Week-in-Review

Exam 1 Review

1. $\int \frac{\cos^3(\ln x)}{x}\,dx$

$t = \ln x$

$dt = \frac{1}{x}\,dx$

$u = \sin t$

$du = \cos t\,dt$

$$\int \cos^3 t\,dt = \int \cos^2 t \cos t\,dt$$

$$= \int (1-\sin^2 t)\cos t\,dt$$

$$= \int (1-u^2)\,du$$

$$= u - \frac{u^3}{3} + C$$

$$= \sin t - \frac{\sin^3 t}{3} + C$$

$$\boxed{\sin(\ln x) - \frac{\sin^3(\ln x)}{3} + C}$$

2. The force required to hold a spring stretched to a length of 7 m is 5 N. Find the work required to stretch the spring from a length of 4 m to 8 m. The natural length of the spring is 3 m.

$-3 \quad -3$

Hooke's Law: $f(x) = kx$

$f(x) =$ force

$x =$ units beyond natural length

Given: $f(4) = 5$

$k4 = 5 \longrightarrow k = \frac{5}{4}$

$f(x) = \frac{5}{4}x$

$w = \int kx\,dx$

$w = \int_1^5 \frac{5}{4}x\,dx$

$\frac{5}{8}x^2\Big|_1^5$

$$\boxed{\frac{5}{8}(24)\text{ J}}$$

3. Find the volume of the solid S whose base is bounded by the region $x^2 + 4y^2 = 4$, and cross-sections perpendicular to the y-axis are isosceles trianges with height equal to the base.

dy

Base of solid is $\boxed{x^2+4y^2=4}$ ellipse

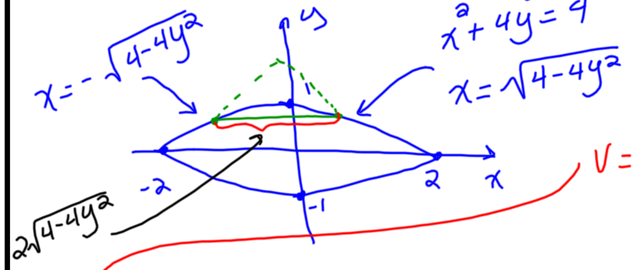
cross-sections are triangles

x-int: $x^2=4$

$x=\pm 2$

y-int: $4y^2=4$

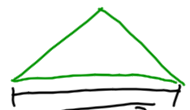
$y=\pm 1$



$$V=\int_{-1}^{1}\left(A_{\text{triangles}}\right)dy$$

$A_{\text{triangle}} = \frac{1}{2}bh$ since $b=h$

$= \frac{1}{2}b^2$, $b =$ base triangle



$b = 2\sqrt{4-4y^2}$

$$V = 2\int_0^1 \frac{1}{2}b^2\,dy$$

$$V=\int_0^1 b^2\,dy$$

$$V=\int_0^1\left(2\sqrt{4-4y^2}\right)^2dy$$

$$=4\int_0^1\left(4-4y^2\right)dy$$

$$=4\left(4y-\frac{4}{3}y^3\right)\Big|_0^1$$

$$=\boxed{4\left(4-\frac{4}{3}\right)}$$

4. Find the area bounded by $x = 3y - y^2$ and $y = -\frac{x}{2}$.

$$\begin{cases} x = 3y - y^2 \\ x = -2y \end{cases}$$

$$3y - y^2 = -2y$$

$$0 = y^2 - 5y$$

$$0 = y(y-5)$$

Test number between $y=0$ & $y=5$

$$y = 2 \begin{cases} x = -2y \to -4 & L \\ x = 3y - y^2 \to 2 & R \end{cases}$$

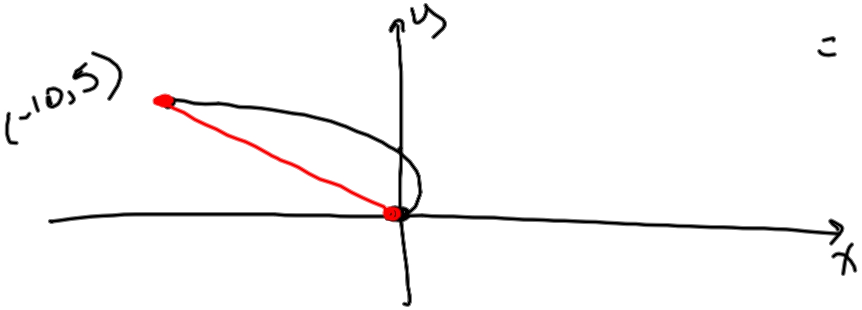
$y = 0 \longrightarrow x = 0$

$y = 5 \longrightarrow x = -10$

$$A = \int (R - L)\, dy$$

$$= \int_0^5 (3y - y^2 + 2y)\, dy$$

$$= \int_0^5 (5y - y^2)\, dy$$



5. $\int \frac{\ln x}{\sqrt{x}}\,dx$

$$\int u\,dv = uv - \int v\,du$$

L = $\ln x$
I = inverse trig
P = polynomial
E = e^x
T = trig

u =

$u = \ln x \qquad dv = \frac{1}{\sqrt{x}}\,dx$

$du = \frac{1}{x}\,dx \qquad dv = x^{-\frac{1}{2}}\,dx$

$v = 2\sqrt{x}$

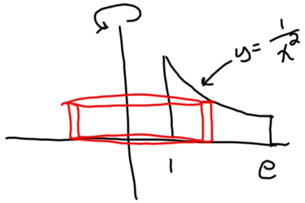
$$\int x^{-\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$$

$$= 2\sqrt{x}\ln x - \int 2\sqrt{x}\,\frac{1}{x}\,dx$$

$$= 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\,dx$$

$$= \boxed{2\sqrt{x}\ln x - 2\cdot 2\sqrt{x} + C}$$

6. The region bounded by $y = \frac{1}{x^2}$, $x = 1$, $x = e$, and $y = 0$ is rotated around the y-axis. Find the volume.



Shells : @ y-axis, $r = x$

$h = T - B$

$$V = \int_1^e 2\pi r h\,dx$$

$$= \int_1^e 2\pi x\left(\frac{1}{x^2}\right)dx$$

$$= 2\pi \int_1^e \frac{1}{x}\,dx$$

$$= 2\pi \ln x \Big|_1^e$$

$$= 2\pi(\ln e - \ln(1))$$

7. The region bounded by $x + y^2 = 4$ and $x - y = 2$ is rotated around the line $x = -1$. Set up but do not evaluate an integral representing the volume of the solid.

$$\begin{cases} x = 4 - y^2 \\ x = 2 + y \end{cases}$$

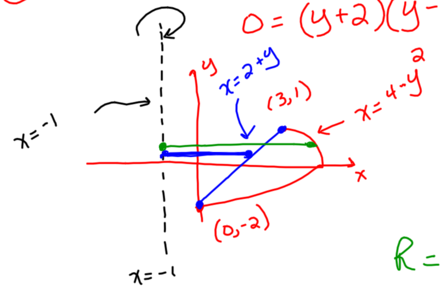
$$4 - y^2 = 2 + y$$

$$0 = y^2 + y - 2$$

$$0 = (y+2)(y-1) = 0$$

$y = 1 \rightarrow x = 3$

$y = -2 \rightarrow x = 0$



washer method

$$V = \int_{-2}^{1} \pi \left(R^2 - r^2 \right) dy$$

$$R = 4 - y^2 - (-1) = 5 - y^2$$

$$r = 2 + y - (-1) = 3 + y$$

$$\boxed{V = \pi \int_{-2}^{1} \left((5-y^2)^2 - (3+y)^2 \right) dy}$$

8. $\displaystyle\int_0^2 x^2 e^{3x}\, dx$

L x
I x
P ✓
E
T

u = polynomial, use tabular!

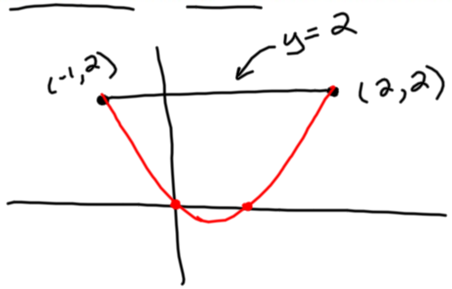
derivative u		integrate dv
x^2	+	e^{3x}
$2x$	−	$\frac{1}{3} e^{3x}$
2	+	$\frac{1}{9} e^{3x}$
0		$\frac{1}{27} e^{3x}$

$$\left(\frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} \right) \Big|_0^2$$

$$\boxed{\frac{4}{3} e^6 - \frac{4}{9} e^6 + \frac{2}{27} e^6 - \frac{2}{27}}$$

9. Find the area bounded by $y = 7 - x^2$ and $y = 2x^2 - 5$.

10. Set up but do not evaluate an integral for the volume of the solid obtained by rotating the region bounded by $y = x^2 - x$ and $y = 2$ rotated around the line $x = 3$.

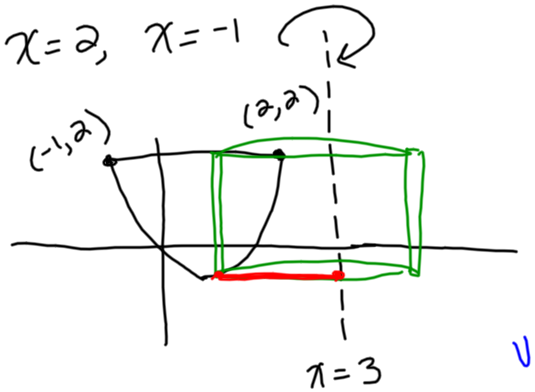


$x^2 - x = 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, \quad x = -1$



@ y-axis, using shells, $r = x$

$r = 3 - x$

$h = T - B = 2 - (x^2 - x)$

$$V = \int_{-1}^{2} 2\pi (3-x)(2-x^2+x)\,dx$$

11. $\int \frac{x^3}{(x^2+1)^8}\,dx$

$u = x^2+1 \longrightarrow x^2 = u-1$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$$\int \frac{x^3}{u^8}\frac{du}{2x} = \frac{1}{2}\int \frac{x^2}{u^8}\,du$$

$$= \frac{1}{2}\int \frac{u-1}{u^8}\,du$$

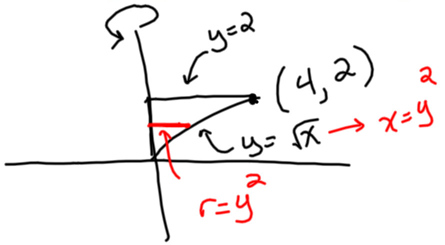
$$= \frac{1}{2}\int \left(\frac{1}{u^7} - \frac{1}{u^8}\right) du$$

$$= \frac{1}{2}\int \left(u^{-7} - u^{-8}\right) du$$

$u = x^2+1$

$$= \frac{1}{2}\left(\frac{u^{-6}}{-6} - \frac{u^{-7}}{-7}\right) + C$$

12. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$ around the y-axis.



Shells: $V = \int_0^4 2\pi x(2-\sqrt{x})\,dx$

Disk: $V = \int_0^2 \pi (y^2)^2\,dy$

13. $\int \tan^6 x \sec^4 dx = \int \tan^6 x \sec^2 x \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int \tan^6 x \left(\tan^2 x + 1 \right) \sec^2 x \, dx$$

$$= \int u^6 (u^2+1) \, du$$

$$= \int (u^8 + u^6) \, du \rightarrow \frac{u^9}{9} + \frac{u^7}{7} + C$$

$$\boxed{\frac{\tan^9 x}{9} + \frac{\tan^7 x}{7} + C}$$

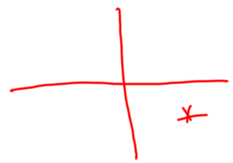
14. $\int_0^{\pi/6} \sin^2(5x) \, dx =$

$\cos^2 \theta = \frac{1}{2}\left(1 + \cos 2\theta \right)$

$\sin^2 \theta = \frac{1}{2}\left(1 - \cos 2\theta \right)$

$$\int_0^{\frac{\pi}{6}} \frac{1}{2}\left(1 - \cos(10x)\right) dx$$

$$\frac{1}{2}\left(x - \frac{1}{10} \sin(10x) \right) \Big|_0^{\frac{\pi}{6}}$$



$$\frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{10} \underbrace{\sin \frac{5\pi}{3}}_{-\frac{\sqrt{3}}{2}} - 0 \right)$$

$$\boxed{\frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{10}\left(-\frac{\sqrt{3}}{2}\right) \right)}$$

15. Find $\int e^x \sin(8x)\,dx$

$u = \sin 8x \qquad dv = e^x\,dx$

$du = 8\cos(8x)\,dx \qquad v = e^x$

$$\int e^x \sin(8x)\,dx = uv - \int v\,du$$

$$= e^x \sin 8x - \boxed{\int 8e^x \cos(8x)\,dx}$$

parts

$u = 8\cos 8x \qquad dv = e^x\,dx$

$du = -64\sin(8x)\,dx \qquad v = e^x$

$uv - \int v\,du$

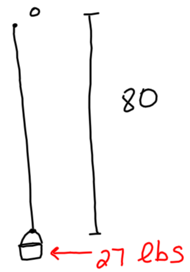
$8e^x \cos 8x + \int 64e^x \sin(8x)\,dx$

$$\int e^x \sin(8x)\,dx = e^x \sin 8x - 8e^x \cos 8x - \int 64 e^x \sin 8x\,dx$$

$$\cancel{6}5\int e^x \sin(8x)\,dx = \boxed{\frac{e^x \sin 8x - 8e^x \cos 8x}{65} + C}$$

16. A bucket attached to a 20 pound rope is used to draw water out of an 80 ft well. The bucket weighs 1 pound and holds 26 pounds of water. How much work is done in drawing up one full bucket of water?

rope weighs $\frac{20 \text{ lbs}}{80 \text{ feet}} = \frac{1 \text{ lb}}{4 \text{ ft}}$



$$W_{total} = W_{rope} + W_{bucket}$$

$$= \int_0^{80} \frac{1}{4} x \, dx + (27 \text{ lbs})(80 \text{ ft})$$

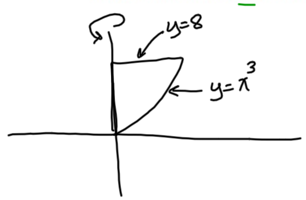
$$= \frac{x^2}{8} \Big|_0^{80} + (27)(80) = 2960 \text{ ft-lbs}$$

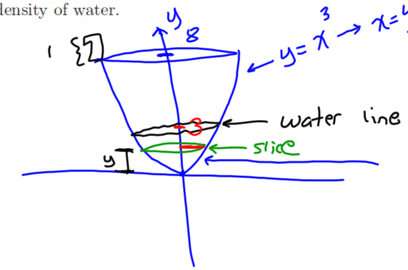
method 2: $\int_0^{80} \left(\text{Total weight} - \frac{1}{4} x \right) dx$

total weight = 20 lbs + 27 lbs = 47

$$W = \int_0^{80} \left(47 - \frac{1}{4} x\right) dx = 2960 \text{ ft-lbs}$$

17. Consider the region R bounded by $y = x^3$, $y = 8$, and $x = 0$. Suppose a tank is in the shape of the region R revolved around the y-axis, and the units are measured in meters. If the tank is filled with water to a depth of 3 m, set up but do not evaluate an integral that gives the work done in pumping all the water out of a 1 m high spout. Use ρg for the weight density of water.





$y = x^3 \rightarrow x = y^{\frac{1}{3}}$

$r = y^{\frac{1}{3}}$

$$V_{slice} = \pi r^2 \, dy$$

$$= \pi \left(y^{2/3} \right) dy$$

$$F_{slice} = \pi \rho g \, y^{\frac{2}{3}} \, dy$$

$$d = 8 - y + 1 = 9 - y$$

$$W_{slice} = \pi \rho g \, y^{\frac{2}{3}} (9 - y) \, dy$$

$$W = \int_0^3 \pi \rho g \, y^{\frac{2}{3}} (9 - y) \, dy$$