Fall 2019 Math 152

Week in Review 3

courtesy: Amy Austin

(covering section 6.3-6.4)

Section 6.3

1. Find the volume of the solid obtained by rotating the region bounded by the given curve(s) about the specified axis.

   a.) $y = 10x - x^2$, $y = 0$ about the y axis.

   b.) $y = x^3$, $y = 0$, $x = 1$, $x = 2$, about the line $x = -1$.

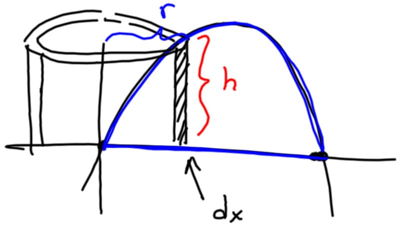
   c.) $y = x^2$ and $y = 4 - x^2$, about the line $x = \sqrt{2}$.

(a) $y = 10x - x^2$ $\quad 0 = x(10 - x)$ $\quad x = 0$

$10 - x = 0 \quad x = 10$

$A_i = 2\pi r \cdot h$ $\quad r = x - 0$

$h = (10x - x^2) - 0$



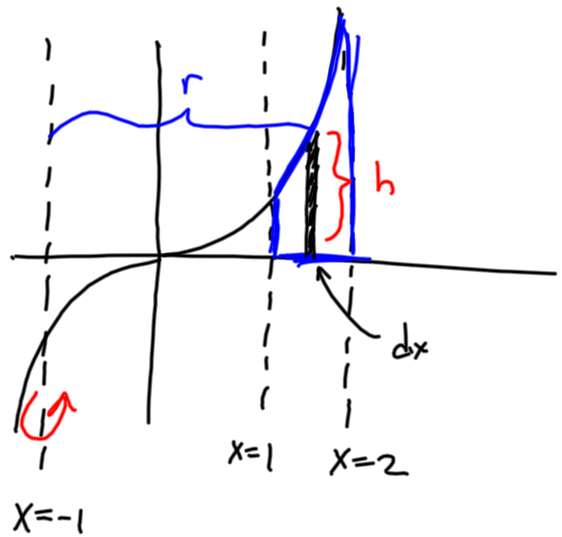
$V_i = 2\pi (x)(10x - x^2)\,dx$

$$\boxed{V = 2\pi \int_0^{10} x\,(10x - x^2)\,dx}$$

b.) $y = x^3$, $y = 0$, $x = 1$, $x = 2$, about the line $x = -1$.

$r = x - (-1) = x + 1$

$h = x^3 - 0 = x^3$



$$\boxed{V = 2\pi \int_1^2 (x+1)\, x^3 \, dx}$$

$$2\pi \int_1^2 x^4 + x^3 \, dx$$

$$2\pi \left[\frac{1}{5}x^5 + \frac{1}{4}x^4\right]_1^2$$

$$\boxed{2\pi \left(\left[\frac{32}{5} + \frac{16}{4}\right] - \left[\frac{1}{5} + \frac{1}{4}\right] \right)}$$

c.) $y = x^2$ and $y = 4 - x^2$, about the line $\underline{x = \sqrt{2}}$.

Bounds: $x^2 = 4 - x^2$

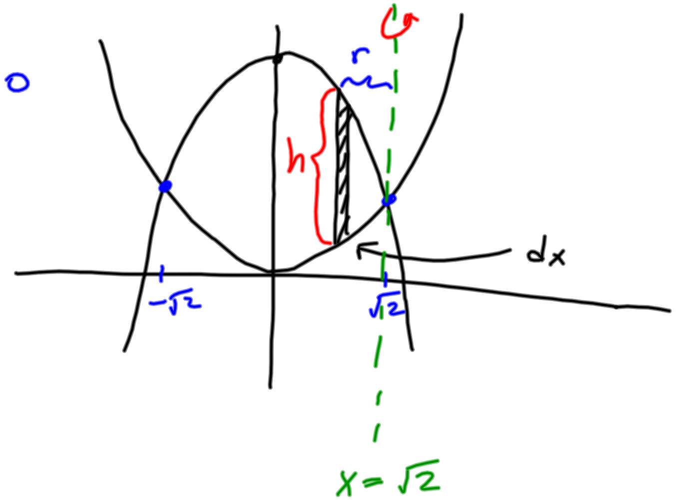
$2x^2 = 4$

$x^2 = 2$

$x^2 - 2 = 0$

$(x+\sqrt{2})(x-\sqrt{2}) = 0$

$x = -\sqrt{2} \quad x = \sqrt{2}$



$r = \sqrt{2} - x$

$h = (4 - x^2) - x^2 = 4 - 2x^2$

$$\boxed{V = 2\pi \int_{-\sqrt{2}}^{\sqrt{2}} (\sqrt{2} - x)(4 - 2x^2)\, dx}$$

2. Using two different methods, set up but do not evaluate the integral that gives the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 3x$, about the y axis.

(a) Shell Method:

$r = x - 0 = x$

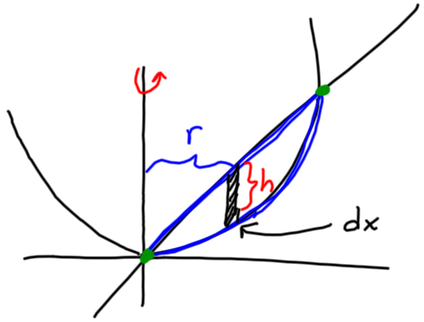
$h = 3x - x^2$

$3x = x^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

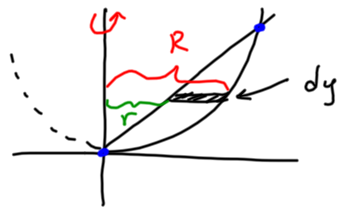
$x = 0 \quad x = 3$



$$\boxed{V = 2\pi \int_0^3 x(3x - x^2)\, dx}$$

$y = 3x \qquad y = x^2$

$x = \frac{y}{3} \qquad x = \sqrt{y}$



$\frac{y}{3} = \sqrt{y}$

$\frac{y^2}{9} = y$

$\frac{y^2}{9} - y = 0$

$y^2 - 9y = 0$

$y(y-9) = 0$

$y = 0 \quad y = 9$

$R = \sqrt{y} - 0 = \sqrt{y}$

$r = \frac{y}{3} - 0 = \frac{y}{3}$

$$\boxed{V = \pi \int_0^9 (\sqrt{y})^2 - \left(\frac{y}{3}\right)^2 \, dy}$$

3. Using two different methods, set up but do not evaluate the integral that gives the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $x = 0$, $x = 4$, $y = 0$, about the line $y = 3$.

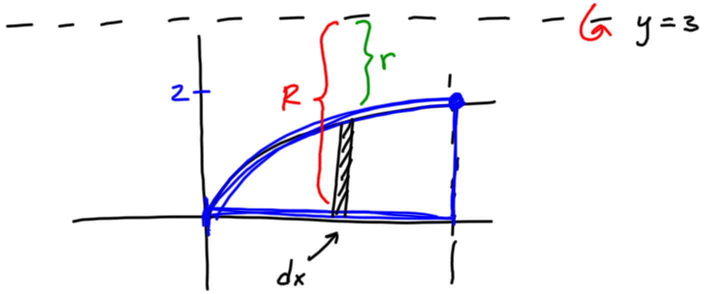
$y = \sqrt{4} = 2$

$R = 3 - 0 = 3$

$r = 3 - \sqrt{x}$

$x = 0$

$x = 4$

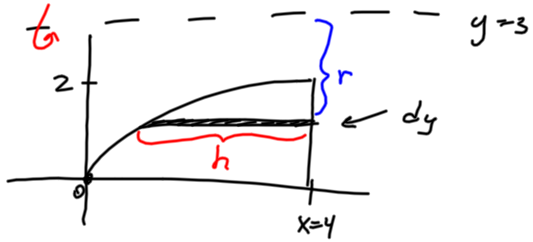


$$\boxed{V = \pi \int_0^4 (3)^2 - (3 - \sqrt{x})^2 \, dx}$$

$y = \sqrt{x} \Rightarrow x = y^2$

$h = 4 - y^2$

$r = 3 - y$



$$\boxed{V = 2\pi \int_0^2 (3 - y)(4 - y^2) \, dy}$$

Section 6.4

4. How much work is done in lifting a 30 lb barbell from the floor to a height of 4 feet?

$$W = F \cdot d = 30 \text{ lbs} \cdot 4 \text{ ft.} = \boxed{120 \text{ ft-lbs}}$$

5. When a particle is at a distance x meters from the origin, a force of $f(x) = 3x^2 + 2$ Newtons acts on it. How much work is done in moving the object from $x = 2$ to $x = 4$?

$$W = \int F \cdot dx$$

$$W = \int_2^4 3x^2 + 2 \quad dx$$

$$x^3 + 2x \Big|_2^4 = (64 + 8) - (8 + 4)$$

$$72 - 12 \quad = 60 \quad N \cdot m = \boxed{60 \text{ J}}$$

6. A spring has a natural length of ~~6 inches~~ 0.5 ft. If a 5-lb force is required to maintain it to a length of ~~18 inches~~ 1.5 ft., how much work is required to stretch it from 1 foot to 3 feet?

$$W = \int f(x)\, dx$$

Spring Force : $\underbrace{f(x) = kx}$

① Find k

$$5 \text{ lbs} = k\,(1.5 - 0.5)$$

② Answer Question

$$5 = k\,(1) \qquad k = 5$$

Spring Force : $f(x) = 5x$

$$W = \int f(x)\, dx = \int_{1-0.5}^{3-0.5} 5x\, dx = \int_{0.5}^{2.5} 5x\, dx = \boxed{\left.\frac{5x^2}{2}\right|_{0.5}^{2.5}}$$

7. Suppose the work needed to stretch a spring from its natural length to a length of 5 feet beyond its natural length is 30 ft-lb.

* Find k

$W = \int \underbrace{f(x)}_{kx}\,dx = \int_0^5 kx\,dx = 30$

$\frac{kx^2}{2}\Big|_0^5 = \frac{25}{2}k = 30 \qquad 25k = 60 \qquad k = \frac{60}{25} = \frac{12}{5}$

a.) How much work is done in stretching the spring from 3 feet beyond its natural length to ~~120 inches~~ 10 ft. beyind its natural length?

$W = \int_3^{10} \frac{12}{5}x\,dx$

$\frac{12}{5}\,\frac{x^2}{2}\Big|_3^{10} = \frac{6}{5}(100-9) = \boxed{\frac{6}{5}(91) \text{ ft-lbs}}$

b.) How far beyond its natural length will a force of 60 lb keep the spring stretched?

$f(x) = \frac{12}{5}x \qquad 60 = \frac{12}{5}(x)$

$\frac{5}{12}(60) = x$

$\boxed{x = 25 \text{ ft.}}$

8. A heavy rope, 50 feet long, weighs 0.5 pounds per foot and hangs over the edge of a building 120 feet high. How much work is done in pulling the rope to the top of the building?

Come up with a force function $f(x)$ to describe the weight remaining on the rope after x feet have been pulled up.

Total weight = 50 ft $\cdot \frac{1}{2}$ lbs/ft = 25 lbs

$$\left.\begin{aligned} f(0) &= 25 \\ f(1) &= 25 - 0.5(1) \\ f(2) &= 25 - 0.5(2) \end{aligned}\right\} \Longrightarrow f(x) = 25 - 0.5x$$

$$W = \int f(x)\,dx = \int_0^{50} 25 - 0.5x \,dx = 25x - \tfrac{1}{4}x^2 \Big|_0^{50}$$

$$\boxed{25(50) - \tfrac{1}{4}(50)^2}$$

9. A 200 pound cable is 300 feet long and hangs vertically from the top of a tall building. How much work is required to pull 20 feet of the cable to the top of the building?

$f(x) = \text{Total Weight} - \underline{\text{rope density}} \cdot (x)$

$d = \frac{200 \text{ lbs}}{300 \text{ ft.}} = \frac{2}{3} \frac{\text{lbs}}{\text{ft.}}$

$f(x) = 200 - \frac{2}{3}x$

$\int_0^{20} 200 - \frac{2}{3}x \, dx = 200x - \frac{1}{3}x^2 \Big|_0^{20}$

$200(20) - \frac{1}{3}(20)^2$

$\boxed{4000 - \frac{400}{3} \text{ ft-lbs}}$

100 ft rope / 5 lbs/ft

Rope Weight = $100 \cdot 5 = 500$ lbs

Holds crate weighs 300 lbs

Crate Weight = 300 lbs

Work to pull first 40 ft of rope up.

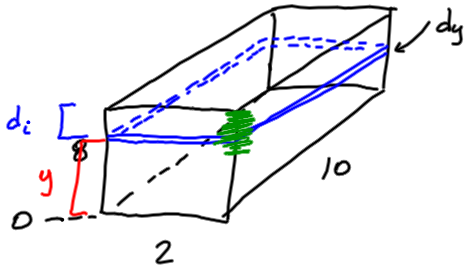
Total Weight = 800 lbs

$f(x) = 800 - 5x$

$\boxed{W = \int_0^{40} 800 - 5x \, dx}$

Water density $= \rho = 1000$ kg/m^3

10. An aquarium 10 m long, 2 m wide and 8 m deep is full of water. Find the work required to pump the top 3 ~~feet~~ meters of water to the top of the aquarium.



$W_i = F_i \cdot d_i$

$F_i = M_i \cdot a_i$

$M_i = V_i \cdot \rho_i$

$V_i = A_i \cdot dy$

$A_i =$ change each question

$W_i = 20 \ \rho g \ (8-y) \ dy$

$F_i = 20 \ \rho g \ dy$

$M_i = 20 \cdot \rho \ dy$

$V_i = 20 \ dy$

$A_i = 2 \cdot 10 = 20$

$$\boxed{W = \rho g \int_5^8 20 \, (8-y) \, dy}$$

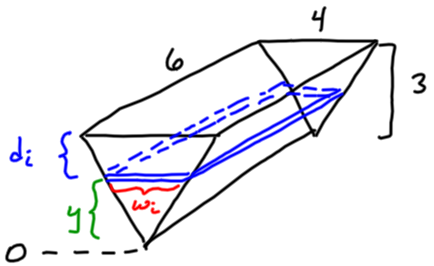
Top 3 feet : 5 - 8

$\rho = 1000$ $g = 9.8$

American units for water
$\rho g = 62.5$

$$W = \rho g \int A_i \cdot d_i \cdot dy$$

11. A tank contains water and has the shape of a trough 6 feet long. The end of the trough is an isosceles triangle with height 3 feet and base length 4 feet. The vertex of the triangle is at the bottom. Find the work required to pump all of the water to the top of the tank.



$A_i = l_i \cdot w_i$

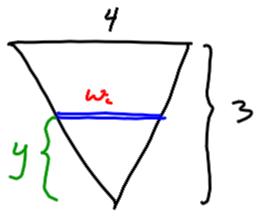
$l_i = 6$

$A_i = 6 \cdot w_i$

$A_i = 6 \cdot \frac{4}{3} y$

$A_i = 8y$

$d_i = 3 - y$

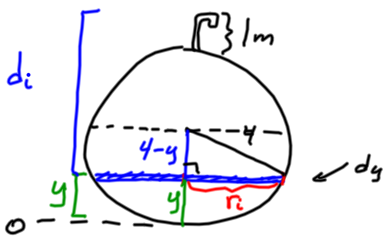


$$\frac{4}{3} = \frac{w_i}{y}$$

$$w_i = \frac{4}{3} y$$

$$\boxed{W = \rho g \int_0^3 8y\,(3-y)\,dy}$$

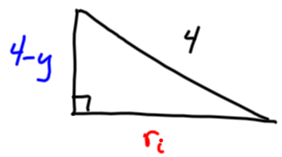
12. A tank in the shape of sphere with radius 4 m is half full of water. The water is pumped from a spout at the top of the tank that is 1 m high. Set up but do not evaluate an integral done in pumping the water through the spout. CLEARLY MARK YOUR AXIS AND WHAT DIRECTION IS POSITIVE!



$A_i = \pi (r_i)^2$

$A_i = \pi \left(16 - (4-y)^2\right)$

$d_i = 9 - y$

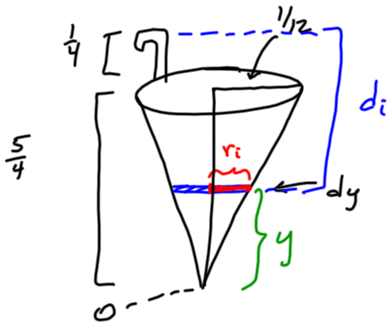


$(r_i)^2 + (4-y)^2 = 4^2$

$r_i^2 = 16 - (4-y)^2$

$$\boxed{W = \rho g \int_0^4 \pi \left(16 - (4-y)^2\right) (9-y) \, dy}$$

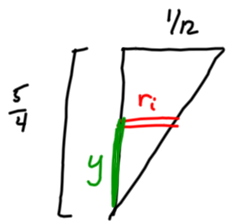
13. A tank in the shape of cone with radius 1 inch ($\frac{1}{12}$ ft) and height 15 inches ($\frac{5}{4}$ ft) is full of water to a depth of 7 inches ($\frac{7}{12}$ ft). Set up but do not evaluate an integral done in pumping the water through the spout. ? 3 inch $\frac{1}{4}$ ft.
CLEARLY MARK YOUR AXIS AND WHAT DIRECTION IS POSITIVE!



$A_i = \pi (r_i)^2$

$$A_i = \pi \left(\frac{1}{15} y\right)^2 = \frac{\pi y^2}{225}$$

$$d_i = \underset{\left(\frac{5}{4}+\frac{1}{4}\right)}{\frac{3}{2}} - y$$



$$\frac{1/12}{5/4} = \frac{r_i}{y} \qquad \frac{1}{15} = \frac{r_i}{y}$$

$$\frac{1}{12} \cdot \frac{4}{5} = \frac{1}{15} \qquad r_i = \frac{1}{15} y$$

$$\boxed{W = \underbrace{\rho g}_{\rho g = 62.5} \int_0^{7/12} \frac{\pi y^2}{225} \left(\frac{3}{2} - y\right) dy}$$