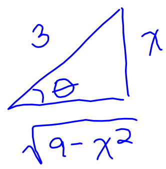
1. Find $\int \frac{x^2}{\sqrt{9-x^2}}\,\boxed{dx}.$

$a^2 - x^2 \longrightarrow x = a\sin\theta$

$\boxed{x = 3\sin\theta} \longrightarrow \frac{O}{H}\frac{x}{3} = \sin\theta$

$dx = 3\cos\theta\, d\theta$



$$\int \frac{9\sin^2\theta}{\sqrt{\underbrace{9-9\sin^2\theta}}} \cdot 3\cos\theta\, d\theta$$

$9(1-\sin^2\theta)$

$9\cos^2\theta$

$$\int \frac{9\sin^2\theta}{\cancel{3\cos\theta}}\, \cancel{3\cos\theta}\, d\theta$$

$$9\int \sin^2\theta\, d\theta$$

$$9\int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$\cancel{2}\sin\theta\cos\theta$

$$\frac{9}{2}\left(\theta - \frac{1}{\cancel{2}}\sin 2\theta\right) + C$$

$$\frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) - \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right) + C$$

2. Evaluate $\displaystyle\int_0^{2/3} \frac{1}{(4+9x^2)^{5/2}}\,dx$

see next page

2. $\int_0^{2/3} \frac{1}{(4+9x^2)^{5/2}}\,dx = \int_0^{\frac{2}{3}} \frac{dx}{\left(4+(3x)^2\right)^{5/2}}$

$3x = 2\tan\theta$: $x=\frac{2}{3} \rightarrow 2 = 2\tan\theta \rightarrow 1=\tan\theta$ $\boxed{\theta=\frac{\pi}{4}}$; $x=0 \rightarrow 0 = 2\tan\theta \rightarrow 0=\tan\theta$ $\boxed{\theta=0}$

$x = \frac{2}{3}\tan\theta$

$dx = \frac{2}{3}\sec^2\theta\, d\theta$

$$\int_0^{\frac{\pi}{4}} \frac{\frac{2}{3}\sec^2\theta\, d\theta}{(\underbrace{4+4\tan^2\theta}_{4\sec^2\theta})^{5/2}}$$

$$\int_0^{\frac{\pi}{4}} \frac{\frac{2}{3}\sec^2\theta\, d\theta}{(4\sec^2\theta)^{5/2}}$$

$$\frac{2}{3}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{4^{\frac{5}{2}}\sec^5\theta}$$

$$\frac{2}{3}\int_0^{\frac{\pi}{4}} \frac{d\theta}{32\sec^3\theta}\, d\theta$$

$$\frac{1}{48}\int_0^{\frac{\pi}{4}} \cos^3\theta\, d\theta$$

$u = \sin\theta$: $\theta=\frac{\pi}{4}, u=\frac{\sqrt{2}}{2}$; $\theta=0, u=0$

$du = \cos\theta\, d\theta$

$$\frac{1}{48}\int_0^{\frac{\pi}{4}} \underbrace{\cos^2\theta}_{1-\sin^2\theta} \cos\theta\, d\theta$$

$$\frac{1}{48}\int_0^{\frac{\sqrt{2}}{2}} (1-u^2)\, du$$

$$\frac{1}{48}\left(u - \frac{u^3}{3}\right)\Big|_0^{\frac{\sqrt{2}}{2}} = \frac{1}{48}\left(\frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3\right)$$

3. After choosing the appropriate trigonometric substitution for $\int (x-4)\sqrt{x^2-8x+7}\,dx$, write the resulting integral in terms of θ. Do NOT integrate.

complete the square:

$$\left(\frac{-8}{2}\right)^2 = 16$$

$$x^2-8x+7 = \underbrace{x^2-8x+16} + 7 - 16$$

$$= (x-4)^2 - 9$$

$$x^2 - a^2$$

$$x = a\sec\theta$$

$$\int (x-4)\sqrt{(x-4)^2-9}\,dx$$

$$x-4 = 3\sec\theta$$

$$dx = 3\sec\theta\tan\theta$$

$$\int 3\sec\theta\sqrt{\underbrace{9\sec^2\theta-9}}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$9(\sec^2\theta-1)$$

$$9\tan^2\theta$$

$$\int 3\sec\theta\cdot 3\tan\theta\cdot 3\sec\theta\tan\theta\,d\theta$$

$$\boxed{27\int \sec^2\theta\tan^2\theta\,d\theta}$$

4. Evaluate $\int \frac{x^4+2x^3-6x^2-2x+12}{x^3+4x^2+4x}\,dx.$

$Q + \frac{R}{D}$

$$\begin{array}{r} x-2 \\ x^3+4x^2+4x \overline{\smash{)}\, x^4+2x^3-6x^2-2x+12} \\ -x^4-4x^3-4x^2 \\ \hline -2x^3-10x^2-2x+12 \\ +2x^3+8x^2+8x \\ \hline -2x^2+6x+12 \end{array}$$

$$\int \left(x-2+\frac{-2x^2+6x+12}{x^3+4x^2+4x} \right) dx$$

PFD: $\frac{-2x^2+6x+12}{x(x^2+4x+4)}$

$$\frac{-2x^2+6x+12}{x(x+2)^2} = \left(\frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \right)\left(x(x+2)^2 \right)$$

$$-2x^2+6x+12 = A(x+2)^2 + Bx(x+2) + Cx$$

$x=0$: $12 = A(4)$ $\boxed{A=3}$

$x=-2$: $-8-12+12 = C(-2)$ $\boxed{C=4}$

$$-2x^2+6x+12 = 3(x+2)^2 + Bx(x+2) + 4x$$

$x=1$: $16 = 3(9) + B(3) + 4$

$16 = 31 + 3B$

$\boxed{B=-5}$

$$\int x-2+\frac{3}{x}-\frac{5}{x+2}+\frac{4}{(x+2)^2}\,dx$$

$$\boxed{\frac{x^2}{2} - 2x + 3\ln|x| - 5\ln|x+2| - \frac{4}{x+2} + C}$$

5. Evaluate $\int_0^2 \frac{x^2+3x}{(x+1)(x^2+4)}\,dx$.

$$\frac{x^2+3x}{(x+1)(x^2+4)} = \left(\frac{A}{x+1} + \frac{Bx+C}{x^2+4}\right)(x+1)(x^2+4)$$

$$x^2+3x = \overset{-\frac{2}{5}}{\cancel{A}}(x^2+4) + (Bx+C)(x+1)$$

$x=-1$: $-2 = A(5) \longrightarrow A = -\frac{2}{5}$

$$x^2+3x = -\frac{2}{5}x^2 - \frac{8}{5} + Bx^2 + Bx + Cx + C$$

$$1\cdot x^2 + 3x + 0 = \left(-\frac{2}{5}+B\right)x^2 + (B+C)x - \frac{8}{5} + C$$

$1 = -\frac{2}{5} + B \rightarrow \boxed{B = 1 + \frac{2}{5} = \frac{7}{5}}$

$3 = B + C \longrightarrow 3 = \frac{7}{5} + C \rightarrow \boxed{C = \frac{8}{5}}$

check : $0 = -\frac{8}{5} + C \rightarrow \boxed{C = \frac{8}{5}}$

$$\int_0^2 \left(\frac{-\frac{2}{5}}{x+1} + \frac{\frac{7}{5}x + \frac{8}{5}}{x^2+4}\right)dx \qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int_0^2 \left(\frac{-\frac{2}{5}}{x+1} + \frac{\frac{7}{5}x}{x^2+4} + \frac{\frac{8}{5}}{x^2+4}\right)dx$$

↑
u-sub
$u = x^2+4$
$du = 2x\,dx$

$$\left(-\frac{2}{5}\ln|x+1| + \frac{7}{5}\cdot\frac{1}{2}\ln|x^2+4| + \frac{8}{5}\cdot\frac{1}{2}\arctan\frac{x}{2}\right)\Bigg|_0^2$$

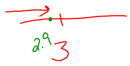
$$-\frac{2}{5}\ln 3 + \frac{7}{10}\ln 8 + \frac{8}{10}\overset{\frac{\pi}{4}}{\cancel{\arctan(1)}} - \left(-\frac{2}{5}\overset{0}{\cancel{\ln(1)}} + \frac{7}{10}\ln 4 + \frac{8}{10}\overset{0}{\cancel{\arctan 0}}\right)$$

6. Evaluate $\int_0^4 \frac{2}{(x-3)^2}\,dx$ or show it diverges.

improper at $x=3$

$$① \int_0^{\boxed{3}\,t} \frac{2}{(x-3)^2}\,dx + ② \int_3^4 \frac{2}{(x-3)^2}\,dx$$

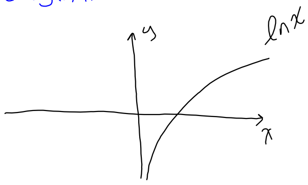
$$① \lim_{t\to 3^-} \int_0^t \frac{2}{(x-3)^2}\,dx = \lim_{t\to 3^-} \left. \frac{-2}{x-3} \right|_0^t$$



$$= \lim_{t\to 3^-} \left[-\frac{2}{t-3} - \left(-\frac{2}{-3}\right) \right]$$

negative

$= +\infty$ Diverges! so original diverges



comparison theorem says:

$\ln(+0) = -\infty$

① if larger integral converges, so does the smaller.

② if smaller integral diverges then so does the larger.

7. Determine whether the improper integral converges or diverges using the comparison theorem.

a.) $\int_1^\infty \frac{\sin(6x)+8}{\sqrt{x}+x^4}\,dx. \le \int_1^\infty \frac{9}{x^4}\,dx$

$-1 \le \sin x \le 1$

$-1 \le \cos x \le 1$

converges by p-test, $p=4>1$

$\int_1^\infty \frac{dx}{x^p}$ converges if $p>1$, diverges if $p \le 1$

larger converges! so original converges also

a.) $\int_1^\infty \frac{\cos x + 7}{x+1}\,dx$

$$\int_1^\infty \frac{\cos x + 7}{x+1}\,dx \le \int_1^\infty \frac{8}{x}$$

diverges so test fails.

$-1 \le \cos x \le 1$

$$\int_1^\infty \frac{\overset{-1}{\cos x} + 7}{x+1}\,dx \ge \int_1^\infty \frac{6}{x+1}\,dx$$

$$6\ln|x+1| \Big|_1^\infty = \infty$$

smaller diverges so does larger!

b.) $\int_1^\infty \frac{e^{1/x}+2}{x}\,dx > \int_1^\infty \frac{2}{x}\,dx$

(p-test) smaller diverges so does larger

8. Find a general formula for the sequence $\left\{2, -\frac{7}{3}, \frac{12}{9}, -\frac{17}{27}, \frac{22}{81}, \cdots\right\}$. Assume the pattern continues, and the sequence begins with $n = 1$.

alternating signs $\rightarrow$ $(-1)^n$ or $\boxed{(-1)^{n+1}}$

numerator: $2, 7, 12, 17, 22$

$2,\ 2+5,\ 2+5+5,\ 2+5+5+5$

$2+5(0),\ 2+5(1),\ 2+5(2),\ 2+5(3), \ldots$

$\boxed{2+5(n-1)}$

$$\boxed{a_n = \frac{(-1)^{n+1}(2+5(n-1))}{3^{n-1}}}$$

denominator: $1, 3, 9, 27, 81,$

$1, 3, 3^2, 3^3, 3^4, \ldots$

$\boxed{3^{n-1}}$

9. Consider the recursive sequence $a_1 = 4$ and $a_{n+1} = \frac{5}{6 - a_n}$.

(a) Find the first three terms of the sequence.

$a_1 = 4$

$a_2 = \frac{5}{6-a_1} = \frac{5}{6-4} = \frac{5}{2}$

$a_3 = \frac{5}{6-a_2} = \frac{5}{6-\frac{5}{2}}$

(b) Assuming the sequence is decreasing and bounded, determine if the sequence converges or diverges. If it converges, find the limit.

$\lim_{n\to\infty} a_n = L$

$\rightarrow \lim_{n\to\infty} a_{n+1} = L$

$a_{n+1} = \frac{5}{6-a_n}$

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{5}{6-a_n}$

$L = \frac{5}{6-L}$

$L(6-L) = 5$

$6L - L^2 = 5$

$0 = L^2 - 6L + 5$

$0 = (L-5)(L-1)$

$L = 5$

$L = 1$

$\boxed{L = 1}$

10. Determine whether the sequence converges or diverges. If it converges, what value does it converge to? Also, which sequences are bounded?

(a) $a_n = \sin n$

$\lim_{n\to\infty} \sin(n)$ dne by oscillation. $\boxed{\text{it diverges!}}$

is bounded (between ± 1.)

(b) $a_n = \frac{\sin n}{n}$

$$\cancel{-\frac{1}{n}}^{\to 0} \le \frac{\sin(n)}{n} \le \cancel{\frac{1}{n}}^{\to 0}$$

$\frac{\sin(n)}{n} \to 0$

$\boxed{\text{converges to } 0 \text{ so it is bounded}}$

(c) $a_n = \cos\left(\frac{5}{n}\right)$

$\lim_{n\to\infty} \cos\left(\frac{5}{n}\right) = \cos(0) = 1$ converges (so bounded)

(d) $a_n = \frac{(-1)^n n}{2n^2+1}$ $\lim_{n\to\infty} \frac{n}{2n^2+1} = 0$

the $(-1)^n \to \pm 0$

so $\lim_{n\to\infty} \frac{(-1)^n n}{2n^2+1} = 0$ converges to 0 (so bounded)

(e) $a_n = \frac{(-1)^{n-1} n}{2+9n}$ $\lim_{n\to\infty} \frac{n}{2+9n} = \frac{1}{9}$

the $(-1)^{n-1} \to \pm\frac{1}{9}$ diverges by oscillation but it is bounded!

11. Determine if the following sequences are increasing, decreasing, or not monotonic. Also, determine if each sequence is bounded.

(a) $a_n = 3 - e^{-2n}$

$a_n = 3 - \frac{1}{e^{2n}}$ → decreasing

$\left(3 - e^{-2n}\right)' = 2e^{-2n} > 0$

increasing!

$3 - \frac{1}{e^{2n}}$ is therefore increasing!

bounded? $\lim_{n\to\infty}\left(3 - e^{-2n}\right) = \lim_{n\to\infty}\left(3 - \frac{1}{e^{2n}}\right)$, where $\frac{1}{e^{2n}} \to 0$

$= 3 \quad \therefore$ Bounded!

(b) $a_n = (-1)^n n$

non monotonic since $(-1)^n n$ is alternating!

not bounded $\lim_{n\to\infty} (-1)^n n \begin{cases} \infty & n \text{ even} \\ -\infty & n \text{ odd} \end{cases}$

Σ sum! series

Asked for sum of a series

geometric series

$\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$

$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

Telescoping

$\Sigma(a_n - a_{n+1})$

Find S_n

$\lim_{n\to\infty} S_n$

given S_n

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

12. For the series $\sum_{n=1}^{\infty} a_n$, the nth partial sum is given by $s_n = \frac{3-2n}{5n+1}$.

(a) Find a_5

$S_n = \cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_{n-1}} + a_n$

$-$

$S_{n-1} = \cancel{a_1} + \cancel{a_2} + \dots + \cancel{a_{n-1}}$

$\boxed{S_n - S_{n-1} = a_n}$

$a_5 = S_5 - S_4$

$a_5 = \frac{3-10}{25+1} - \frac{3-8}{20+1}$

$\boxed{a_5 = \frac{-7}{26} + \frac{5}{21}}$

(b) Find $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

$= \lim_{n\to\infty} \frac{3-2n}{5n+1} = \boxed{-\frac{2}{5}}$

(c) What is $\lim_{n\to\infty} a_n$?

T.D. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

$\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

So since our series converges to $-\frac{2}{5}$,

$\lim_{n\to\infty} a_n = 0$.

13. Determine if the following series converges or diverges. If the series converges, find its sum.

(a) $\sum_{n=1}^{\infty} \left[\frac{1}{2^n} - \frac{1}{2^{n+1}}\right]$

$S_n = a_1 + a_2 + \cdots + a_n$

$S_n = \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{8}} + \cancel{\frac{1}{8}} - \cancel{\frac{1}{16}} + \cdots + \cancel{\frac{1}{2^n}} - \frac{1}{2^{n+1}}$

$\boxed{S_n = \frac{1}{2} - \frac{1}{2^{n+1}}}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{2^{n+1}}\right) = \boxed{\frac{1}{2}}$

(b) $\sum_{n=1}^{\infty} \frac{6}{n(n+2)}$

PFD of $\frac{6}{n(n+2)} = \frac{3}{n} - \frac{3}{n+2}$

telescoping!

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n+1}}{5 \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2 \cdot 2^n}{5 \cdot 3^n}$

$= \sum_{n=1}^{\infty} \frac{2}{5}\left(\frac{-2}{3}\right)^n$ $\quad r = -\frac{2}{3}$

$= \sum_{n=1}^{\infty} \frac{2}{5}\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)^{n-1}$ $\quad |r| < 1$ will converge

$= \frac{a}{1-r}$ $\quad a = -\frac{4}{15}$ $\quad r = -\frac{2}{3}$

or $\frac{\text{First term}}{1-r} = \frac{\frac{(-1)(4)}{15}}{1+\frac{2}{3}}$

$= \frac{-\frac{4}{15}}{1+\frac{2}{3}}$

(d) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^3}\right)$

T.D. $\lim_{n\to\infty} a_n \neq 0$, Σa_n diverges!

$\lim_{n\to\infty} \cos\left(\frac{1}{n^3}\right) = \cos(0) = 1 \neq 0$

14. Determine whether the following series converge or diverge. Support your answer.

(a) $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ diverges by T.D.

(b) $\sum_{n=3}^{\infty} \frac{5}{n\sqrt{\ln n}}$

$\lim_{n\to\infty} \frac{5}{n\sqrt{\ln n}} = 0$ no conclusion!!!

$\frac{5}{n\sqrt{\ln n}}$ is positive & decreasing, use integral test

$$\int_3^{\infty} \frac{5}{x\sqrt{\ln x}}\,dx$$

$$\lim_{t\to\infty} \int_3^{t} \frac{5}{x\sqrt{\ln x}}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\lim_{t\to\infty} 10\sqrt{\ln x}\,\Big|_3^t = \infty$$

$\int \frac{du}{\sqrt{u}} = 2\sqrt{u}$

integral diverges so does series!

15. Using the remainder for the integral test, find an upper bound for the remainder if we use s_8 to approximate $\sum_{n=1}^{\infty} \frac{1}{n^5}$

if S_n is used to approximate $\sum_{n=1}^{\infty} a_n$,

then error $= R_n < \int_n^{\infty} f(x)\,dx$

we used S_8, $R_8 < \int_8^{\infty} \frac{dx}{x^5} = \left. \frac{-1}{4x^4} \right|_8^{\infty}$

$= 0 + \frac{1}{4(8^4)}$

$$\boxed{R_8 < \frac{1}{4(8^4)}}$$

16. Using the remainder for the integral test, what is the smallest value of n that ensures s_n to approximates $\sum_{n=1}^{\infty} \frac{3}{n^4}$ with error less than $\frac{1}{100}$?

$\int_n^{\infty} \frac{3}{x^4}\,dx = \left. -\frac{1}{x^3} \right|_n^{\infty}$

$= 0 + \frac{1}{n^3}$

$R_n < \frac{1}{100}$

$\frac{1}{n^3} < \frac{1}{100}$

$100 < n^3$

$$\boxed{n = 5}$$

17. Determine if the following statements are true or false. If the statement is false, give a counter example

(a) If a sequence converges, then it is bounded.

True

(b) If a sequence is bounded, then it converges.

False $a_n = (-1)^n$ is bounded but diverges

(c) If a sequence is increasing, then it converges.

False $a_n = n$

(d) If $\lim_{n\to\infty} s_n = 4$, then $\sum_{n=1}^{\infty} a_n$ converges.

true

(e) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

true

(f) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p \leq 1$.

False, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(g) The geometric series $\sum_{n=2}^{\infty} ar^{n-1}$ converges if $|r| < 1$.

true