n 8^n$

Diverges

$\sum_{n=0}^{\infty} c_n 5^n$

Inconclusive

$\sum_{n=0}^{\infty} c_n 3^n$ Converges

6. For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+3)!}$, Use the Alternating Series Estimation Theorem to find an upper bound for the error if we used s_5 to estimate the sum.

$$|R_n| \leq b_{n+1} \qquad b_n = \frac{n^2}{(n+3)!}$$

$$|R_5| \leq b_6 = \boxed{\frac{6^2}{9!}}$$

7. Using The Alternating Series Estimation Theorem, what is the smallest value of n that guarantees s_n approximates $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+3}$ with error less than $\frac{1}{20}$?

$b_n = \frac{1}{2n+3}$

$b_{n+1} = \frac{1}{2(n+1)+3}$

$\frac{1}{2n+5}$

$|R_n| \leq \underbrace{\frac{1}{2n+5} \leq \frac{1}{20}}$

$20 \leq 2n+5$

$15 \leq 2n$

$n \geq 7.5$

$\boxed{n = 8}$

Maclaurin Series

8. Find a power series centered at 0 for the following functions:

Relate to $\frac{1}{1-\boxed{x}} = \sum_{n=0}^{\infty} \boxed{x}^n$

a.) $\frac{4}{6-x^2} = 4 \cdot \frac{1}{6-x^2} =$

$= 4 \frac{1}{6\left(1-\frac{x^2}{6}\right)} = \frac{4}{6} \cdot \frac{1}{1-\boxed{\frac{x^2}{6}}} = \frac{4}{6} \sum_{n=0}^{\infty} \left(\frac{x^2}{6}\right)^n$

$\frac{4}{6} \cdot \sum_{n=0}^{\infty} \frac{x^{2n}}{6^n} = \boxed{\sum_{n=0}^{\infty} \frac{4x^{2n}}{6^{n+1}}}$

b.) $\frac{8x}{(6-x^2)^2}$, by using the result from above.

$$\frac{4}{6-x^2} = \sum_{n=0}^{\infty} \frac{4\ x^{2n}}{6^{n+1}}$$

$$= \underbrace{\frac{4x^0}{6}}_{n=0} + \underbrace{\frac{4x^2}{6^2}}_{n=1} + \underbrace{\frac{4x^4}{6^3}}_{n=2} \cdots$$

$$-\int \frac{-4 \cdot 2x}{(6-x^2)^2}\,dx \qquad u = 6-x^2 \qquad du = -2x\,dx \qquad -4\int u^{-2}\,du$$

$$-4\left(-u^{-1}\right) = \frac{4}{u} = \frac{4}{6-x^2}$$

$$\frac{d}{dx}\left[\int \frac{8x}{(6-x^2)^2}\,dx\right] = \frac{d}{dx}\left[\frac{4}{6-x^2}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{4\ x^{2n}}{6^{n+1}}\right]$$

$$\frac{8x}{(6-x^2)^2} = \sum_{n=1}^{\infty} \frac{4 \cdot 2n\ x^{2n-1}}{6^{n+1}}$$

Shift index back to $n=0$

$$= \sum_{n=0}^{\infty} \frac{4 \cdot 2(n+1)\ x^{2(n+1)-1}}{6^{(n+1)+1}}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{8(n+1)\ x^{2n+1}}{6^{n+2}}}$$

c.) $\int x^4 \arctan(5x)\,dx$

$$\arctan(5x) = \sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+1}}{2n+1}$$

$$\int x^4 \cdot \arctan(5x)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+5}}{2n+1}\,dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\frac{1}{1-\boxed{x}} = \sum_{n=0}^{\infty} \boxed{x}^n$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \quad ??$$

$$\boxed{\int x^4 \arctan(5x)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+6}}{(2n+1)(2n+6)}}$$

9. Evaluate $\int_0^1 x^4 \ln(2-x^3)\,dx$

Goal: Relate $\frac{1}{1-x}$

Power Series $\ln(2-x^3)$

$$\frac{d}{dx}\left[\ln(2-x^3)\right] = \frac{1}{2-x^3}\cdot -3x^2 = \frac{-3x^2}{2}\,\frac{1}{1-\boxed{\frac{x^3}{2}}} = \frac{-3x^2}{2}\sum_{n=0}^{\infty}\left(\frac{x^3}{2}\right)^n$$

$$= \frac{-3x^2}{2}\sum_{n=0}^{\infty}\frac{x^{3n}}{2^n} = \sum_{n=0}^{\infty}\frac{-3\cdot x^{3n+2}}{2^{n+1}}$$

$$\int \frac{d}{dx}\left[\ln(2-x^3)\right]dx = \int \sum_{n=0}^{\infty}\frac{-3\,x^{3n+2}}{2^{n+1}}\,dx$$

$$\ln(2-x^3) = C + \sum_{n=0}^{\infty}\frac{-3\,x^{3n+3}}{2^{n+1}\,(3n+3)}$$

$3(n+1)$

Let $x=0$

$\ln(2) = C + \Sigma 0$

$$\int x^4 \ln(2-x^3)\,dx = \int x^4 \ln(2)\,dx + \int \sum_{n=0}^{\infty}\frac{-x^{3n+3+4}}{2^{n+1}\cdot(n+1)}\,dx$$

$$\int_0^1 x^4 \ln(2-x^3)\,dx = \frac{\ln(2)}{5}x^5\Big|_0^1 + \sum_{n=0}^{\infty}\frac{-x^{3n+8}}{2^{n+1}(n+1)(3n+8)}\Big|_0^1$$

$$= \boxed{\frac{\ln(2)}{5} + \sum_{n=0}^{\infty}\frac{-1}{2^{n+1}(n+1)(3n+8)}}$$

$$\ln(1-x) = \sum_{n=0}^{\infty}\frac{-x^{n+1}}{n+1}$$

$$\ln(2-x^3) = \ln\left[2\left(1-\frac{x^3}{2}\right)\right] = \ln(2) + \ln\left(1-\frac{x^3}{2}\right)$$

$$\ln(2) + \sum_{n=0}^{\infty}\frac{-\left(\frac{x^3}{2}\right)^{n+1}}{n+1} = \ln 2 + \sum_{n=0}^{\infty}\frac{-x^{3n+3}}{2^{n+1}(n+1)}$$

10. Find $f^{(26)}(2)$ if $f(x) = \sum_{n=0}^{\infty} \frac{3^{n+1}(x-2)^n}{(n+8)!}$ is the Taylor Series for $f(x)$ centered at $a = 2$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{3^{n+1}(x-2)^n}{(n+8)!}$$

$$\frac{f^{(n)}(2)}{n!} = \frac{3^{n+1}}{(n+8)!}$$

Plug in $n = 26$

$$\frac{f^{(26)}(2)}{26!} = \frac{3^{27}}{34!}$$

$$f^{(26)}(2) = \frac{3^{27} \cdot 26!}{34!}$$

$$\boxed{f^{(26)}(2) = \frac{3^{27} \cdot \cancel{26!}}{34 \cdot 33 \cdot 32 \cdot 31 \cdot 30 \cdot 29 \cdot 28 \cdot 27 \cdot \cancel{26!}}}$$

11. Find the Taylor Series centered at 4 for $f(x) = \dfrac{1}{(x+1)^2}$.

$$(n+1)\cdot n!$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \overset{\uparrow}{(n+1)!}}{5^{n+2}\; n!}(x-4)^n$$

n	$f^{(n)}(x)$	$f^{(n)}(4)$
$n=0$	$(x+1)^{-2}$	$\frac{1}{5^2}$
$n=1$	$-2\,(x+1)^{-3}$	$\frac{-2}{5^3}$
$n=2$	$2\cdot 3\,(x+1)^{-4}$	$\frac{2\cdot 3}{5^4}$
$n=3$	$-2\cdot 3\cdot 4\,(x+1)^{-5}$	$\frac{-2\cdot 3\cdot 4}{5^5}$

$$\boxed{\frac{1}{(x+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n\cdot(n+1)}{5^{n+2}}(x-4)^n}$$

$$f^{(n)}(4) = \frac{(-1)^n\,(n+1)!}{5^{n+2}}$$

12. Find a Maclaurin series for e^{3x^2}.

Taylor Series centered @ $a=0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$e^{3x^2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!}}$$

e^{3x^2} Centered @ $x=1$

13. Express $\int x^4 \cos(5x^3)\, dx$ as a power series about 0.

$$\cos(5x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (5x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{6n}}{(2n)!}$$

$$\int x^4 \cos(5x^3)\, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{6n+4}}{(2n)!}\, dx$$

$$\boxed{\int x^4 \cos(5x^3)\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{6n+5}}{(2n)!\,(6n+5)}}$$

14. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!}$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \boxed{\sin(2)}$$

$(-1)^n \quad 5^n$

15. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-5)^n 2^{2n+1}}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{(-5)^n \; 2^{2n} \cdot 2}{n!} = 2 \cdot \sum_{n=0}^{\infty} \frac{(-5)^n \cdot 4^n}{n!}$$

$$= 2 \sum_{n=0}^{\infty} \frac{(-5\cdot 4)^n}{n!} = 2 \underbrace{\sum_{n=0}^{\infty} \frac{(-20)^n}{n!}}_{e^{-20}}$$

$$= \boxed{2 \cdot e^{-20}} = \frac{2}{e^{20}}$$

Homework

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

$$e^3 = \sum_{n=0}^{\infty} \frac{3^n}{n!}$$

$$e^3 = \underset{(n=0)}{1} + \sum_{n=1}^{\infty} \frac{3^n}{n!}$$

$$= \underset{(n=0)}{1} + \underset{(n=1)}{3} + \sum_{n=2}^{\infty} \frac{3^n}{n!}$$

16. Find the third degree Taylor Polynomial for $f(x) = e^{-x}$ at $x = 2$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \underset{(n=0)}{\frac{e^{-2}}{0!}(x-2)^0} - \underset{n=1}{\frac{e^{-2}}{1!}(x-2)^1} + \underset{(n=2)}{\frac{e^{-2}}{2!}(x-2)^2} \underset{(n=3)}{- \frac{e^{-2}}{3!}(x-2)^3}$$

n	$f^{(n)}(x)$	$f^{(n)}(2)$
$n=0$	e^{-x}	e^{-2}
$n=1$	$-e^{-x}$	$-e^{-2}$
$n=2$	e^{-x}	e^{-2}
$n=3$	$-e^{-x}$	$-e^{-2}$

$f^{(n)}(2) = (-1)^n e^{-2}$

$$\boxed{T_3(x) = e^{-2} - e^{-2}(x-2) + \frac{e^{-2}}{2}(x-2)^2 - \frac{e^{-2}}{6}(x-2)^3}$$