## Section 11.1

1. Find the limit of the following sequences, if it exists. If the sequence diverges, state why.

a.) $a_n = \dfrac{n}{\sqrt{n+2}}$

$\lim_{n\to\infty} \frac{n}{\sqrt{n+2}} \quad \frac{\infty}{\infty} = \infty$

or $\overset{L}{=} \lim_{n\to\infty} \dfrac{1}{\frac{1}{\sqrt{n+2}}}$

$= \lim_{n\to\infty} \sqrt{n+2}$

$= \infty \longrightarrow$ diverges

b.) $a_n = \ln(n) - \ln(3n+1)$ $\quad \ln\infty = \infty$

$\lim_{n\to\infty} [\ln n - \ln(3n+1)] = \infty - \infty$

$\lim_{n\to\infty} \ln\left(\frac{n}{3n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{3n+1}\right)$

$= \boxed{\ln\left(\frac{1}{3}\right)}$ converges

c.) $a_n = \dfrac{(-1)^n n}{n^2+1}$

Suppose $(-1)^n a_n$ is an alternating sequence.

If $\lim_{n\to\infty} |(-1)^n a_n| \neq 0$ then $(-1)^n a_n$ diverges by oscillation.

converges to 0.

Since $\lim_{n\to\infty} \left|\frac{(-1)^n n}{n^2+1}\right| = 0$

d.) $a_n = \dfrac{(-1)^n n^2}{n^2+1}$

① $a_n = \frac{n}{n+1} \longrightarrow 1$

③ $a_n = \frac{(-1)^n n}{n^2+1} \longrightarrow 0$

② $a_n = \frac{(-1)^n n}{n+1} \longrightarrow$ diverges by oscillation

$\lim_{n\to\infty} \left|\frac{(-1)^n n^2}{n^2+1}\right| = 1$

thus $\frac{(-1)^n n^2}{n^2+1}$ diverges by oscillation.

what if:

① $a_n = \cos(n)$

$\lim_{n\to\infty} \cos(n)$ diverges by oscillation

② $a_n = \cos\left(\frac{1}{n}\right)$

$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{n}\right)$

$= \cos(0)$

$= \boxed{1}$

converges

③ $a_n = \frac{\cos(n)}{n}$

$\cos(n)$ is bounded by $\pm 1$

bottom is growing without bound

converges

$\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$

$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$

e.) $a_n = \dfrac{\ln n}{n}$

$\lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty}$

$\overset{L}{=} \lim_{n\to\infty} \dfrac{\frac{1}{n}}{1} = 0$

sequence converges

$a_n = \tan\left(\frac{5n\pi}{3+20n}\right)$

$\lim_{n\to\infty} \tan\left(\frac{5n\pi}{3+20n}\right)$

$\tan\left(\lim_{n\to\infty} \frac{5n\pi}{3+20n}\right)$

$\tan\left(\frac{5\pi}{20}\right) = \tan\left(\frac{\pi}{4}\right)$

$= \boxed{1}$ converges

$a_n = \frac{(-1)^n n^3}{4n^3+n^2}$ $\to \frac{\pm 1}{4}$ diverges by oscillation!

$\frac{(-1)^n}{4\sqrt{n}}$ converges to zero.

2. Suppose $\{a_n\}$ is a ~~decreasing~~ bounded sequence, $a_1 = 2$, and $a_{n+1} = \frac{1}{3 - a_n}$, find:

a.) $a_4 = \frac{2}{5}$

$a_1 = 2$

$a_2 = \frac{1}{3-a_1} = \frac{1}{3-2} = 1$

$a_3 = \frac{1}{3-a_2} = \frac{1}{3-1} = \frac{1}{2}$

$a_4 = \frac{1}{3-a_3} = \frac{1}{3-\frac{1}{2}} = \frac{1}{\frac{5}{2}} = \frac{2}{5}$

b.) the limit of the sequence.

L = limit of the sequence

$\lim_{n\to\infty} a_n = L$ $\lim_{n\to\infty} a_{n+1} = L$

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{3-a_n}$

$L = \frac{1}{3-L}$

$L(3-L) = 1$

$3L - L^2 = 1$

$0 = L^2 - 3L + 1$

by Quad formula

$L = \frac{3 \pm \sqrt{5}}{2}$

$= \frac{3+\sqrt{5}}{2}, \boxed{\frac{3-\sqrt{5}}{2}}$

3. Determine whether the following sequences are increasing, decreasing, or non monotonic.

a.) $a_n = \frac{1}{n^5}$ decreasing!

b.) $a_n = \frac{\ln n}{n}$

$a_n' = \frac{\frac{1}{n}(n) - \ln(n)(1)}{n^2} = \frac{1-\ln n}{n^2} < 0$ (eventually)

$\boxed{\text{decreasing}}$

c.) $a_n = \cos(n\pi)$

$n=1$, $a_1 = \cos\pi = -1$

$a_2 = \cos 2\pi = 1$

$a_3 = \cos 3\pi = -1$

$\vdots$

neither inc nor dec, so non monotonic.

4. Determine whether the following sequences are bounded.

a.) $a_n = \left\{ \frac{1}{n^2} \right\}_{n=1}^{\infty}$

$a_n$ is bounded if

$\underline{N} \le a_n \le \underline{M}$ for all $n$.

($N$ & $M$ finite)

$\lim_{n\to\infty} \frac{1}{n^2} = 0.$

what is it bounded by?

$a_n = \{ 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots \}$

all convergent sequences are bounded. But not all sequences are bounded convergent.

$0 \lneq a_n \le 1$

b.) $a_n = \left\{ \frac{n^2}{n+1} \right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \frac{n^2}{n+1} = \infty$ not a bounded sequence

(but is bounded below by its first term, $\frac{1}{2}$)

**Definition**: If we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, ...\}$ together, we will get an expression of the form $a_1 + a_2 + a_3 + ... + a_n + ... + ...$ which is called a **series** and is denoted, for short, by the symbol $\sum_{n=1}^{\infty} a_n$, or $\Sigma a_n$. Does it even make sense to talk about the sum of infinitely many terms?

$$\sum_{n=1}^{\infty}(n) = 1+2+3+4+\cdots+\cdots = \infty \quad (\text{Diverges})$$

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\cdots$$

Definition: **The sequence of partial sums** is the sequence whose terms are the cumulative sums of the series.

Consider $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...a_n + ....$

$s_1$, $s_2$, $s_3$

We will construct the **sequence of partial sums**, $\{s_n\} = \{s_1, s_2, s_3, ..., ...\}$, as follows:

$s_1 = a_1$ **Called the first partial sum**

$S_n = \{a_1, a_1+a_2, a_1+a_2+a_3, \ldots\}$

$s_2 = a_1 + a_2$ **Called the second partial sum**

$s_3 = a_1 + a_2 + a_3$ **Called the third partial sum**

Therefore a general formula for $s_n$, the $n^{th}$ term of the sequence, is

$\boxed{s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n}$ **Called the** ***nth*** **partial sum**

$\boxed{\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n}$

This is the definition of the sum of a series! **LEARN IT!!**

$$\rightarrow \boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\sum_{i=1}^{n} a_i} \leftarrow$$

## Section 11.2

5. Find the first 5 terms in the sequence of partial sums the series $\sum_{n=1}^{\infty}(1)$. Does the series coverge?

$a_n = 1$

$\cancel{\boxed{\sum_{n=1}^{\infty}}(1) = 1} \leftarrow$ BAD!! $\Sigma$ = sum!!

$S_1 = a_1 = 1$

$S_2 = a_1 + a_2 = 1+1 = 2$

$S_3 = a_1 + a_2 + a_3 = 1+1+1 = 3$

$S_4 = 4$

$S_5 = 5$

$S_n = \{1, 2, 3, 4, 5, \ldots\}$

$S_n = \{n\}_{n=1}^{\infty}$

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

$= \lim_{n\to\infty} n$

$= \infty$

series diverges

6. Find the first 5 terms in the sequence of partial sums the series $\sum_{n=1}^{\infty}(-1)^n$. Does the series coverge?

$a_n = (-1)^n$

$S_1 = a_1 = (-1)^1 = -1$

$S_2 = a_1 + a_2 = (-1)^1 + (-1)^2 = -1+1 = 0$

$S_3 = a_1 + a_2 + a_3 = (-1)^1 + (-1)^2 + (-1)^3 = -1+1-1 = -1$

$S_4 = 0$

$S_5 = -1$

$S_n = \{-1, 0, -1, 0, -1, \ldots\}$

$\lim_{n\to\infty} S_n$ dne by oscillation.

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$

Since $S_n$ does not have a limit, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^n$ diverges.

$\boxed{\phantom{x}} \leftarrow$ series!

7. Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series and sequence of partial sums

$s_n = 5 + \frac{n}{2n+3}$ is a formula for the nth partial sum. What is the sum of the series? "the sum"

Partial sum = "part of"

$S_n$ = "sum of first $n$ terms"

$S_n = a_1 + a_2 + \cdots + a_n$

$S_n = 5 + \frac{n}{2n+3}$ is the $n^{th}$ partial sum for $\sum_{n=1}^{\infty} a_n$.

① what is $a_7$?

$S_7 = a_1 + a_2 + \cdots + a_6 + a_7$

$-\; S_6 = a_1 + a_2 + \cdots + a_6$

$a_7 = S_7 - S_6$

$a_7 = 5 + \frac{7}{17} - \left(5 + \frac{6}{15}\right) = \frac{7}{17} - \frac{6}{15}$

② what is $a_n$?

$a_n = S_n - S_{n-1} = 5 + \frac{n}{2n+3} - \left(5 + \frac{n-1}{2(n-1)+3}\right)$

③ what is the sum of the series $\sum_{n=1}^{\infty} a_n$?

$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(5 + \frac{n}{2n+3}\right)$

$= 5 + \frac{1}{2} = \boxed{\frac{11}{2}}$

(TD)

8. What is the Test For Divergence and explain why the series $\sum_{n=1}^{\infty}\frac{n}{n+1}$ diverges.

T.D. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty}$ an diverges!

If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ may or may not converge! in which case T.D. Fails!

#8. $\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$, $\sum_{n=1}^{\infty}\frac{n}{n+1}$ diverges by T.D

$\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ T.D. $\lim_{n\to\infty}\sin\left(\frac{1}{n}\right) = \sin(0) = 0$

T.D. Fails! (will learn later)

9. Find the sum of the following series. If it diverges, support your answer.

a.) $\sum_{n=1}^{\infty}\left(\frac{1}{n+5}-\frac{1}{n+6}\right)$

① Find $S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$

$$S_n = \underbrace{\frac{1}{6}-\cancel{\frac{1}{7}}}_{a_1} + \underbrace{\cancel{\frac{1}{7}}-\cancel{\frac{1}{8}}}_{a_2} + \underbrace{\cancel{\frac{1}{8}}-\cancel{\frac{1}{9}}}_{a_3} + \cdots + \underbrace{\cancel{\frac{1}{n+4}}-\cancel{\frac{1}{n+5}}}_{a_{n-1}} + \underbrace{\cancel{\frac{1}{n+5}}-\frac{1}{n+6}}_{a_n}$$

$$\boxed{S_n = \frac{1}{6} - \frac{1}{n+6}}$$

② $\sum_{n=1}^{\infty}\left(\frac{1}{n+5}-\frac{1}{n+6}\right) = \lim_{n\to\infty} S_n$

$$= \lim_{n\to\infty}\left(\frac{1}{6} - \cancel{\frac{1}{n+6}}^{\,0}\right)$$

$$= \boxed{\frac{1}{6}} \leftarrow \text{sum of series}$$

b.) $\sum_{n=2}^{\infty} \ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$

c.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{\frac{1}{2}}{n} - \frac{\frac{1}{2}}{n+2}$$

$1 = A(n+2) + Bn$

$n=0: \quad 1 = A(2) \quad A = \frac{1}{2}$

$n=-2: \quad 1 = B(-2) \quad B = -\frac{1}{2}$

$$\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \boxed{\frac{3}{4}}$$

Telescoping!

$S_n$ for $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$S_n = a_1 + a_2 + a_3 + a_4 + \cdots a_{n-2} + a_{n-1} + a_n$

$S_n = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \cdots + \cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n-1}} - \frac{1}{n+1} + \cancel{\frac{1}{n}} - \frac{1}{n+2}$

$S_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n+2}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n+2}\right) = 1 + \frac{1}{2} = \frac{3}{2}$$

**Definition:** A **geometric series** is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n + ...$$

The geometric series $\sum\limits_{n=1}^{\infty} ar^{n-1}$ will converge if $|r| < 1$ and will diverge if $|r| \geq 1$. Moreover, if $|r| < 1$, then $\sum\limits_{n=1}^{\infty} ar^{n-1} = \dfrac{a}{1-r}$.

d.) $\sum\limits_{n=1}^{\infty} 2\left(\dfrac{1}{7}\right)^{n-1}$

e.) $\sum_{n=1}^{\infty} (-5)\left(\frac{2}{3}\right)^n$

f.) $\sum_{n=0}^{\infty} \frac{(-1)^n + 3^{n+1}}{5^n}$

g.) $\sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}}$

h.) $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{7^{n+1}}$

i.) $4+\frac{8}{5}+\frac{16}{25}+\frac{32}{125}+...$