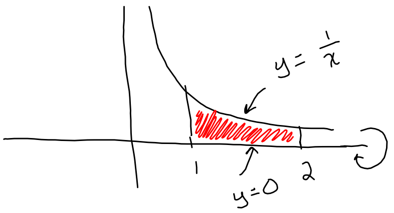
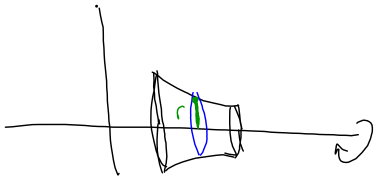
Section 6.2

1. Find the volume of the solid obtained by revolving the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 2$ about the x-axis.



Disk method :



$r = T - B$

$r = \frac{1}{x} - 0$

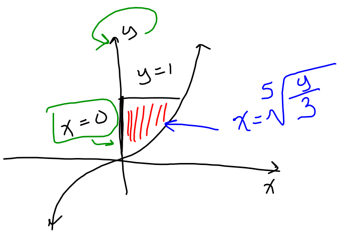
$V = \int_1^2 \pi r^2 \, dx$

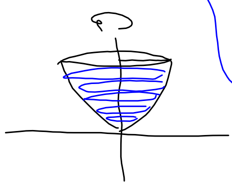
$V = \int_1^2 \pi \left(\frac{1}{x}\right)^2 dx = \pi\left(-\frac{1}{x}\right)\Big|_1^2$

$= \pi\left(-\frac{1}{2}+1\right)$

$= \boxed{\frac{\pi}{2}}$

2. Find the volume of the solid obtained by revolving the region bounded by $y = 3x^5$, $y = 1$ and $x = 0$ about the y-axis.





Disk method dy

$y = 3x^5$

$\frac{y}{3} = x^5$

$\sqrt[5]{\frac{y}{3}} = x$

$r = R - L$

$r = \sqrt[5]{\frac{y}{3}} - 0$

$V = \int_0^1 \pi r^2 \, dy$

$= \pi \int_0^1 \left(\sqrt[5]{\frac{y}{3}}\right)^2 dy$

$= \pi \int_0^1 \left(\frac{y}{3}\right)^{2/5} dy$

$= \pi \int_0^1 \frac{y^{2/5}}{3^{2/5}} \, dy$

$= \frac{\pi}{3^{2/5}} \int_0^1 y^{\frac{2}{5}} \, dy$

$= \frac{\pi}{3^{2/5}} \frac{5}{7} y^{\frac{7}{5}} \Big|_0^1 = \boxed{\frac{\pi}{3^{2/5}} \cdot \frac{5}{7}}$

3. Find the volume of the solid obtained by revolving the region bounded by $y = x^2$ and $y = 4x$ about the x-axis, then the y axis.

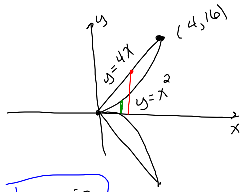
intersect: $x^2 = 4x$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0 \longrightarrow y = 0$

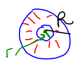
$x = 4 \longrightarrow y = 16$



$R = \text{"T"} = 4x$

$r = \text{"B"} = x^2$

washer method:



$A = \pi R^2 - \pi r^2$

$A = \pi(R^2 - r^2)$

① x-axis

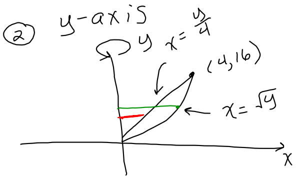
$$V = \int_0^4 \pi (R^2 - r^2)\,dx$$

$$= \pi \int_0^4 (16x^2 - x^4)\,dx$$

$$= \pi\left(\frac{16}{3}x^3 - \frac{x^5}{5}\right)\Big|_0^4$$

$$= \pi\left(\frac{16}{3}(4^3) - \frac{1}{5}(4^5)\right)$$

② y-axis



$R = \text{right} = \sqrt{y}$

$r = \text{left} = \frac{y}{4}$

$y = 4x \rightarrow x = \frac{1}{4}y$

$y = x^2 \rightarrow x = \sqrt{y}$

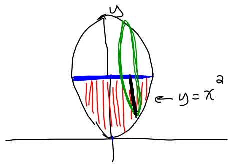
$$V = \int_0^{16} \pi (R^2 - r^2)\,dy$$

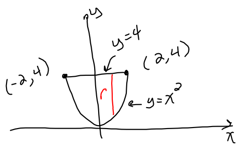
$$= \pi \int_0^{16} \left(y - \frac{y^2}{16}\right) dy$$

$$= \pi\left(\frac{y^2}{2} - \frac{y^3}{48}\right)\Big|_0^{16}$$

$$= \boxed{\pi\left(\frac{16^2}{2} - \frac{16^3}{48}\right)}$$

4. Find the volume of the solid obtained by revolving the region bounded by $y = x^2$, $y = 4$, about the line $y = 4$. ← horizontal line → dx





$r = T - B$

$= 4 - x^2$

disk! with dx

$$V = \int_{-2}^{2} \pi r^2\,dx$$

OR use symmetry

$$= 2\int_0^2 \pi r^2\,dx$$

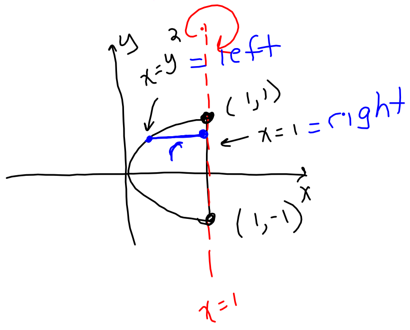
$$= 2\int_0^2 \pi(4 - x^2)^2\,dx$$

$$= 2\pi \int_0^2 (16 - 8x^2 + x^4)\,dx$$

$$= 2\pi\left(16x - \frac{8}{3}x^3 + \frac{x^5}{5}\right)\Big|_0^2$$

$$= \boxed{2\pi\left(32 - \frac{64}{3} + \frac{32}{5}\right)}$$

5. Find the volume of the solid obtained by revolving the region bounded by $x = y^2$, $x = 1$, about the line $x = 1$.



r= right - left

revolved around a vertical line → dy

disk

$$V = \int_{-1}^{1} \pi r^2 \, dy \quad \text{or} \quad 2\int_0^1 \pi r^2 \, dy$$

symmetry

$r = 1 - y^2$

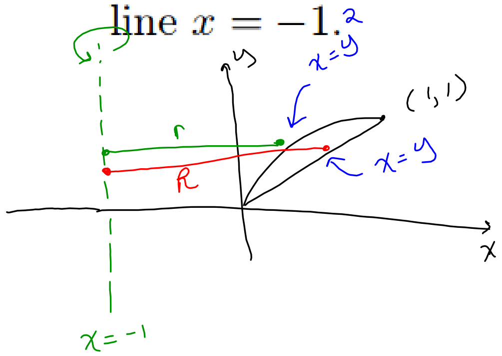
$$2\pi \int_0^1 (1-y^2)^2 \, dy$$

$$2\pi \int_0^1 (1 - 2y^2 + y^4) \, dy$$

$$2\pi \left(y - \frac{2}{3}y^3 + \frac{y^5}{5} \right)\Big|_0^1$$

$$2\pi \left(1 - \frac{2}{3} + \frac{1}{5} \right)$$

6. Find the volume of the solid obtained by revolving the region bounded by $y = x$, $y = \sqrt{x}$, about the line $x = -1$.



vertical line rotation → dy!

$$V = \int_0^1 \pi (R^2 - r^2) \, dy$$

solve all equations for x!

R = right - left = $y - (-1) = y+1$

r = right - left = $y^2 - (-1) = y^2 + 1$

$$V = \pi \int_0^1 \left((y+1)^2 - (y^2+1)^2 \right) dy$$

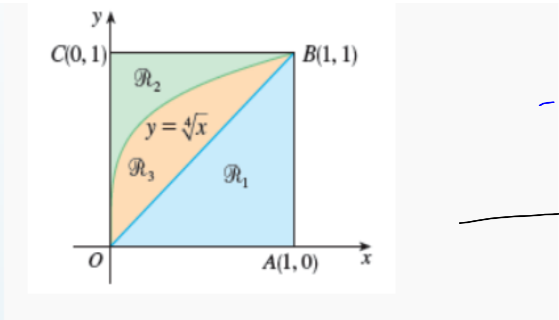
$$= \pi \int_0^1 \left(y^2 + 2y + 1 - (y^4 + 2y^2 + 1) \right) dy$$

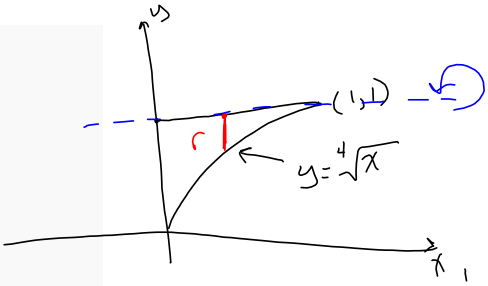
$$= \pi \int_0^1 \left(-y^4 - y^2 + 2y \right) dy$$

$$= \pi \left(-\frac{y^5}{5} - \frac{y^3}{3} + y^2 \right)\Big|_0^1$$

7. Refer to the figure below to set up but do not evaluate an integral that finds the volume generated by rotating the given region about the specified line.

(a) $\mathcal{R}_2$ about BC





rotate about the line $y=1$

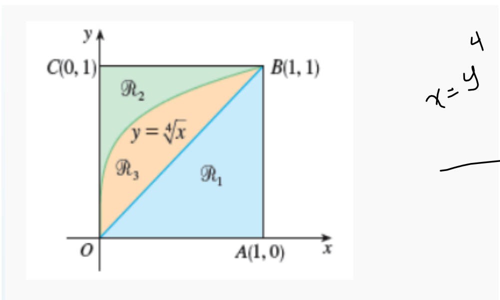
$r = T - B$

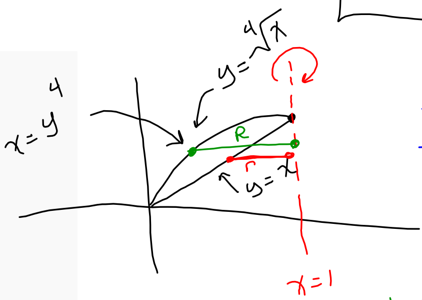
$r = 1 - \sqrt[4]{x}$

disk $\quad V = \int_0^1 \pi r^2 \, dx$

$$\boxed{V = \pi \int_0^1 \left(1 - \sqrt[4]{x}\right)^2 dx}$$

(b) $\mathcal{R}_3$ about AB





washer

$V = \int_0^1 \pi (R^2 - r^2) \, dy$

$R = \text{right} - \text{left}$

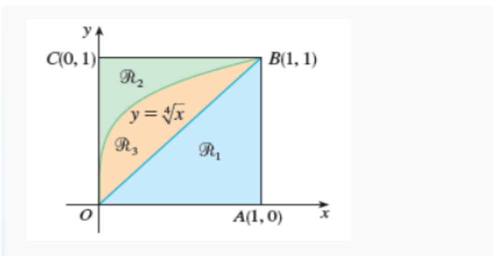
$R = 1 - y^4$

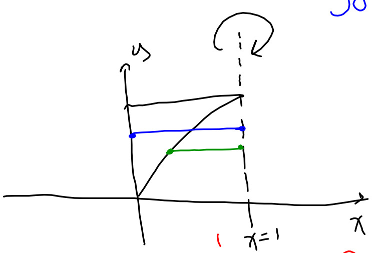
$r = \text{right} - \text{left}$

$r = 1 - y$

$V = \int_0^1 \pi \left((1-y^4)^2 - (1-y)^2 \right) dy$

(c) $\mathcal{R}_2$ about AB





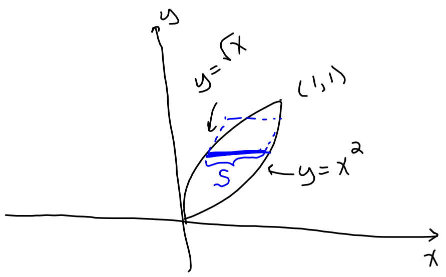
$R = 1$

$r = 1 - y^4$

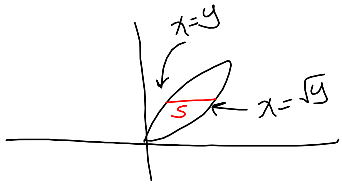
washer: $\quad V = \int_0^1 \pi (R^2 - r^2) \, dy$

$= \pi \int_0^1 \left(1 - (1-y^4)^2 \right) dy$

9. Find the volume of S where the base of S is the region bounded by $y = x^2$ and $y = \sqrt{x}$. The cross sections perpendicular to the y-axis are squares.



$$V = \int A_{\text{square}}\, dy$$



$A_{\text{square}} = S^2$

$S = \text{right} - \text{left}$

$= \sqrt{y} - y^2$

$$V = \int_0^1 \left(\sqrt{y} - y^2\right)^2 dy$$

$$= \int_0^1 \left(y - 2\sqrt{y}\,y^2 + y^4\right) dy$$

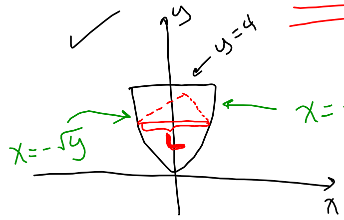
$$= \int_0^1 \left(y - 2y^{\frac{5}{2}} + y^4\right) dy$$

$$= \frac{1}{2}y^2 - \frac{4}{7}y^{\frac{7}{2}} + \frac{y^5}{5}\Big|_0^1$$

$$= \boxed{\frac{1}{2} - \frac{4}{7} + \frac{1}{5}}$$

10. Find the volume of the solid S where the base of S is the region bounded by $y = x^2$ and $y = 4$. The cross-sections perpendicular to the y axis are equilateral triangles.

$\hookrightarrow dy$



$$V = \int A_{\text{triangle}}\, dy$$

$L = \text{right} - \text{left}$

$L = \sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}$

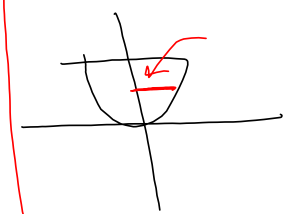
$$\boxed{A_{\text{equilateral triangle}} = \frac{\sqrt{3}}{4} L^2}$$

Find L in terms of y

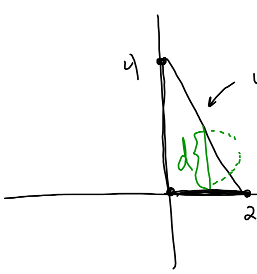
$$V = \int_0^4 \frac{\sqrt{3}}{4}\left(2\sqrt{y}\right)^2 dy$$

$$= \int_0^4 \sqrt{3}\, y\, dy = \sqrt{3}\,\frac{y^2}{2}\Big|_0^4 = \boxed{8\sqrt{3}}$$

use symmetry



11. Find the volume of the solid S where the base of S is the triangular region with vertices $(0,0)$, $(2,0)$ and $(0,4)$. Cross-sections perpendicular to the x $\rightarrow dx$ axis are semi-circles.



$y=-2x+4$

$$V = \int_0^2 A_{semicircle}\, dx$$

$d = -2x+4$

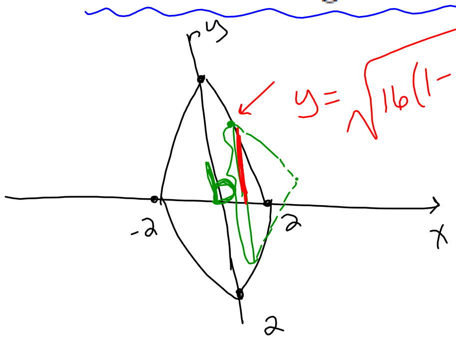
$r = \frac{1}{2}d = \frac{1}{2}(-2x+4) = -x+2$

$$A_{semicircle} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(-x+2)^2$$

$$V = \int_0^2 \frac{\pi}{2}(-x+2)^2\, dx$$

$$= \frac{\pi}{2}\int_0^2 (x^2-4x+4)\, dx$$

12. Find the volume of the solid S where the base of S is the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{16}=1$. Cross sections perpendicular to the x-axis are isosceles triangles where the base and height are equal. $b=h$



$y=\sqrt{16\left(1-\frac{x^2}{4}\right)}$

$b=2y$

$b^2=4y^2$

$b^2=4\left(16\left(1-\frac{x^2}{4}\right)\right)$

$\frac{x^2}{4}+\frac{y^2}{16}=1 \longrightarrow \frac{y^2}{16}=1-\frac{x^2}{4}$

$y^2=16\left(1-\frac{x^2}{4}\right)$

$y=\sqrt{16\left(1-\frac{x^2}{4}\right)}$

x-int: $\frac{x^2}{4}=1$

$x=\pm 2$

y-int: $\frac{y^2}{16}=1$

$y=\pm 4$

$V=\int_{-2}^{2} A_{\text{isosceles triangle}}\,dx$

$=2\int_0^2 A_{\text{isosceles triangle}}\,dx$

$=2\cdot\frac{1}{2}\int_0^2 64\left(1-\frac{x^2}{4}\right)dx$

$A_{\text{isosceles triangle}}=\frac{1}{2}bh$

since $b=h$ $\quad=\frac{1}{2}b^2$