Final exam review
wednesday may 1
1-3 pm HELD 100

1. Determine if the series converges or diverges. FULLY explain your reasoning.

(a) $\sum_{n=1}^{\infty} \frac{2+3\cos n}{n^3+4n^2} \leq \sum_{n=1}^{\infty} \frac{5}{n^3}$ ← converges by p-series $p=3$

$-1 \leq \cos(n) \leq 1$

larger converges so does smaller by C.T.

(b) $\sum_{n=2}^{\infty} \frac{n+1}{5n^2-2} \geq \sum_{n=2}^{\infty} \frac{n}{5n^2} = \sum_{n=2}^{\infty} \frac{1}{5n}$ ← diverges p-series $p=1$

smaller diverges so does larger by CT

(c) $\sum_{n=3}^{\infty} \frac{5+\sin n}{n-4\sqrt{n}} \geq \sum_{n=3}^{\infty} \frac{4}{n}$

$\underbrace{-1 \leq \sin n}$ ≤ 1

smaller diverges so does larger by CT (p-series $p=1$)

(d) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n}$

AST

show $\frac{1}{n}$ decreases ✓

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\boxed{\text{converges}}$

2. Determine if the series converges absolutely, converges conditionally, or diverges. FULLY explain your reasoning.

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(3+\ln n)^3}$ converges by AST

$\frac{1}{n(3+\ln n)^3}$ decreases to zero.

Absolute convergence is determined by

- Ratio Test [only if terms contain a factorial or exponential]
- Test $\Sigma|a_n|$ for convergence

$\sum_{n=1}^{\infty} \frac{1}{n(3+\ln n)^3}$ positive, decreasing, use I.T.

$$\int_1^{\infty} \frac{1}{x(3+\ln x)^3}\,dx = \left.\frac{-1}{2(3+\ln x)^2}\right|_1^{\infty}$$

$$= 0 + \frac{1}{2(3)^2} < \infty$$

integral converges $\rightarrow$ original series converges absolutely.

b.) $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$ AC since $\sum_{n=1}^{\infty}\left|\frac{2}{n\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{2}{n\sqrt{n}}$ convergent p-series $p=\frac{3}{2}>1$

c.) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n+1}$ TD $\lim_{n\to\infty} \frac{(-1)^n n}{2n+1}$ dne by oscillation between $\pm\frac{1}{2}$

$\boxed{\text{diverges by T.D.}}$

d.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$ converges by AST since $\frac{1}{5n+1}$ decreases to zero.

AC? Test $\sum_{n=1}^{\infty}\frac{1}{5n+1} \le \sum_{n=1}^{\infty}\frac{1}{5n}$

bigger diverges, CT fails!

could use I.T. or L.C.T.

LCT If $\lim_{n\to\infty}\frac{a_n}{b_n} = c>0$ and finite

both series do the same thing!

$$\lim_{n\to\infty}\frac{\frac{1}{5n+1}}{\frac{1}{5n}} = \lim_{n\to\infty}\frac{5n}{5n+1} = 1 > 0$$

$\therefore$ since $\Sigma\frac{1}{5n}$ diverges by p-series $p=1$ so does $\Sigma\frac{1}{5n+1}$

so $\Sigma\frac{(-1)^n}{5n+1}$ converges conditionally

e.) $\sum_{n=1}^{\infty} \frac{(-10)^n n!}{(2n+3)!}$ $\ln n < n^p < a^n < n! < n^n$

RT

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-10)^{n+1}(n+1)!}{(2n+5)!}\cdot\frac{(2n+3)!}{(-10)^n n!}\right|$$

$[2(n+1)+3]!$

$$= \lim_{n\to\infty}\left|\frac{(-10)\cancel{(-10)^n}(n+1)\cancel{n!}}{(2n+5)(2n+4)\cancel{(2n+3)!}}\cdot\frac{\cancel{(2n+3)!}}{\cancel{(-10)^n}\cancel{n!}}\right|$$

$= 0 < 1$ so it converges Absolutely by RT

3. Find the radius and interval of convergence for $\sum_{n=2}^{\infty} \frac{(x+3)^n}{5^n\sqrt{n-1}}$. FULLY explain your reasoning.

$$\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{5^{n+1}\sqrt{n}}\cdot\frac{5^n\sqrt{n-1}}{(x+3)^n}\right| = \lim_{n\to\infty}\left|\frac{(x+3)\cancel{(x+3)^n}}{5\cdot\cancel{5^n}\sqrt{n}}\,\frac{\cancel{5^n}\sqrt{n-1}}{\cancel{(x+3)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+3)\sqrt{n-1}}{5\sqrt{n}}\right| \quad \frac{\sqrt{n-1}}{\sqrt{n}} \to 1$$

$|x-a| < R$

$$= \left|\frac{x+3}{5}\right| < 1 \qquad \boxed{R=5}$$

$$\boxed{|x+3| < 5}$$

$$-5 < x+3 < 5$$

$$-8 < x < 2$$

Test endpoints

$$\sum_{n=2}^{\infty}\frac{(x+3)^n}{5^n\sqrt{n-1}}$$

$x=2$: $\sum_{n=2}^{\infty}\frac{5^n}{5^n\sqrt{n-1}} =$

$$\sum_{n=2}^{\infty}\frac{1}{\sqrt{n-1}} \geq \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}$$

smaller diverges p-series $p=\frac{1}{2}$

larger diverges by CT

$x=-8$: $\sum_{n=2}^{\infty}\frac{(-5)^n}{5^n\sqrt{n-1}} = \sum_{n=2}^{\infty}\frac{(-1)^n 5^n}{5^n\sqrt{n-1}}$

$$= \sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n-1}}$$

converges by AST since $\frac{1}{\sqrt{n-1}}$ decreases to zero

$I = [-8, 2)$

$-8 \leq x < 2$

4. Find the radius and interval of convergence for $\sum_{n=0}^{\infty}\frac{(2x-3)^{n+1}n!}{100^n}$. ~~FULLY~~ explain your reasoning.

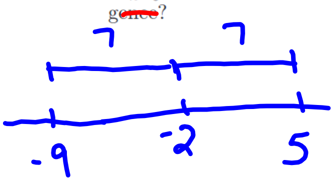
$R=0$

$I = \left\{\frac{3}{2}\right\}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(2x-3)^{n+2}}(n+1)!}{\cancel{100^{n+1}}\,100}\,\frac{\cancel{100^n}}{n!\,\cancel{(2x+3)^{n+1}}}\right|$$

$$= \infty < 1 \text{ if } x = \frac{3}{2}$$

5. If $\sum_{n=0}^{\infty}c_n(x+2)^n$ converges at $x=5$, on what interval are we guaranteed convergence?



$(-9, 5]$

6. For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+3)!}$, Use the Alternating Series Estimation Theorem to find an upper bound for the error if we used s_5 to estimate the sum.

ASET Given $\sum (-1)^n a_n, \quad a_n > 0$

then $|R_n| = |\text{error}| \leq a_{n+1}$

here, $n = 5$ $\quad |R_5| \leq a_6 = \frac{36}{9!}$

7. Using The Alternating Series Estimation Theorem, what is the smallest value of n that guarantees s_n approximates $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+3}$ with error less than $\frac{1}{20}$?

Know $|R_n| \leq a_{n+1} = \frac{1}{2n+5} < \frac{1}{20}$

$20 < 2n+5$

$n = 8$ smallest value that makes this true

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

8. Find a power series centered at 0 for the following functions:

a.) $\frac{4}{6-x^2}$

$$= \frac{4}{6\left(1-\frac{x^2}{6}\right)} = \frac{4}{6}\sum_{n=0}^{\infty}\left(\frac{x^2}{6}\right)^n, \quad \left|\frac{x^2}{6}\right| < 1$$

$$|x^2| < 6$$

$$= \frac{4}{6}\sum_{n=0}^{\infty}\frac{x^{2n}}{6^n} \qquad |x| < \sqrt{6}$$

$$\boxed{R = \sqrt{6}}$$

$$\frac{4}{6-x^2} = \sum_{n=0}^{\infty}\frac{4x^{2n}}{6^{n+1}}$$

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

b.) $\frac{8x}{(6-x^2)^2}$, by using the result from above.

$$\frac{d}{dx}\int\frac{8x}{(6-x^2)^2}\,dx = \frac{4}{6-x^2} = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{4\cdot x^{2n}}{6^{n+1}}$$

$u = 6-x^2$

$du = -2x\,dx$

$$-\frac{8}{2}\int\frac{1}{u^2}\,du$$

$$\frac{4}{u} = \frac{4}{6-x^2}$$

$$\frac{8x}{(6-x^2)^2} = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{4x^{2n}}{6^{n+1}}$$

$$= \sum_{n=1}^{\infty}\frac{4}{6^{n+1}}(2n)x^{2n-1}$$

change index

$$= \sum_{\boxed{n=0}}^{\infty}\frac{4}{6^{n+2}}(2n+2)x^{2n+1}$$

re-index

c.) $\int x^4\arctan(5x)\,dx$

$$\arctan(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\int x^4\arctan(5x)\,dx = \int x^4\sum_{n=0}^{\infty}\frac{(-1)^n(5x)^{2n+1}}{2n+1}\,dx$$

$$= \int x^4\sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}x^{2n+1}}{2n+1}\,dx$$

$$= \int\sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}x^{2n+5}}{2n+1}\,dx$$

$$= C + \sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}x^{2n+6}}{(2n+1)(2n+6)}$$

9. Evaluate $\int_0^1 x^4 \ln(2-x^3)\,dx$

① $\int \frac{d}{dx}\ln(2-x^3) = \frac{-3x^2}{2-x^3} = \frac{-3x^2}{2\left(1-\frac{x^3}{2}\right)}$

$$= \frac{-3x^2}{2}\sum_{n=0}^{\infty}\left(\frac{x^3}{2}\right)^n$$

$$= \frac{-3x^2}{2}\sum_{n=0}^{\infty}\frac{x^{3n}}{2^n}$$

$$\ln(2-x^3) = \int \sum_{n=0}^{\infty}\frac{-3x^{3n+2}}{2^{n+1}}\,dx$$

$$= \xrightarrow{\ln 2}\cancel{C} + \sum_{n=0}^{\infty}\frac{-3x^{3n+3}}{2^{n+1}(3n+3)}$$

let $x=0$ to find C

$$\ln 2 = C + \Sigma\, 0$$

step 2: multiply by x^4

$$\int_0^1 x^4\ln(2-x^3) = \int_0^1\left(x^4\ln 2 + \sum_{n=0}^{\infty}\frac{-3x^{3n+7}}{2^{n+1}(3n+3)}\right)dx$$

$$= \left[\frac{x^5}{5}\ln 2 + \sum_{n=0}^{\infty}\frac{-3x^{3n+8}}{2^{n+1}(3n+3)(3n+8)}\right]\Bigg|_0^1$$

$$= \frac{1}{5}\ln 2 + \sum_{n=0}^{\infty}\frac{-3}{2^{n+1}(3n+3)(3n+8)}$$

10. Find $f^{(26)}(2)$ if $f(x) = \sum_{n=0}^{\infty} \frac{3^{n+1}(x-2)^n}{(n+8)!}$ is the Taylor Series for $f(x)$ centered at $a = 2$.

Taylor series centered at 2 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(2)}{n!}(x-2)^n \overset{\text{given}}{=} \sum_{n=0}^{\infty} \frac{3^{n+1}(x-2)^n}{(n+8)!}$$

$$\frac{f^n(2)}{n!} = \frac{3^{n+1}}{(n+8)!}$$

$$f^n(2) = \frac{3^{n+1}\,n!}{(n+8)!}$$

$$\boxed{f^{26}(2) = \frac{3^{27}(26)!}{(34)!}}$$

11. Find the Taylor Series centered at 4 for $f(x) = \frac{1}{(x+1)^2}$.

$$\sum_{n=0}^{\infty} \frac{f^n(4)}{n!}(x-4)^n$$

$$f(x) = \frac{1}{(x+1)^2} = (x+1)^{-2}$$

$$f'(x) = \frac{-2}{(x+1)^3} = -2(x+1)^{-3}$$

$$f''(x) = \frac{3\cdot 2}{(x+1)^4}$$

$$f'''(x) = \frac{-4\cdot 3\cdot 2}{(x+1)^5}$$

$$f^n(x) = \frac{(-1)^n(n+1)!}{(x+1)^{n+2}}$$

$$f^n(4) = \frac{(-1)^n(n+1)!}{5^{n+2}}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \boxed{(n+1)!}}{5^{n+2}\cancel{n!}}(x-4)^n$$

$(n+1)\cancel{n!}$

12. Find a Maclaurin series for e^{3x^2}.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \longrightarrow e^{3x^2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!}$$

13. Express $\int x^4 \cos(5x^3)\,dx$ as a power series about 0.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$5^{2n} x^{6n}$$

$$\int x^4 \cos(5x^3)\,dx = \int x^4 \sum_{n=0}^{\infty} \frac{(-1)^n (5x^3)^{2n}}{(2n)!}$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{6n+4}}{(2n)!}\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{6n+5}}{(2n)!\,(6n+5)}$$

14. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!}$ $= \boxed{\sin(2)}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

15. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-5)^n 2^{2n+1}}{n!}$ $= \sum_{n=0}^{\infty} \frac{(-5)^n 2^{2n} \cdot 2}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^{\boxed{n}}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-5)^n 4^n\, 2}{n!}$$

$$= 2 \sum_{n=0}^{\infty} \frac{(-20)^n}{n!}$$

$$= 2e^{-20}$$

16. Find the third degree Taylor Polynomial for $f(x) = e^{-x}$ at $x = 2$.

$$T_3(x) = \cancel{f(2)}^{\,e^{-2}} + \frac{\cancel{f'(2)}^{\,-e^{-2}}}{1!}(x-2) + \frac{\cancel{f''(2)}^{\,e^{-2}}}{2!}(x-2)^2 + \frac{\cancel{f'''(2)}^{\,-e^{-2}}}{3!}(x-2)^3$$

$f(2) = e^{-2}$

$f'(2) = -e^{-2}$

$f''(2) = e^{-2}$

$f'''(2) = -e^{-2}$