Fall 2019 Math 152

Week in Review 4

courtesy: Amy Austin

(covering section 7.3-7.8)

Section 7.3 Trigonometric Substitutions

1. $\int x^3\sqrt{4-x^2}\,dx$

Pretend this is $1-x^2$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

Since we square this, we need to introduce a coefficient that will match with the 4 after squaring

$$\int (2\sin\theta)^3 \cdot \sqrt{4-(2\sin\theta)^2}\; 2\cos\theta\, d\theta$$

$$\sqrt{4-4\sin^2\theta}$$

$$\sqrt{4(1-\sin^2\theta)}$$

$$\sqrt{4\cos^2\theta}$$

$$\int 2^3\sin^3\theta \cdot 2\cos\theta \cdot 2\cos\theta\, d\theta$$

$$\int 2^5 \sin^3\theta\cos^2\theta\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$x = 2\sin\theta$

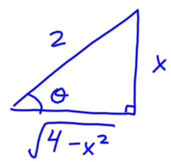
$\sin\theta = \frac{x}{2} = \frac{opp.}{hyp.}$

$$\int 2^5\, \underline{\sin^2\theta}\; \underline{\cos^2\theta}\; \underline{\sin\theta\, d\theta}$$

$$-\int 2^5 (1-\cos^2\theta)\cos^2\theta(-\sin\theta)\,d\theta$$

$$-2^5\left[\frac{\cos^3\theta}{3}-\frac{\cos^5\theta}{5}\right]$$

$\cos\theta = \frac{adj}{hyp.} = \frac{\sqrt{4-x^2}}{2}$



$$-2^5\int (1-u^2)u^2\, du$$

$$-2^5\int u^2-u^4$$

$$-2^5\left[\frac{u^3}{3}-\frac{u^5}{5}\right]$$

$$\boxed{-2^5\left[\frac{(4-x^2)^{3/2}}{3\cdot 2^3}-\frac{(4-x^2)^{5/2}}{5\cdot 2^5}\right]+C}$$

2. $\int_{5\sqrt{2}}^{10} \frac{dx}{x^3\sqrt{x^2-25}}$

$x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

Bounds:

Top: $x = 10$ $\quad x = 5\sec\theta$

$10 = 5\sec\theta \quad 2 = \sec\theta$

$\frac{1}{2} = \cos\theta \quad \theta = \frac{\pi}{3}$

Bot: $x = 5\sqrt{2} \quad x = 5\sec\theta$

$5\sqrt{2} = 5\sec\theta \quad \sqrt{2} = \sec\theta$

$\frac{1}{\sqrt{2}} = \cos\theta \quad \frac{\sqrt{2}}{2} = \cos\theta$

$\theta = \frac{\pi}{4}$

$$\int_{\pi/4}^{\pi/3} \frac{5\sec\theta\tan\theta\,d\theta}{5^3\sec^3\theta\sqrt{25\sec^2\theta - 25}}$$

$$\int_{\pi/4}^{\pi/3} \frac{5\sec\theta\tan\theta\,d\theta}{5^3\sec^3\theta \cdot 5\tan\theta}$$

$$\frac{1}{5^3}\int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta}\,d\theta = \frac{1}{5^3}\int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta$$

$$\frac{1}{5^3}\int_{\pi/4}^{\pi/3} \frac{1}{2}\left(1+\cos(2\theta)\right)d\theta = \frac{1}{2\cdot 5^3}\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Bigg|_{\pi/4}^{\pi/3}$$

$$\frac{1}{2\cdot 5^3}\left[\left(\frac{\pi}{3} + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\right) - \left(\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right)\right]$$

$$\frac{1}{2\cdot 5^3}\left[\left(\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) - \left(\frac{\pi}{4} + \frac{1}{2}(1)\right)\right]$$

$$\boxed{\frac{1}{250}\left(\frac{\pi}{12} + \frac{\sqrt{3}}{4} - \frac{1}{2}\right)}$$

3. $\int \frac{1}{\sqrt{x^2+4x+8}}\,dx$

$(x \pm a)^2 = x^2 \pm 2ax + a^2$

$(x^2 + 4x + 4) + 8 - 4$

$\Rightarrow a = 2$

$(x+2)^2 + 4$

$$\int \frac{1}{\sqrt{(x+2)^2+4}}\,dx$$

$x+2 = 2\tan\theta$

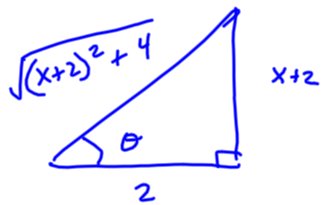
$dx = 2\sec^2\theta\,d\theta$

$$\int \frac{1}{\sqrt{4\tan^2\theta+4}}\,2\sec^2\theta\,d\theta = \int \frac{1}{2\sec\theta}\cdot 2\sec^2\theta\,d\theta$$

$$\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$$

$x+2 = 2\tan\theta$

$\tan\theta = \frac{x+2}{2} = \frac{\text{opp.}}{\text{adj.}}$



$\sec\theta = \frac{\sqrt{(x+2)^2+4}}{2}$

$$\boxed{\ln\left|\frac{\sqrt{(x+2)^2+4}}{2} + \frac{x+2}{2}\right| + C}$$

Techniques of Integration

1. Take the Antiderivative

5. u-sub

6. Int. by parts

2. Trig Integrals

4. Trig Substitution $\rightarrow$ $x^2 \pm 1$

   $1 \pm x^2$

3. Partial Fractions

4. $\int_0^{2/3} \frac{1}{(4+9x^2)^{5/2}}\,dx$

$x = \frac{2}{3}\tan\theta$

$dx = \frac{2}{3}\sec^2\theta\, d\theta$

Bounds:

Top: $x = \frac{2}{3} = \frac{2}{3}\tan\theta$

$1 = \tan\theta$

$1 = \frac{\sin\theta}{\cos\theta}$ $\quad \theta = \frac{\pi}{4}$

Bot: $x = 0 = \frac{2}{3}\tan\theta$

$0 = \tan\theta$

$0 = \frac{\sin\theta}{\cos\theta}$ $\quad \theta = 0$

$$\int_0^{\pi/4} \frac{\frac{2}{3}\sec^2\theta}{\left[4 + 9\left(\frac{4}{9}\tan^2\theta\right)\right]^{5/2}}\,d\theta$$

$$\int_0^{\pi/4} \frac{\frac{2}{3}\sec^2\theta\, d\theta}{\left(4\sec^2\theta\right)^{5/2}}$$

$$\int_0^{\pi/4} \frac{\frac{2}{3}\sec^2\theta\, d\theta}{2^5 \cdot \sec^5\theta} = \frac{1}{3\cdot 2^4}\int_0^{\pi/4} \frac{1}{\sec^3\theta}\,d\theta$$

$$\frac{1}{3\cdot 2^4}\int_0^{\pi/4} \cos^3\theta\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

Bounds

$u\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$u(0) = \sin(0) = 0$

$$\frac{1}{3\cdot 2^4}\int_0^{\pi/4} \left(1 - \sin^2\theta\right)\cos\theta\, d\theta$$

$$\frac{1}{3\cdot 2^4}\int_0^{\frac{\sqrt{2}}{2}} 1 - u^2\, du = \frac{1}{3\cdot 2^4}\left[u - \frac{u^3}{3}\right]\Bigg|_0^{\sqrt{2}/2}$$

$$\frac{1}{3\cdot 2^4}\left[\frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right] - 0$$

Section 7.4

Factor Denominator : $x\,(2x+3)\,(x+1)$

5. $\displaystyle\int \frac{4x+5}{2x^3+5x^2+3x}\,dx$

$$\left[\frac{4x+5}{x\,(2x+3)\,(x+1)} = \frac{A}{x} + \frac{B}{2x+3} + \frac{C}{x+1}\right] x\cdot(2x+3)(x+1)$$

$$4x+5 = A\,(2x+3)(x+1) + B\,x\,(x+1) + C\,x\,(2x+3)$$

Plug in smart x-values to solve for A, B, C,

$\underline{x=-1}$: $\quad 4(-1)+5 = 0 + 0 + C\,(-1)(1)$

$$1 = -C \qquad C = -1$$

$\underline{x=0}$: $\quad 4(0)+5 = A\,(3)(1) + 0 + 0 \qquad 5 = 3A \Rightarrow A = \frac{5}{3}$

$x = \frac{-3}{2}$: $\quad 4\left(\frac{-3}{2}\right)+5 = 0 + B\left(\frac{-3}{2}\right)\left(\frac{-1}{2}\right) + 0$

$$-1 = \frac{3}{4}B \qquad B = -\frac{4}{3}$$

$$\int \frac{5/3}{x} + \frac{-4/3}{2x+3} + \frac{-1}{x+1}\,dx$$

$$\boxed{\frac{5}{3}\ln|x| - \frac{4}{3}\cdot\frac{1}{2}\ln|2x+3| - \ln|x+1| + C}$$

6. $\int \frac{x^3+2x+1}{x^2+4x}\,dx$

$$\begin{array}{r l}
 & x - 4 \\
x^2 + 4x + 0 & \overline{\big)\, x^3 + 0x^2 + 2x + 1} \\
 & -(x^3 + 4x^2 + 0x) \\
\hline
 & -4x^2 + 2x + 1 \\
 & -(-4x^2 - 16x + 0) \\
\hline
 & 18x + 1
\end{array}$$

$$= \int x - 4 + \frac{18x+1}{x^2+4x}$$

Concentrate on this

$$\left[\frac{18x+1}{x(x+4)} = \frac{A}{x} + \frac{B}{x+4}\right] \times (x+4)$$

$$18x+1 = A(x+4) + Bx$$

$x=0$: $\quad 1 = 4A + 0 \quad \Rightarrow A = \frac{1}{4}$

$x=-4$: $\quad -71 = 0 + -4B \quad \Rightarrow B = \frac{71}{4}$

$$\int x - 4 + \frac{1/4}{x} + \frac{71/4}{x+4}\,dx$$

$$\boxed{\frac{x^2}{2} - 4x + \frac{1}{4}\ln|x| + \frac{71}{4}\ln|x+4| + C}$$

7. $\int_1^2 \frac{dx}{x(x^2+2x+1)}$

$$\left[\frac{1}{x\,(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\right] \times (x+1)^2$$

$$1 = A(x+1)^2 + Bx\cdot(x+1) + Cx$$

$x=0:$ $\quad 1 = A + 0 + 0 \qquad A=1$

$x=-1:$ $\quad 1 = 0 + 0 + -C \qquad C=-1$

$x=1:$ $\quad 1 = (1)\,4 + B(1)(2) + (-1)(1) \qquad B=-1$

$1 = 4 + 2B - 1 \qquad 2B = -2$

$1 = 3 + 2B \qquad B = -1$

$$\int \frac{1}{x} + \frac{-1}{x+1} + \frac{-1}{(x+1)^2}\,dx \qquad -1\int (x+1)^{-2}\,dx$$

$$(-1)\cdot -(x+1)^{-1}$$

$$\boxed{\left[\ln|x| - \ln|x+1| + \frac{1}{x+1}\right]_1^2}$$

8. $\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}\,dx$

$$\left[\frac{3x^2-4x+5}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}\right](x-1)(x^2+1)$$

$$3x^2-4x+5 = A(x^2+1) + (Bx+C)(x-1)$$

$\underline{x=1}$ $\quad 3-4+5 = 2A$

$4 = 2A$

$A = 2$

$$\underline{3x^2} \; \underline{-4x} \; \underline{+5} = \underline{Ax^2} + \underline{A} + \underline{Bx^2} + \underline{Cx} \; \underline{-Bx} \; \underline{-C}$$

Examine the coefficients:

$A=2$

x^2 : $\quad 3 = A + B \qquad 3 = 2 + B \qquad B=1$

x : $\quad -4 = C - B$

1 : $\quad 5 = A - C \qquad 5 = 2 - C \qquad 3 = -C \qquad C = -3$

$$\int \frac{2}{x-1} + \frac{x-3}{x^2+1}\,dx$$

$$\int \frac{2}{x-1} + \underset{(u\text{-sub})}{\frac{x}{x^2+1}} + \underset{(\text{antiderivative})}{\frac{-3}{x^2+1}}\,dx$$

$$\boxed{2\ln|x-1| + \frac{1}{2}\ln(x^2+1) - 3\arctan(x) + C}$$

Suppose $5\int \frac{1}{x^2+15} = 5 \cdot \frac{1}{\sqrt{15}} \arctan\left(\frac{x}{\sqrt{15}}\right)$

Section 7.8

9. Determine whether the following improper integrals converge or diverge. If it converges, find the value of the integral. If it diverges, explain why.

a.) $\int_e^\infty \frac{1}{x(\ln x)^4}\,dx$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^4}\,dx$$

$$\int \frac{1}{u^4}\,du = \int u^{-4}\,du = \frac{u^{-3}}{-3}$$

$$\lim_{t\to\infty} \left.\frac{-1}{3(\ln x)^3}\right|_e^t = \lim_{t\to\infty} \frac{-1}{3(\ln t)^3} + \frac{1}{3(\ln e)^3}$$

$$0 \qquad \boxed{\frac{1}{3}} \qquad \underline{\text{Converges}}$$

b.) $\int_0^\infty xe^{-x}\,dx$

Tabular Method

$$\begin{array}{cc} x & e^{-x} \\ 1 & -e^{-x} \\ 0 & e^{-x} \end{array}$$

$$\lim_{t\to\infty} \int_0^t xe^{-x}\,dx$$

(Integration by parts)

$$\lim_{t\to\infty} -xe^{-x} - e^{-x}\Big|_0^t$$

$$\lim_{t\to\infty} e^{-x}(-x-1)\Big|_0^t$$

$$\lim_{t\to\infty} e^{-t}(-t-1) - e^{-0}(-0-1) = 0 - (-1) = \boxed{1}$$

$$\lim_{t\to\infty} \frac{-t-1}{e^t} \overset{L'H}{=} \lim_{t\to\infty} \frac{-1}{e^t} = 0$$

Converges

c.) $\int_{-\infty}^{0} \frac{1}{1-2x}\,dx$

$$\lim_{t\to-\infty} \int_t^0 \frac{1}{1-2x}\,dx = \lim_{t\to-\infty} \frac{-1}{2} \ln|1-2x| \Big|_t^0$$

$$\lim_{t\to-\infty} \underbrace{\frac{-1}{2} \ln|1-2(0)|}_{0} + \frac{1}{2} \ln|1-2t|$$

$$\lim_{t\to-\infty} \frac{1}{2} \ln|1-2t| = \frac{1}{2} \ln|1-2(-\infty)| = \frac{1}{2} \ln|\infty| = \infty$$

$\boxed{\text{Diverges}}$

d.) $\int_{-\infty}^{\infty} \frac{dx}{x^2+9}$

$$\underbrace{\lim_{t\to-\infty} \int_t^0 \frac{dx}{x^2+9}}_{\frac{\pi}{6}} + \underbrace{\lim_{s\to\infty} \int_0^s \frac{dx}{x^2+9}}_{\frac{\pi}{6}} = \boxed{\frac{\pi}{3}}$$

$$\lim_{t\to-\infty} \frac{1}{3}\arctan\left(\frac{x}{3}\right)\Big|_t^0$$

$$\lim_{t\to-\infty} \underbrace{\frac{1}{3}\arctan(0)} - \frac{1}{3}\arctan\left(\frac{t}{3}\right) = \frac{1}{3}(0) - \frac{1}{3}\left(\frac{-\pi}{2}\right) = \frac{\pi}{6}$$

$\tan\theta = 0$

$\frac{\sin\theta}{\cos\theta} = 0$

$\theta = 0$

$\arctan 0 = 0$

$\lim_{t\to-\infty} \arctan \frac{t}{3}$

$\arctan(-\infty) = \theta$

$\tan\theta = -\infty$

$\frac{\sin\theta}{\cos\theta} = -\infty$

Need $\cos\theta = 0$ and $\sin\theta$ negative

$\theta = \frac{-\pi}{2}$

$\arctan(-\infty) = \frac{-\pi}{2}$

$+$

$$\lim_{s\to\infty} \frac{1}{3}\arctan\frac{x}{3}\Big|_0^s$$

$$\lim_{s\to\infty} \frac{1}{3}\underbrace{\arctan\left(\frac{s}{3}\right)}_{\pi/2} - \cancel{\frac{1}{3}\arctan 0}^{\,0}$$

$$\frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

$\arctan(\infty) = \theta$

$\tan\theta = \infty$

$\frac{\sin\theta}{\cos\theta} = \infty$

Need $\cos\theta = 0$ and $\sin\theta$ positive

$\theta = \frac{\pi}{2}$

$\arctan(\infty) = \frac{\pi}{2}$

e.) $\int_{-3}^{0} \frac{dx}{(x+3)^2}$ $= \lim_{t \to -3^+} \int \frac{dx}{(x+3)^2}$

Diverges

f.) $\int_0^3 \frac{1}{2x-1}\,dx$

$$\lim_{t \to \frac{1}{2}^-} \int_0^t \frac{1}{2x-1}\,dx + \lim_{s \to \frac{1}{2}^+} \int_s^3 \frac{1}{2x-1}\,dx$$

10. Determine whether the following integrals converge or diverge using the comparison theorem:

a.) $\int_0^\infty \frac{x-1}{x^{10}+e^{5x}}\,dx \leq \int_0^\infty \frac{x}{x^{10}+e^{5x}}\,dx \leq \int_0^\infty \frac{x}{e^{5x}}\,dx \Rightarrow$ Converges

$\int_0^\infty \frac{x-1}{x^{10}+e^{5x}}\,dx$ also converges by comparison theorem b.) $\int_2^{\infty} \frac{x}{x^{3/2} - x - 1}\,dx \geq \int_2^{\infty} \frac{x}{x^{3/2}}\,dx$

$$\int_2^{\infty} \frac{1}{x^{1/2}}\,dx \Rightarrow \text{Diverges}$$

(larger)

$\int_2^{\infty} \frac{x}{x^{3/2} - x - 1}\,dx$ also diverges by comparison theorem c.) $\int_1^\infty \frac{\cos^2 x}{x^4}\,dx$

$0 \le \cos^2 x \le 1$

$$\int_1^\infty \frac{\cos^2 x}{x^4}\,dx \le \int_1^\infty \frac{1}{x^4}\,dx \Rightarrow$$ Converges

p-integral $p=4>1$

$\boxed{\int_1^\infty \frac{\cos^2 x}{x^4}\,dx \text{ also converges by comparison theorem}}$

$$\int_1^{\infty} \frac{(\text{constant})}{x^p}\,dx \qquad \text{p-integral}$$

$p > 1$ integral converges

$p \leq 1$ integral diverges