Fall 2019 Math 152

Week in Review 7

courtesy: Amy Austin

(covering section 11.6, 11.8)

Section 11.6

1. Determine whether the following series converge absolutely, convege conditionally, or diverge.

a.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (Not Absolutely Convergent)

Check Absolute Convergence First!

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ Diverges p-series $p = \frac{1}{2} \ngtr 1$

Now check (regular) convergence

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Alternating!

$b_n = \frac{1}{\sqrt{n}}$

(1) $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

(2) Decreasing? ✓

Converges by A.S.T.

$\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ is conditionally convergent}}$

b.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$ Diverges by Test for Divergence

$$\lim_{n\to\infty} \frac{(-1)^n n}{n+1} = DNE$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

c.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+n}$

Absolute Convergence

$$\sum_{n=1}^{\infty} \frac{1}{n^3+n} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Converges

p-series $p = 3 > 1$ ✓

Converges by comparison test.

$\boxed{\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+n} \text{ is absolutely convergent}}$

d.) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

Absolute Convergence:

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \qquad f(x) = \frac{1}{x\cdot(\ln x)^2}$$

(1) Continuous? ✓
(2) Positive? ✓
(3) Decreasing? ✓

$$\int_2^{\infty} \frac{1}{x\cdot(\ln x)^2}\,dx \qquad \begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned} \qquad \int u^{-2}\,du = -u^{-1}$$

$$\frac{-1}{\ln x}\Bigg|_2^{\infty} = \frac{-1}{\ln(\infty)} - \frac{-1}{\ln(2)} = 0 + \frac{1}{\ln(2)} = \frac{1}{\ln(2)}$$

Integral Converges $\Rightarrow$ Series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Converges by Integral Test

$$\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2} \text{ is absolutely convergent}}$$

e.) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos n}{n^4}$

$0 \leq |\cos n| \leq 1$

Absolute Convergence

$$\sum_{n=2}^{\infty} \frac{|\cos n|}{n^4} \leq \sum_{n=2}^{\infty} \frac{1}{n^4}$$

Converges
p-series $p=4>1$ ✓

↑

Converges by comparison test.

$\boxed{\displaystyle\sum_{n=2}^{\infty} \frac{\cos n}{n^4} \text{ is absolutely convergent}}$

f.) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{6n+2}$

Absolute Convergence

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{6n+2} \sim \sum_{n=2}^{\infty} \frac{1}{6n}$ Diverges p-series $p = 1 \not> 1$ X

$\displaystyle\lim_{n\to\infty} \frac{1}{6n+2} \cdot \frac{6n}{1} = \lim_{n\to\infty} \frac{6n}{6n+2} = 1 = L$

$0 < L < \infty$ ✓

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{6n+2}$ Diverges by Limit comparison Test

Check regular convergence

$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{6n+2}$ Alternating!

(1) $\displaystyle\lim_{n\to\infty} \frac{1}{6n+2} = 0$ ✓

(2) Decreasing? ✓

$b_n = \dfrac{1}{6n+2}$

$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{6n+2}$ converges by A.S.T.

$\boxed{\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{6n+2} \text{ is conditionally convergent}}$

g.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-9)^n}{5^n n^{10}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right|$$

$$\lim_{n\to\infty} \left| \frac{9^{n+1}}{5^{n+1} \cdot (n+1)^{10}} \cdot \frac{5^n \cdot n^{10}}{9^n} \right| = \lim_{n\to\infty} \left| \frac{9^{n+1}}{9^n} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{n^{10}}{(n+1)^{10}} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cancel{9^n} \cdot 9^1}{\cancel{9^n}} \cdot \frac{\cancel{5^n}}{\cancel{5^n} \cdot 5^1} \cdot \frac{n^{10}}{(n+1)^{10}} \right| = \lim_{n\to\infty} \left| \frac{9}{5} \frac{n^{10}}{(n+1)^{10}} \right|$$

$$\frac{9}{5}(1) = \frac{9}{5}$$

< 1 Absolutely Convergent

> 1 Diverges

$= 1$ Inconclusive

$$\boxed{\sum_{n=1}^{\infty} \frac{(-9)^n}{5^n n^{10}} \text{ Diverges by Ratio Test}}$$

h.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-7)^n}{(n+2)!}$

$$\lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right|$$

$$5! = 5 \cdot \underbrace{4 \cdot 3 \cdot 2 \cdot 1} = 5 \cdot 4! = 5 \cdot 4 \cdot 3!$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

$$\lim_{n\to\infty} \left| \frac{7^{n+1}}{[(n+1)+2]!} \cdot \frac{(n+2)!}{7^n} \right| = \lim_{n\to\infty} \left| \frac{7^{n+1}}{7^n} \cdot \frac{(n+2)!}{(n+3)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cancel{7^n} \cdot 7^1}{\cancel{7^n}} \cdot \frac{\cancel{(n+2)!}}{(n+3)\,\cancel{(n+2)!}} \right| = \lim_{n\to\infty} \left| \frac{7}{n+3} \right| = \frac{7}{\infty} = 0 < 1$$

$$\boxed{\sum_{n=1}^{\infty} \frac{(-7)^n}{(n+2)!} \text{ is Absolutely Convergent by Ratio Test}}$$

2. Explain why the Ratio Test is or is not conclusive (you decide which) for the following series:

a.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}$

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)+2} \cdot \frac{n+2}{1} \right| = \lim_{n\to\infty} \left| \frac{n+2}{n+3} \right| = 1$$

Ratio Test is Inconclusive.

b.) $\sum_{n=1}^{\infty} \frac{(-7)^n n!}{(2n+1)!}$

$$\lim_{n\to\infty} \left| \frac{7^{n+1}\,(n+1)!}{[2(n+1)+1]!} \cdot \frac{(2n+1)!}{7^n\, n!} \right| = \lim_{n\to\infty} \left| \frac{7^{n+1}}{7^n} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n+1)!}{(2n+3)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{7^n \cdot 7}{7^n} \cdot \frac{(n+1)\cdot n!}{n!} \cdot \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{7(n+1)}{(2n+3)(2n+2)} \right| = 0 < 1$$

$\sum_{n=1}^{\infty} \frac{(-7)^n n!}{(2n+1)!}$ is absolutely convergent by Ratio Test

Section 11.8

3. For the following power series, find the radius and interval of convergence.

a.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n x^n}{n^2+5}$

Center of this series? $a = 0$

Goal: $|x-a| < R$

$|x| < R$

$$\lim_{n\to\infty} \left| \frac{4^{n+1} x^{n+1}}{(n+1)^2+5} \cdot \frac{n^2+5}{4^n x^n} \right| = \lim_{n\to\infty} \left| \frac{4^n \cdot 4}{4^n} \cdot \frac{x^n \cdot x}{x^n} \cdot \frac{n^2+5}{(n+1)^2+5} \right|$$

$$\lim_{n\to\infty} \left| 4x \cdot \frac{n^2+5}{(n+1)^2+5} \right| = |4x \cdot (1)| = |4x|$$

< 1 Absolutely Convergent

> 1 Divergent

$= 1$ Inconclusive

$|4x| < 1$ $\qquad |x| < \frac{1}{4} = R$ Radius of Convergence

$|4x| < 1$ $\qquad -1 < 4x < 1$

$-1/4 < x < 1/4$

Check $x = \frac{-1}{4}$

$$\sum_{n=1}^{\infty} \frac{(-4)^n \left(\frac{-1}{4}\right)^n}{n^2+5} = \sum_{n=1}^{\infty} \frac{1}{n^2+5} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$$ Converges p-series $p = 2 > 1$ ✓

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ Converges by Comparison Test

Check $x = \frac{1}{4}$

$$\sum_{n=1}^{\infty} \frac{(-4)^n \left(\frac{1}{4}\right)^n}{n^2+5} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+5}$$

Alternating! $b_n = \frac{1}{n^2+5}$

(1) $\displaystyle\lim_{n\to\infty} \frac{1}{n^2+5} = 0$ ✓

(2) Decreasing? ✓

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+5}$ Converges by A.S.T.

$$\boxed{\begin{array}{l} R = \frac{1}{4} \\ I : \left[\frac{-1}{4}, \frac{1}{4}\right] \end{array}}$$

b.) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(3x-1)^n}{\sqrt{n}}$

Center: $a = \frac{1}{3}$

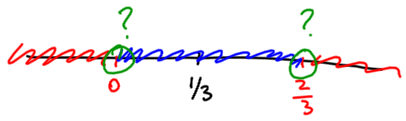
Goal: $\left|x - \frac{1}{3}\right| < R$

$$\lim_{n\to\infty} \left| \frac{(3x-1)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(3x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{(3x-1)^n \cdot (3x-1)}{(3x-1)^n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \right|$$

$$\lim_{n\to\infty} \left| (3x-1) \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \right| = |3x-1| < 1$$

$R = \frac{1}{3}$

$$3\left|x - \frac{1}{3}\right| < 1 \qquad \left|x - \frac{1}{3}\right| < \frac{1}{3}$$



Check $x=0$

$$\sum_{n=1}^{\infty} \frac{(-1)^n(3(0)-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

Diverges p-series $p = 1/2 \not> 1$

Check $x = 2/3$

$$\sum_{n=1}^{\infty} \frac{(-1)^n\left(3\left(\frac{2}{3}\right)-1\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n\, 1^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$$

Alternating! $b_n = \frac{1}{n^{1/2}}$

(1) $\lim_{n\to\infty} \frac{1}{n^{1/2}} = 0$ ✓

(2) Decreasing? ✓

Converges by A.S.T.

$$\boxed{\begin{array}{l} R = \frac{1}{3} \\ I: (0, 2/3] \end{array}}$$

c.) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!(x+2)^n}{100^n}$

Center: $a = -2$

Goal: $|x - (-2)| < R$

$$\lim_{n\to\infty}\left|\frac{[2(n+1)]!\,(x+2)^{n+1}}{100^{n+1}}\cdot\frac{100^n}{(2n)!\,(x+2)^n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(2n)!}{(2n)!}\cdot\frac{(x+2)^n(x+2)}{(x+2)^n}\cdot\frac{100^n}{100^n\cdot 100}\right|$$

$$\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\,(x+2)}{100}\right| = \infty < 1 \quad X$$

"Never" less than 1

"Never" converges

$$\boxed{\begin{array}{l} R = 0 \\ I: \{-2\} \end{array}}$$

d.) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{(2n+1)!}$

Center: $a=1$

Goal: $|x-1| < R$

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{[2(n+1)+1]!} \cdot \frac{(2n+1)!}{(x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^n \cdot (x-1)}{(x-1)^n} \cdot \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{x-1}{(2n+3)(2n+2)} \right| = 0 < 1$$

Always < 1

Always converges

$$\boxed{\begin{array}{l} R = \infty \\ I: (-\infty, \infty) \end{array}}$$

Center: $a=0$

4. Suppose it is known that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = 4$ and diverges when $x = -10$. What can be said about the convergence or divergence of the following series, if anything?

a.) $\sum_{n=0}^{\infty} c_n (8)^n$ $\quad x = 8 \quad$ Inconclusive

b.) $\sum_{n=0}^{\infty} c_n (-3)^n$ $\quad x = -3 \quad$ Converges

c.) $\sum_{n=0}^{\infty} c_n (12)^n$ $\quad x = 12 \quad$ Diverges

d.) $\sum_{n=0}^{\infty} c_n (-4)^n$ $\quad x = -4 \quad$ Inconclusive

e.) $\sum_{n=0}^{\infty} c_n (10)^n$ $\quad x = 10 \quad$ Inconclusive

f.) $\sum_{n=0}^{\infty} \frac{c_n (-1)^n 2^n}{3^n} = \sum_{n=0}^{\infty} c_n \cdot \left(\frac{-2}{3}\right)^n$ $\quad x = \frac{-2}{3} \quad$ Converges