Form	Substitution	Identity Used	Domain
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$\cos^2\theta = 1 - \sin^2\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$\sec^2\theta = \tan^2\theta + 1$	$0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$\tan^2\theta = \sec^2\theta - 1$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Section 7.3

1. $\displaystyle\int x^3\sqrt{4 - x^2}\,dx$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$$\int 8\sin^3\theta\sqrt{4 - 4\sin^2\theta}\;2\cos\theta\,d\theta$$

$$16\int \sin^3\theta\sqrt{4(\underbrace{1 - \sin^2\theta}_{\cos^2\theta})}\;\cos\theta\,d\theta$$

$$16\int \sin^3\theta \cdot 2\cos\theta\cos\theta\,d\theta$$

$$32\int \sin^3\theta\cos^2\theta\,d\theta$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$32\int \underbrace{\sin^2\theta}_{\substack{\downarrow\\ 1-\cos^2\theta\\ \downarrow\\ 1-u^2}}\underbrace{\cos^2\theta}_{u^2}\underbrace{\sin\theta\,d\theta}_{-du}$$

$$-32\int (1-u^2)u^2\,du$$

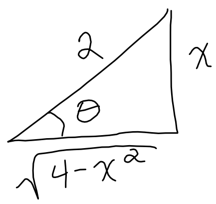
$$-32\int (u^2 - u^4)\,du$$

$$-32\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C$$

$$-32\left(\frac{\cos^3\theta}{3} - \frac{\cos^5\theta}{5}\right) + C$$

Recall: $x = 2\sin\theta$

$\frac{O}{H}$ $\quad \frac{x}{2} = \sin\theta$



$\cos\theta = \frac{A}{H} = \frac{\sqrt{4-x^2}}{2}$

$$-32\left[\frac{\left(\frac{\sqrt{4-x^2}}{2}\right)^3}{3} - \frac{\left(\frac{\sqrt{4-x^2}}{2}\right)^5}{5}\right] + C$$

2. $\displaystyle\int_{5\sqrt{2}}^{10} \frac{dx}{x^3\sqrt{x^2-25}}$

$x = 5\sec\theta$

$x = 10 \to 10 = 5\sec\theta \to 2 = \sec\theta \to \frac{1}{2} = \cos\theta$ $\boxed{\theta = \frac{\pi}{3}}$

$x = 5\sqrt{2} \to 5\sqrt{2} = 5\sec\theta \to \sqrt{2} = \sec\theta \to \frac{1}{\sqrt{2}} = \cos\theta$

$\cos\theta = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\boxed{\theta = \frac{\pi}{4}}$

$dx = 5\sec\theta\tan\theta\,d\theta$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{5\sec\theta\tan\theta\,d\theta}{125\sec^3\theta\sqrt{\underbrace{25\sec^2\theta - 25}_{\substack{25(\sec^2\theta-1)\\ 25\cdot\tan^2\theta}}}}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cancel{5}\,\cancel{\sec\theta}\,\cancel{\tan\theta}\,d\theta}{125\sec^{\cancel{3}\,2}\theta\cdot\cancel{5}\,\cancel{\tan\theta}} = \frac{1}{125}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{125}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\,d\theta$$

$$= \frac{1}{125}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2}\left(1+\cos 2\theta\right)d\theta$$

$$= \frac{1}{250}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{250}\left(\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3} - \left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{250}\left(\frac{\pi}{3} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{4} - \frac{1}{2}\right)$$

3. $\int \frac{1}{x^2\sqrt{16x^2-9}}\,dx = \int \frac{dx}{x^2\sqrt{(4x)^2-9}}$

$4x = \boxed{3\sec\theta}$

$x = \frac{3}{4}\sec\theta$

$dx = \frac{3}{4}\sec\theta\tan\theta\,d\theta$

$$= \int \frac{\frac{3}{4}\cancel{\sec\theta}\tan\theta\,d\theta}{\frac{9}{16}\sec^{\cancel{2}}\theta\sqrt{9\sec^2\theta-9}}$$

$\hookrightarrow 9(\sec^2\theta-1)=9\tan^2\theta$

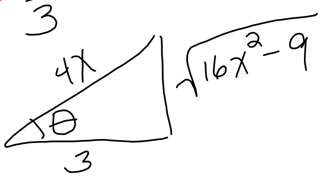
$$\frac{\cancel{3}}{\cancel{4}}\,\frac{\overset{4}{\cancel{16}}}{\cancel{9}\,3}\int \frac{\tan\theta\,d\theta}{\sec\theta\,3\tan\theta}$$

$$\frac{4}{9}\int\frac{d\theta}{\sec\theta} = \frac{4}{9}\int\cos\theta\,d\theta$$

$$= \frac{4}{9}\sin\theta + C$$

$$= \frac{4}{9}\left(\frac{\sqrt{16x^2-9}}{4x}\right)+C$$

$\frac{H}{A}$ $\frac{4x}{3}=\sec\theta$



$\sin\theta = \frac{O}{H}$

4. $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+8}}$

$$\left(\frac{4}{2}\right)^2 = 2^2 = 4$$

$$\underbrace{x^2+4x+4}+\underbrace{8-4}$$

$$(x+2)^2+4$$

$$\int \frac{dx}{\sqrt{(x+2)^2+4}}$$

$$\boxed{x+2 = 2\tan\theta}$$

$$dx + 0 = 2\sec^2\theta \, d\theta$$

$$\int \frac{2\sec^2\theta \, d\theta}{\sqrt{4\tan^2\theta+4}} = \int \frac{2\sec^2\theta \, d\theta}{2\sec\theta}$$

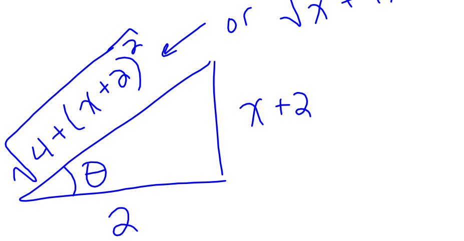
$$4(\tan^2\theta+1)$$

$$4\sec^2\theta$$

$$= \int \sec\theta \, d\theta$$

$$= \ln\left|\sec\theta+\tan\theta\right| + C$$

$$\frac{x+2}{2} = \tan\theta$$



$$\boxed{= \ln\left|\frac{\sqrt{x^2+4x+8}}{2} + \frac{x+2}{2}\right| + C}$$

5. $\displaystyle\int \frac{1}{(x^2+6x+13)^{3/2}}\,dx$

$\underline{x^2+6x+9} \quad + \quad \frac{13-9}{4}$

$(x+3)^2 + 4$

$x+3 = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{dx}{\left[(x+3)^2+4\right]^{3/2}}$$

$$\int \frac{2\sec^2\theta\, d\theta}{(4\tan^2\theta+4)^{3/2}}$$

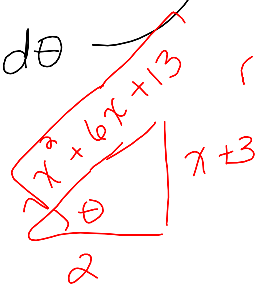
$4^{\frac{3}{2}} = 2^3 = 8$

$$\int \frac{2\sec^2\theta\, d\theta}{(4\sec^2\theta)^{3/2}}$$

$$\int \frac{2\sec^2\theta\, d\theta}{8\sec^3\theta}$$

$$\frac{1}{4}\int \frac{1}{\sec\theta}\, d\theta \longrightarrow \frac{1}{4}\int \cos\theta\, d\theta$$

$$\frac{1}{4}\sin\theta + C$$



recall $x+3 = 2\tan\theta$

$\frac{x+3}{2} = \tan\theta$

$$\frac{1}{4}\left(\frac{x+3}{\sqrt{x^2+6x+13}}\right) + C$$

6. $\displaystyle\int_0^{2/3} \frac{1}{(4+9x^2)^{5/2}}\,dx = \int_0^{\frac{2}{3}} \frac{dx}{\left[4+(3x)^2\right]^{5/2}}$

$3x = 2\tan\theta$

$x = \frac{2}{3}$: $3\left(\frac{2}{3}\right) = 2\tan\theta \rightarrow 1 = \tan\theta$, $\boxed{\frac{\pi}{4} = \theta}$

$x = 0$: $3(0) = 2\tan\theta \rightarrow 0 = \tan\theta$, $\boxed{0 = \theta}$

$x = \frac{2}{3}\tan\theta$

$dx = \frac{2}{3}\sec^2\theta\,d\theta$

$$\int_0^{\frac{\pi}{4}} \frac{\frac{2}{3}\sec^2\theta\,d\theta}{\left[4+4\tan^2\theta\right]^{5/2}}$$

$$\frac{2}{3}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\,d\theta}{\left[4\sec^2\theta\right]^{5/2}}$$

$$\frac{2}{3}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\,d\theta}{4^{5/2}\cdot\sec^5\theta}$$

$$\frac{2}{3}\cdot\frac{1}{32}\int_0^{\frac{\pi}{4}} \frac{1}{\sec^3\theta}\,d\theta$$

$$\frac{1}{48}\int_0^{\frac{\pi}{4}} \cos^3\theta\,d\theta$$

$$\frac{1}{48}\int_0^{\frac{\pi}{4}} \cos^2\theta\,\underbrace{\cos\theta\,d\theta}_{du}$$

$\rightarrow 1-\sin^2\theta = 1-u^2$

$u = \sin\theta$

$\theta = \frac{\pi}{4}$, $u = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\theta = 0$, $u = \sin 0 = 0$

$du = \cos\theta\,d\theta$

$$\frac{1}{48}\int_0^{\frac{\sqrt{2}}{2}} (1-u^2)\,du$$

$$\frac{1}{48}\left(u - \frac{u^3}{3}\right)\Bigg|_0^{\frac{\sqrt{2}}{2}}$$

$$\frac{1}{48}\left(\frac{\sqrt{2}}{2} - \frac{\left(\frac{\sqrt{2}}{2}\right)^3}{3}\right)$$

#9 $\int \tan^2\theta \, d\theta = \int (\sec^2\theta - 1)\, d\theta$

$= \tan\theta - \theta + C$