Fall 2019 Math 152

Week in Review 5

courtesy: Amy Austin

(covering sections 11.1, 11.2)

Section 11.1

a_n terms of sequence

1. Find the limit of the following sequences, if it exists. If the sequence diverges, state why.

a.) $a_n = \dfrac{n}{\sqrt{n+2}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n+2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n\left(1+\frac{2}{n}\right)}} = \lim_{n\to\infty} \frac{n}{\sqrt{n}\cdot\sqrt{1+\frac{2}{n}}}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{1+\frac{2}{n}}} = \frac{\sqrt{\infty}}{\sqrt{1+\underbrace{\cancel{\tfrac{2}{\infty}}}_{0}}} = \infty \Rightarrow \boxed{\text{Diverges}}$$

b.) $a_n = \ln(n) - \ln(3n+1)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln(n) - \ln(3n+1) = \ln(\infty) - \ln(\infty) \ ????$$

$$= \lim_{n\to\infty} \ln\left(\frac{n}{3n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{3n+1}\right) = \ln\left(\frac{1}{3}\right)$$

$$= \boxed{\ln \frac{1}{3}}$$

Converges

$$\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \qquad \lim_{n\to\infty} \frac{6-5n^2}{3n+1} = -\infty \qquad \lim_{n\to\infty} \frac{3n+1}{6-5n^2} = 0$$

c.) $a_n = \frac{(-1)^n n}{n^2+1}$

Alternating!

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

$$|a_n| = \frac{n}{n^2+1}$$

Converges, $L = 0$

$$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

d.) $a_n = \dfrac{(-1)^n n^2}{n^2+1}$

Alternating!

$\boxed{\text{Diverges, by oscillation}}$

$|a_n| = \dfrac{n^2}{n^2+1}$

$\lim_{n\to\infty} \dfrac{1n^2}{1n^2+1} = 1$

e.) $a_n = \dfrac{\ln n}{n}$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1}$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$\boxed{\text{Converges, } L=0}$

Technically, we can't take the derivative of a sequence, so we should introduce a function $f(x) = \dfrac{\ln x}{x}$, but we are lazy, so we will skip that in the future. The function and the sequence have the same limit at infinity.

2. Suppose $\{a_n\}$ is a decreasing bounded sequence, $a_1 = 2$, and $a_{n+1} = \dfrac{1}{3 - a_n}$, find:

a.) a_4

b.) the limit of the sequence.

Note: Can't just evaluate $\lim_{n\to\infty} a_n$

$a_1 = 2$

$a_2 = \dfrac{1}{3-2} = \dfrac{1}{1} = 1$

$a_3 = \dfrac{1}{3-1} = \dfrac{1}{2}$

$a_4 = \dfrac{1}{3-\frac{1}{2}} = \dfrac{1}{5/2} = \dfrac{2}{5}$

$\boxed{a_4 = \frac{2}{5}}$

If a sequence is bounded and monotonic {either increasing or decreasing} then the sequence converges.

$\lim_{n\to\infty} a_n = L$

$\lim_{n\to\infty} a_{n+1} = L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3-a_n}$$

$$L = \frac{1}{3 - \lim_{n\to\infty} a_n} = \frac{1}{3-L}$$

$$(3-L)\, L = \frac{1}{\cancel{3-L}} \cdot \cancel{(3-L)}$$

$3L - L^2 = 1$

$0 = L^2 - 3L + 1$

$$L = \frac{3 \pm \sqrt{9 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2}$$

Recall: $a_1 = 2$ and sequence is decreasing

$L = \cancel{\dfrac{3+\sqrt{5}}{2}} > \dfrac{5}{2}$ or $\boxed{L = \dfrac{3-\sqrt{5}}{2}}$

3. Determine whether the following sequences are increasing, decreasing, or non monotonic.

a.) $a_n = \dfrac{1}{n^5}$

$$f(x) = \frac{1}{x^5} = x^{-5}$$

$$f'(x) = -5x^{-6} = \frac{-5}{x^6} \quad \text{Always positive} \quad = \frac{-}{+} = - < 0$$

f(x) is decreasing

$$\boxed{a_n \text{ is decreasing}}$$

b.) $a_n = \dfrac{\ln n}{n}$

Eventually, the function is decreasing,

so eventually $\boxed{a_n \text{ is also decreasing}}$

$$f(x) = \frac{\ln x}{x}$$

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln(x)\,(1)}{x^2} = \frac{1 - \ln(x)}{x^2}$$ ← Always positive

$$1 - \ln(x) = 0$$

$$1 = \ln(x)$$

$$x = e$$

	1	e	e^2
$f'(x)$	+		−
$f(x)$	Inc.		Dec.

c.) $a_n = \cos(n\pi) = (-1)^n$

$a_1 = \cos(\pi) = -1$

$a_2 = \cos(2\pi) = 1$

$a_3 = \cos(3\pi) = -1$

$a_4 = \cos(4\pi) = 1$

$\boxed{\text{Not monotonic}}$

4. Determine whether the following sequences are bounded.

Note: If a sequence converges, then it is bounded.

a.) $a_n = \left\{ \dfrac{1}{n^2} \right\}_{n=1}^{\infty}$

b.) $a_n = \left\{ \dfrac{n^2}{n+1} \right\}_{n=1}^{\infty}$

(a) $\lim_{n\to\infty} \frac{1}{n^2} = 0$ Converges $\Rightarrow$ $\boxed{\text{Bounded}}$

(b) $\lim_{n\to\infty} \frac{n^2}{n+1} = \infty$

Continues to grow

$\boxed{\text{Not Bounded}}$

(c). $a_n = \cos(n\pi) = (-1)^n$

Doesn't converge,

but <u>is</u> $\boxed{\text{bounded}}$

Section 11.2

5. Find the first 5 terms in the sequence of partial sums the series $\sum_{n=1}^{\infty}(1)$. Does the series coverge?

$S_n =$ Sum of the first n terms

$a_n = 1$

$S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 1 + 1 = 2$

$S_3 = a_1 + a_2 + a_3 = 1 + 1 + 1 = 3$

$S_4 = a_1 + a_2 + a_3 + a_4 = 1 + 1 + 1 + 1 = 4$

$S_5 = a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 1 + 1 + 1 + 1 = 5$

What is $S_n = n$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty} n = \infty$$

$$\boxed{\sum_{n=1}^{\infty} 1 \quad \text{Diverges}}$$

6. Find the first 5 terms in the sequence of partial sums S_n the series $\sum_{n=1}^{\infty} (-1)^n$. Does the series coverge?

$a_n = (-1)^n$

Is this going to converge?

No, $\boxed{\text{Diverges}}$

$S_1 = a_1 = -1$

$S_2 = a_1 + a_2 = -1 + 1 = 0$

$S_3 = a_1 + a_2 + a_3 = -1 + 1 - 1 = -1$

$S_4 = a_1 + a_2 + a_3 + a_4 = -1 + 1 - 1 + 1 = 0$

$S_5 = a_1 + a_2 + a_3 + a_4 + a_5 = -1 + 1 - 1 + 1 - 1 = -1$

7. Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series and

$s_n = 5 + \dfrac{n}{2n+3}$ is a formula for the nth partial sum. What is the sum of the series?

$$\sum_{n=1}^{\infty} a_n = \underbrace{\lim_{n\to\infty} S_n}_{S_\infty} = \lim_{n\to\infty} 5 + \frac{n}{2n+3} = 5 + \frac{1}{2} = \boxed{\frac{11}{2}}$$

8. What is the Test For Divergence and explain why the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ Diverges

Note: $\lim_{n\to\infty} a_n = 0$, we can't yet make a conclusion on whether $\sum_{n=1}^{\infty} a_n$ converges or diverges

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \qquad a_n = \frac{n}{n+1} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

$\sum_{n=1}^{\infty} \frac{n}{n+1}$ Diverges by Test for Divergence

T.D.

9. Find the sum of the following series. If it diverges, support your answer.

a.) $\sum_{n=1}^{\infty}\left(\frac{1}{n+5}-\frac{1}{n+6}\right)$

$$S_n = \overbrace{\left(\frac{1}{6}-\cancel{\frac{1}{7}}\right)}^{a_1} + \overbrace{\left(\cancel{\frac{1}{7}}-\cancel{\frac{1}{8}}\right)}^{a_2} + \overbrace{\left(\cancel{\frac{1}{8}}-\cancel{\frac{1}{9}}\right)}^{a_3} + \cancel{\cdots}$$

$$\cancel{\cdots} + \overbrace{\left(\cancel{\frac{1}{n+3}}-\cancel{\frac{1}{n+4}}\right)}^{a_{n-2}} + \overbrace{\left(\cancel{\frac{1}{n+4}}-\cancel{\frac{1}{n+5}}\right)}^{a_{n-1}} + \overbrace{\left(\cancel{\frac{1}{n+5}}-\frac{1}{n+6}\right)}^{a_n}$$

$$S_n = \frac{1}{6} - \frac{1}{n+6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n+5} - \frac{1}{n+6} = \lim_{n\to\infty} \frac{1}{6} - \frac{1}{n+6} = \frac{1}{6} - \frac{1}{\infty} = \frac{1}{6}$$

$$\boxed{\text{Converges}, \quad S = \frac{1}{6}}$$

b.) $\sum_{n=2}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=2}^{\infty} \ln(n) - \ln(n+1)$

$$S_n = \overset{a_2}{[\ln 2 - \cancel{\ln 3}]} + \overset{a_3}{[\cancel{\ln 3} - \cancel{\ln 4}]} + \overset{a_4}{[\cancel{\ln 4} - \cancel{\ln 5}]} + \cancel{\cdots}$$

$$\cancel{\cdots} + \overset{a_{n-2}}{[\cancel{\ln(n-2)} - \cancel{\ln(n-1)}]} + \overset{a_{n-1}}{[\cancel{\ln(n-1)} - \cancel{\ln n}]} + \overset{a_n}{[\cancel{\ln n} - \ln(n+1)]}$$

$$S_n = \ln 2 - \ln(n+1)$$

$$\sum_{n=2}^{\infty} \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \ln(2) - \ln(n+1) = \ln(2) - \ln(\infty) = -\infty$$

$$\boxed{\sum_{n=2}^{\infty} \ln\left(\frac{n}{n+1}\right) \text{ Diverges}}$$

c.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

$$\left[\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}\right] n(n+2)$$

$$1 = A(n+2) + Bn$$

$$n=-2 \qquad 1 = 0 + -2B \qquad B = -1/2$$

$$n=0 \qquad 1 = A(2) + 0 \qquad A = 1/2$$

$$\sum_{n=1}^{\infty} \frac{1/2}{n} - \frac{1/2}{n+2}$$

$$S_n = \overset{a_1}{\left[\frac{1/2}{1} - \cancel{\frac{1/2}{3}}\right]} + \overset{a_2}{\left[\frac{1/2}{2} - \cancel{\frac{1/2}{4}}\right]} + \overset{a_3}{\left[\cancel{\frac{1/2}{3}} - \cancel{\frac{1/2}{5}}\right]} + \cdots$$

$$\cdots + \overset{a_{n-2}}{\left[\cancel{\frac{1/2}{n-2}} - \cancel{\frac{1/2}{n}}\right]} + \overset{a_{n-1}}{\left[\cancel{\frac{1/2}{n-1}} - \frac{1/2}{n+1}\right]} + \overset{a_n}{\left[\cancel{\frac{1/2}{n}} - \frac{1/2}{n+2}\right]}$$

$$S_n = \frac{1/2}{1} + \frac{1/2}{2} - \frac{1/2}{n+1} - \frac{1/2}{n+2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n\to\infty} \frac{1}{2} + \frac{1}{4} - \cancelto{0}{\frac{1/2}{n+1}} - \cancelto{0}{\frac{1/2}{n+2}} = \frac{3}{4}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+2)} \text{ Converges}, \quad S = \frac{3}{4}}$$

d.) $\sum_{n=1}^{\infty} 2\left(\frac{1}{7}\right)^{n-1}$

Geometric!

$= \sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{7}\right)^{n} \cdot \left(\frac{1}{7}\right)^{-1} = \sum_{n=1}^{\infty} 14 \left(\frac{1}{7}\right)^{n}$

$r = \frac{1}{7}$ $\quad |r| = \frac{1}{7} \Rightarrow$ Converges

First term: $a = 14\left(\frac{1}{7}\right)^{1} = 2$

$S = \frac{2}{1 - 1/7} = \frac{2}{6/7} = 2 \cdot \frac{7}{6} = \frac{14}{6} = \frac{7}{3}$

$\boxed{\sum_{n=1}^{\infty} 2\left(\frac{1}{7}\right)^{n-1} \text{ Converges, } S = \frac{7}{3}}$

e.) $\sum_{n=1}^{\infty}(-5)\left(\frac{2}{3}\right)^n$

Ratio

$r = \frac{2}{3}$

$|r| = \frac{2}{3} < 1$

Series converges

If $|r| < 1$, series converges

If $|r| \geq 1$, series diverges

First term: $a = -5\left(\frac{2}{3}\right)^1 = \frac{-10}{3}$

If a Geometric Series Converges, the sum is

$$S = \frac{a}{1-r}$$

$$S = \frac{-10/3}{1 - 2/3} = \frac{-10/3}{1/3} = \frac{-10}{3} \cdot \frac{3}{1} = \boxed{-10}$$

$$\boxed{\sum_{n=1}^{\infty}(-5)\left(\frac{2}{3}\right)^n \text{ Converges, } S = -10}$$

f.) $$\sum_{n=0}^{\infty} \frac{(-1)^n + 3^{n+1}}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} + \sum_{n=0}^{\infty} \frac{3^{n+1}}{5^n}$$

$$3^{n+1} = 3^n \cdot 3^1$$

$$\sum_{n=0}^{\infty} \left(\frac{-1}{5}\right)^n + \sum_{n=0}^{\infty} \frac{3 \cdot 3^n}{5^n}$$

$$\sum_{n=0}^{\infty} \left(\frac{-1}{5}\right)^n + \sum_{n=0}^{\infty} 3\left(\frac{3}{5}\right)^n$$

$r_1 = \frac{-1}{5}$ $\qquad$ $r_2 = \frac{3}{5}$

$|r_1| = \frac{1}{5} < 1$ ✓ $\qquad$ $|r_2| = \frac{3}{5} < 1$ ✓

$a_1 = 1$ $(n=0)$ $\qquad$ $a_2 = 3$ $(n=0)$

$S_1 = \frac{1}{1-(-1/5)}$ $\qquad$ $S_2 = \frac{3}{1-3/5}$

$\frac{1}{6/5} = 5/6$ $\qquad$ $\frac{3}{2/5} = 3 \cdot \frac{5}{2} = \frac{15}{2} = \frac{45}{6}$

$S = \frac{5}{6} + \frac{45}{6}$

$= \frac{50}{6} = \frac{25}{3}$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n + 3^{n+1}}{5^n} \text{ Converges, } S = \frac{25}{3}}$$

g.) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}} = \sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{3^n \cdot 3} = \sum_{n=2}^{\infty} \frac{1}{3} \cdot \left(\frac{-2}{3}\right)^n$

$r = \frac{-2}{3}$ $\quad |r| = \frac{2}{3} < 1$ ✓

$a = \frac{1}{3}\left(\frac{-2}{3}\right)^2 = \frac{1}{3} \cdot \frac{4}{9} = \frac{4}{27}$

$(n=2)$

$$S = \frac{4/27}{1 - -2/3} = \frac{4/27}{5/3} = \frac{4}{27} \cdot \frac{3}{5} = \frac{4}{45}$$

$$\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}} \text{ Converges, } S = \frac{4}{45}}$$

h.) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{7^{n+1}}$ $(3^2)^n$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (3^2)^n}{7^n \cdot 7^1} = \sum_{n=0}^{\infty} \frac{(-1)^n \; 9^n}{7^n \cdot 7}$$

$$\sum_{n=0}^{\infty} \frac{1}{7} \cdot \left(\frac{-9}{7}\right)^n$$

$r = \frac{-9}{7}$

$|r| = \frac{9}{7} \geq 1$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \; 3^{2n}}{7^{n+1}} \quad \text{Diverges}}$$

i.) $4 + \frac{8}{5} + \frac{16}{25} + \frac{32}{125} + \ldots$

$\sum_{n=0}^{\infty} 4\left(\frac{2}{5}\right)^n$

$r = \frac{a_2}{a_1} = \frac{8/5}{4} = \frac{8}{5} \cdot \frac{1}{4} = \frac{2}{5}$

$|r| = \frac{2}{5} < 1$

$a = 4$

$S = \frac{4}{1 - 2/5} = \frac{4}{3/5} = 4 \cdot \frac{5}{3} = \frac{20}{3}$

$\boxed{\text{Series Converges, } S = \frac{20}{3}}$