Section 7.4

1. $\displaystyle\int \frac{4x+5}{2x^3+5x^2+3x}\,dx$

① degree on bottom is higher than degree on top ✓

② Factor bottom!

$$\frac{4x+5}{x(2x^2+5x+3)}$$

$$\frac{4x+5}{\cancel{x(2x+3)(x+1)}} = \left(\frac{A}{x} + \frac{B}{2x+3} + \frac{C}{x+1}\right)\left(x(2x+3)(x+1)\right)$$

$A \to \frac{5}{3}$, $B \to -\frac{4}{3}$, $C \to -1$

$$4x+5 = A(2x+3)(x+1) + Bx(x+1) + Cx(2x+3)$$

$x=0$: $5 = A(3)(1) \longrightarrow \boxed{A = \frac{5}{3}}$

$x=-1$: $1 = C(-1)(1) \longrightarrow \boxed{C=-1}$

$x = -\frac{3}{2}$: $4\left(-\frac{3}{2}\right) + 5 = B\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)$

$-1 = \frac{3}{4}B \longrightarrow \boxed{B = -\frac{4}{3}}$

$$\int \left(\frac{5/3}{x} - \frac{4/3}{2x+3} - \frac{1}{x+1}\right) dx$$

$$= \frac{5}{3}\ln|x| - \frac{4}{3}\frac{1}{2}\ln|2x+3| - \ln|x+1| + C$$

2. $\displaystyle\int \frac{x^3+2x+1}{x^2+4x}\,dx$ since power on bottom is <u>NOT</u> higher, must long divide

Q

D

$$\begin{array}{r} x-4 \\ x^2+4x \overline{\big)\, x^3+0x^2+2x+1} \\ -x^3 \mp 4x^2 \\ \hline -4x^2+2x+1 \\ +4x^2+16x \\ \hline 18x+1 \end{array}$$

R

$Q+\frac{R}{D}$

$$\int\left(x-4+\frac{18x+1}{x^2+4x}\right)dx$$

PFD : $\dfrac{18x+1}{\cancel{x(x+4)}} = \left(\dfrac{\cancel{A}}{x}+\dfrac{\cancel{B}}{x+4}\right)\cancel{(x(x+4))}$ $\quad A \to \frac{1}{4}, \quad B \to \frac{71}{4}$

$18x+1 = A(x+4)+Bx$

$x=0$: $1=A(4)$ $\quad A=\frac{1}{4}$

$x=-4$: $-71=B(-4)$ $\quad B=\frac{71}{4}$

$\overset{3}{18}$
$\frac{4}{72}$

$$\int\left(x-4+\frac{\frac{1}{4}}{x}+\frac{\frac{71}{4}}{x+4}\right)dx$$

$$\boxed{\frac{x^2}{2}-4x+\frac{1}{4}\ln|x|+\frac{71}{4}\ln|x+4|+C}$$

3. $\displaystyle\int_1^2 \frac{dx}{x(x^2+2x+1)}$

$$\frac{1}{x(x+1)^2} = \left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\right) x(x+1)^2$$

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$

$x=0$: $\boxed{1 = A}$

$x=-1$: $1 = C(-1)$ $\boxed{C=-1}$

$$1 = (x+1)^2 + Bx(x+1) - x$$

$x=1$: $1 = 4 + 2B - 1$

$1 = 3 + 2B$

$-2 = 2B$ $\boxed{B=-1}$

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\right) dx$$

$u = x+1$

$du = dx$

$\int -\frac{1}{u^2}\,du$

$\frac{1}{u} = \frac{1}{x+1}$

$$\left(\ln|x| - \ln|x+1| + \frac{1}{x+1}\right)\Bigg|_1^2$$

$$\underbrace{\ln 2 - \ln 3} + \frac{1}{3} - \left(\ln 1 - \ln 2 + \frac{1}{2}\right)$$

$$\ln\frac{2}{3} + \frac{1}{3} + \ln 2 - \frac{1}{2}$$

$$\boxed{\ln\frac{4}{3} - \frac{1}{6}}$$

4. $\displaystyle\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}\,dx$

irreducible quadratics have a "$Ax+B$" above it.

$$\frac{3x^2-4x+5}{\cancel{(x-1)(x^2+1)}} = \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+1}\right)(x-1)(x^2+1)$$

$$3x^2-4x+5 = A(x^2+1)+(Bx+C)(x-1)$$

$x=1$: $\quad 4 = A(2) \quad \boxed{A=2}$

$$3x^2-4x+5 = 2x^2+2+Bx^2-Bx+Cx-C$$

$$3x^2-4x+5 = (2+B)x^2+(C-B)x+2-C$$

$3 = 2+B \rightarrow \boxed{B=1}$

$-4 = C-B \rightarrow -4 = C-1 \rightarrow \boxed{C=-3}$

check : $\quad 5 = 2-C \quad 5=5$ ✓

$$\int\left(\frac{2}{x-1} + \frac{1\cdot x-3}{x^2+1}\right)dx$$

$$\int\frac{dx}{x^2+1} = \arctan x$$

$$\int\frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a}$$

$$\int\left(\frac{2}{x-1} + \frac{x}{x^2+1} - \frac{3}{x^2+1}\right)dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}\int\frac{1}{u}\,du$

$$\boxed{2\ln|x-1| + \frac{1}{2}\ln(x^2+1) - 3\arctan x + C}$$

5. $\displaystyle\int \frac{x+6}{(x^2+1)(x^2+4)}\,dx$

$$\frac{x+6}{\cancel{(x^2+1)(x^2+4)}} = \left(\frac{Ax+B}{x^2+1} + \frac{Cx+d}{x^2+4}\right)(x^2+1)(x^2+4)$$

$$x+6 = (Ax+B)(x^2+4) + (Cx+d)(x^2+1)$$

$$= Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + dx^2 + d$$

$$0x^3 + 0x^2 + x + 6 = (A+C)x^3 + (B+d)x^2 + (4A+C)x + 4B + d$$

$$0 = A + C \longrightarrow C = -A$$

$$d = -B \longleftarrow 0 = B + d$$

$$1 = 4A + C \longrightarrow 1 = 4A - A \longrightarrow 1 = 3A$$

$$\boxed{A = \tfrac{1}{3} \quad C = -\tfrac{1}{3}}$$

$$6 = 4B - B \longleftarrow 6 = 4B + d$$

$$6 = 3B$$

$$\boxed{B = 2 \quad d = -2}$$

$$\int \left(\frac{\frac{1}{3}x + 2}{x^2+1} + \frac{-\frac{1}{3}x - 2}{x^2+4}\right) dx$$

$$\int \left(\frac{\frac{1}{3}x}{x^2+1} + \frac{2}{x^2+1} - \frac{\frac{1}{3}x}{x^2+4} - \frac{2}{x^2+4}\right) dx$$

$$\frac{1}{3}\cdot\frac{1}{2}\ln|x^2+1| + 2\arctan x - \frac{1}{3}\cdot\frac{1}{2}\ln|x^2+4| - 2\cdot\frac{1}{2}\arctan\frac{x}{2}$$

$$+ C$$

Section 7.8

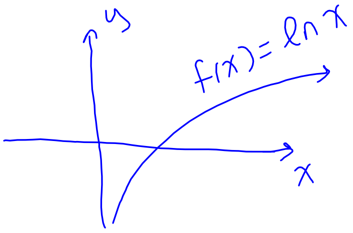
6. Determine whether the following improper integrals converge or diverge. If it converges, find the value of the integral. If it diverges, explain why.

a.) $\int_2^{\boxed{\infty}\, t} \frac{x}{x^2+1}\, dx$

$$\lim_{t\to\infty} \int_2^t \frac{x}{x^2+1}\, dx$$

$$\lim_{t\to\infty} \frac{1}{2} \ln(x^2+1) \Big|_2^t$$

$$\lim_{t\to\infty} \frac{1}{2}\left[\ln(\cancel{t^2+1})^{\infty} - \ln 5\right] = \infty$$



$\ln \infty = \infty$!

$\boxed{\text{integral diverges}}$

b.) $\int_e^{\boxed{\infty}\, t} \frac{1}{x(\ln x)^4}\, dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^4}\, dx$

$u = \ln x$ — $x = t,\ u = \ln t$; $x = e,\ u = \ln e = 1$

$du = \frac{1}{x} dx$

$\int u^{-4}\, du = \frac{u^{-3}}{-3} = -\frac{1}{3u^3}$

$$= \lim_{t\to\infty} \int_1^{\ln t} \frac{du}{u^4}$$

$$= \lim_{t\to\infty} \left(-\frac{1}{3u^3}\right)\Big|_1^{\ln t}$$

$$= \lim_{t\to\infty} \left(-\frac{1}{3(\ln t)^3} + \frac{1}{3}\right) = \boxed{\frac{1}{3}\ \text{converges}}$$

$$-\cancel{\frac{1}{3(\infty)^3}}^{\,0}$$

$\ln \infty = \infty$

c.) $\int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$

u	dv
x $+$	e^{-x}
1 $-$	$-e^{-x}$
0	e^{-x}

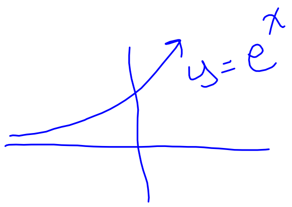
$$= \lim_{t\to\infty}\left(-xe^{-x} - e^{-x}\right)\Big|_0^t$$

$$= \lim_{t\to\infty}\left(-te^{-t} - e^{-t} + 1\right) = \boxed{1}$$

converges!

$-te^{-t} \to \infty \cdot 0$

$e^{-t} \to e^{-\infty} = 0$



$e^{\infty} = \infty$

$e^{-\infty} = 0$

$$\lim_{t\to\infty} -te^{-t} = \lim_{t\to\infty} \frac{-t}{e^t} \quad \frac{-\infty}{\infty}$$

$$\stackrel{L}{=} \lim_{t\to\infty} \frac{-1}{e^t} = \frac{-1}{e^{\infty}} = 0$$

d.) $\int_{\boxed{-\infty}}^{0} \frac{1}{1-2x}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1-2x}\,dx$

t

$= \lim_{t\to-\infty} -\frac{1}{2}\ln|1-2x| \Big|_t^0$

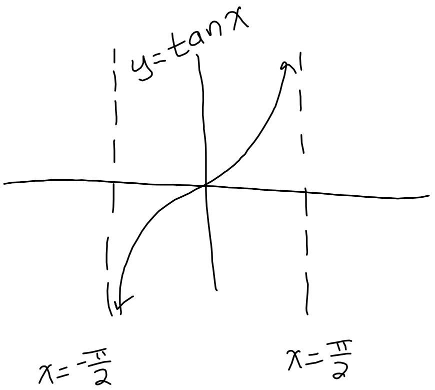
$\ln\left(\frac{1}{1-2t}\right)$

$= \lim_{t\to-\infty} -\frac{1}{2}\Big[\ln(1) - \underbrace{\ln|\underbrace{1-2t}_{\infty}|}_{\ln\infty\,=\,\infty}\Big]$

$= -\frac{1}{2}(0-\infty) = \boxed{\infty}$

e.) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+9} = \overset{1}{\int_{-\infty}^{0}} \frac{dx}{x^2+9} + \overset{2}{\int_{0}^{\infty}} \frac{dx}{x^2+9}$

1 $\displaystyle\lim_{t\to-\infty}\int_t^0 \frac{dx}{x^2+9} = \lim_{t\to-\infty} \frac{1}{3}\arctan\frac{x}{3}\Big|_t^0$



$$= \lim_{t\to-\infty}\frac{1}{3}\Big(\underbrace{\arctan 0}_{0} - \underbrace{\arctan\frac{t}{3}}_{\arctan(-\infty)=-\frac{\pi}{2}}\Big)$$

$$= \frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

2 $\displaystyle\lim_{t\to\infty}\int_0^t \frac{dx}{x^2+9} = \lim_{t\to\infty}\frac{1}{3}\arctan\frac{x}{3}\Big|_0^t$

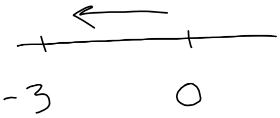
$$= \lim_{t\to\infty}\frac{1}{3}\Big[\underbrace{\arctan\frac{t}{3}}_{\arctan\infty=\frac{\pi}{2}} - \cancelto{0}{\arctan 0}\Big]$$

$$= \frac{\pi}{6}$$

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+9} = \frac{\pi}{6}+\frac{\pi}{6} = \boxed{\frac{\pi}{3}}$$

f.) $\displaystyle\int_{\boxed{-3}}^{0} \frac{dx}{(x+3)^2}$

t



$$\lim_{t\to -3^+} \int_t^0 \frac{dx}{(x+3)^2}$$

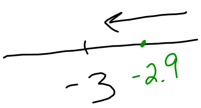
$u = x+3$ — $x=0,\ u=3$; $x=t,\ u=t+3$

$du = dx$

$$\lim_{t\to -3^+} \int_{t+3}^{3} \frac{du}{u^2} = \lim_{t\to -3^+} \left.\frac{-1}{u}\right|_{t+3}^{3}$$

$$= \lim_{t\to -3^+} \left(-\frac{1}{3} + \frac{1}{t+3}\right)$$

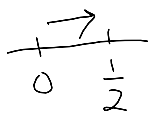
positive for $t\to -3^+$



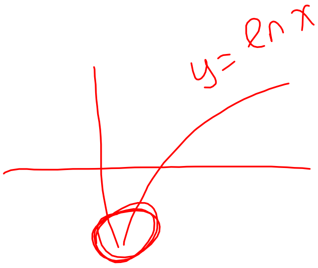
$$= -\frac{1}{3} + \infty = \boxed{\infty}$$

g.) $\displaystyle\int_0^3 \frac{1}{2x-1}\,dx = ① \int_0^{\frac{1}{2}} \frac{dx}{2x-1} + ② \int_{\frac{1}{2}}^{3} \frac{dx}{2x-1}$

① $\displaystyle\int_0^{\boxed{\frac{1}{2}}\,t} \frac{dx}{2x-1} = \lim_{t\to\frac{1}{2}^-} \int_0^t \frac{dx}{2x-1}$



$\displaystyle= \lim_{t\to\frac{1}{2}^-} \frac{1}{2} \ln\left|2x-1\right| \Big|_0^t$



$\displaystyle= \lim_{t\to\frac{1}{2}^-} \frac{1}{2}\left[\ln\left|\underbrace{2t-1}_{\downarrow\, 0}\right| - \ln(1)\right]$ $\to \ln 0 = -\infty$

$\displaystyle= \frac{1}{2}(-\infty) = \boxed{-\infty}$ DONE!

7. Determine whether the following integrals converge or diverge using the comparison theorem:

a.) $\int_0^\infty \frac{1}{x^{10}+e^{5x}}\,dx \leq \int_0^\infty \frac{1}{e^{5x}} = \int_0^\infty e^{-5x}\,dx$

$$= -\frac{1}{5}e^{-5x}\Big|_0^\infty$$

$$= -\frac{1}{5}\left(\cancel{e^{-\infty}}^{\to 0} - 1\right)$$

$$= \frac{1}{5}$$

larger converges, so does smaller.

b.) $\displaystyle\int_2^\infty \frac{x}{x^{3/2}-x-1}\,dx \geq \int_2^\infty \frac{x}{x^{3/2}}\,dx = \int_2^\infty x^{-\frac{1}{2}}\,dx$

$$= 2\sqrt{x}\,\Big|_2^\infty$$

$$= \infty$$

smaller diverges so does larger!

c.) $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^4}\,dx$

$\cos x \le 1$

$\sin x \le 1$

$\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$

converge if $p>1$

$$\int_1^\infty \frac{\cos^2 x}{x^4}\,dx \le \int_1^\infty \frac{1}{x^4}\,dx$$

will converge, $p=4>1$

d.) $\displaystyle\int_1^\infty \frac{\sin(4x)+9}{x^2+x+1}\,dx \le \int_1^\infty \frac{10}{x^2}$

p-test converges