Always try T.O. First.

If $\lim_{n\to\infty} a_n = 0$ < a_n positive / a_n alternating ↳ AST

Section 11.4 The Comparison Tests

The Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{i=1}^{\infty} b_n$ are series of **positive terms** (Note: series need not start at 1.)

- If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent. In other words, "if the larger series converges, so does the smaller series".

- If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent. In other words, "if the smaller series diverges, so does the larger series".

Note: if the larger series diverges, no conclusion can be made about the smaller series. Likewise, if the smaller series converges no conclusion can be made about the larger series.

Section 11.4

1. Determine whether the following series converge or diverge.

a.) $\sum_{n=1}^{\infty} \frac{n^4}{10n^4 + n^2 + 1}$

T.O. $\lim_{n\to\infty} \frac{n^4}{10n^4+n^2+1} = \frac{1}{10} \neq 0$

series diverges

b.) $\sum_{n=1}^{\infty} \frac{n^2}{n^5 + 10n + 1}$

positive terms

T.O. Fails

Try comparison Test (CT)

$a_n = \frac{n^2}{n^5+10n+1} \; \boxed{\le} \; \frac{n^2}{n^5}$

$b_n = \frac{n^2}{n^5} = \frac{1}{n^3}$

since $a_n \le b_n$

and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$

converges by p-series $p=3>1$

$\therefore$ Σa_n also converges by CT

$2-\frac{5}{2} = -\frac{1}{2}$

c.) $\sum_{n=3}^{\infty} \frac{n^2+n+9}{\sqrt{n^5 - n^2 - 1}}$

T.O. Fails

$\ge \Sigma \frac{n^2}{\sqrt{n^5}} = \Sigma \frac{n^2}{n^{5/2}} = \Sigma \frac{1}{n^{1/2}}$

divergent p-series, $p=\frac{1}{2}<1$

larger series also diverges by C.T.

d.) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos^2 n + 5}{n^3+\sqrt{n}} \le \sum_{n=2}^{\infty} \frac{6}{n^3}$

convergent p-series, $p=3>1$

larger converges, so does smaller by CT

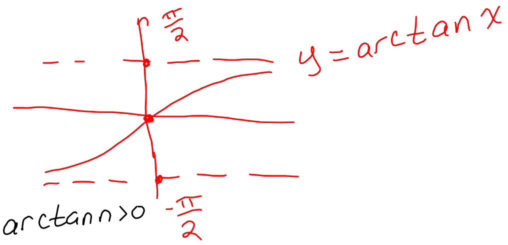
e.) $\displaystyle\sum_{n=1}^{\infty} \frac{5+\sin(7n)}{\sqrt{n}} \le \sum_{n=1}^{\infty} \frac{6}{\sqrt{n}}$

does nothing since larger diverging tells nothing about smaller!

$$\sum_{n=1}^{\infty} \frac{4}{\sqrt{n}} \le \sum_{n=1}^{\infty} \frac{5+\sin(7n)}{\sqrt{n}}$$

smaller divergent p-series, $p=\frac{1}{2}<1$, larger must also diverge

f.) $\displaystyle\sum_{n=1}^{\infty} \frac{5+\arctan(n)}{n^3}$



$0 \le \arctan n \le \frac{\pi}{2}$

since n is a positive integer, $\arctan n > 0$

$$\sum_{n=1}^{\infty} \frac{5+\arctan n}{n^3} \le \sum_{n=1}^{\infty} \frac{5+\frac{\pi}{2}}{n^3}$$

convergent p-series $p=3>1$

larger converges so does smaller.

g.) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{4^n+n} \le \sum_{n=1}^{\infty} \frac{3}{n}$ gives nothing!

or

$$\sum_{n=1}^{\infty} \frac{3}{4^n+n} \le \sum_{n=1}^{\infty} \frac{3}{4^n} = \sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^n$$

geometric series! converges since $r=\frac{1}{4}$, $|r|<1$.

larger converges so does smaller by CT.

h.) $\sum_{n=5}^{\infty} \frac{n}{8n^2+6n+1} \leq \sum \frac{n}{8n^2} = \sum \frac{1}{8n}$

a divergent p-series $p=1$.

CT Fails :(

The Limit Comparison Test: If the Comparison Test is inconclusive, then we may apply the Limit Comparison Test, which determines whether two series behave similarly for vert large values of n.

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of positive terms. $a_n \sim c\, b_n$

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ and finite, then either both series converge or both diverge.

Note: If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ or $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then the test fails and therefore we need to apply another test.

h.) $\sum_{n=5}^{\infty} \frac{n}{8n^2+6n+1}$

CT Failed, try LCT

$a_n = \frac{n}{8n^2+6n+1}$, $b_n = \frac{1}{8n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\frac{n}{8n^2+6n+1}}{\frac{1}{8n}} \right) = \lim_{n\to\infty} \frac{8n^2}{8n^2+6n+1} = \frac{8}{8} = 1 > 0$$

Both series diverge because $\sum \frac{1}{8n}$ diverges (by p-series) + LCT

i.) $\sum_{n=1}^{\infty} \frac{n^4-n^3+1}{\sqrt{n^{10}-n^6+3}}$ a_n

LCT $b_n = \frac{n^4}{\sqrt{n^{10}}} = \frac{n^4}{n^5} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\frac{n^4-n^3+1}{\sqrt{n^{10}-n^6+3}}}{\frac{1}{n}} \right)$$

$$= \lim_{n\to\infty} \frac{n^5-n^4+n}{\sqrt{n^{10}-n^6+3}}$$

$$= \lim_{n\to\infty} \frac{n^5}{\sqrt{n^{10}}} = \lim_{n\to\infty} \frac{n^5}{n^5} = 1 > 0$$

$\therefore$ both series diverge

because $\sum \frac{1}{n}$ diverges! p-series $p=1$

j.) $\sum_{n=1}^{\infty} \frac{n^2+5n-2}{(3n+1)^4+2n}$ a_n

$b_n = \frac{n^2}{(3n)^4} = \frac{n^2}{81n^4} = \frac{1}{81n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+5n-2}{(3n+1)^4+2n}}{\frac{1}{81n^2}}$$

$$= \lim_{n\to\infty} \frac{81n^2(n^2-5n-2)}{(3n+1)^4+2n} = \frac{81}{81} = 1 > 0$$

both series converge since $\sum \frac{1}{81n^2}$ converges p-series ($p=2>1$)

by LCT

The Alternating Series Test: The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$, where $a_n > 0$, converges if it satisfies both conditions given below:

- $a_{n+1} \leq a_n$ for all n (ie the sequence $\{a_n\}$ is decreasing).
- $\lim_{n\to\infty} a_n = 0$ "if absolute value of the terms decrease to 0, then the series converges"

Section 11.5

4. Use the alternating series test to determine whether the following series converge.

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ TD $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n+1}} = 0$ Fails.

Ast: prove $\left|\frac{(-1)^n}{\sqrt{n+1}}\right|$ decreases to 0

$= \frac{1}{\sqrt{n+1}}$ decreases ✓

$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$ ✓

$\therefore \sum \frac{(-1)^n}{\sqrt{n+1}}$ converges by AST

b.) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{1+n^2}$ T.D.

$\lim_{n\to\infty} \frac{(-1)^n n^2}{1+n^2}$ dne (by oscillation) $\neq 0$

series diverges by T.D

c.) $\sum_{n=1}^{\infty}(-1)^{n-1}2^{-n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n}$ (Fails TD)

① convergent geometric series $r = -\frac{1}{2}$, $|r|<1$

② do AST

$\frac{1}{2^n}$ decreases ✓

$\lim_{n\to\infty} \frac{1}{2^n} = 0$ ✓

converges by AST

d.) $\sum_{n=1}^{\infty}(-1)^n 2^{3/n}$

TD $\lim_{n\to\infty} (-1)^n 2^{\frac{3}{n}}$ $2^0 = 1$

dne by oscillation $\neq 0$

$\therefore$ series diverges!

Recall remainder estimate for integral test

$R_n \le \int_n^\infty f(x)\,dx$

5. Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

a.) Prove the series is convergent.

AST show $\frac{1}{n^2}$ decreases to 0

converges by AST

b.) Use s_6 to approximate the sum of the series and use the Alternating Series Estimation Theorem to estimate the error in using the 6th partial sum to approximate the sum of the series.

how big is $|R_6|$? $S_6 = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2}$

Remainder Estimate and The Alternating Series Estimation Theorem

If $\sum_{n=1}^{\infty}(-1)^n a_n$, $a_n > 0$, is a convergent alternating series, and a partial sum

$s_n = \sum_{i=1}^{n}(-1)^i a_i$ is used to approximate the sum of the series with remainder R_n, then

$|R_6| \le |a_7| = \frac{1}{7^2}$ $\qquad |R_n| \le a_{n+1}$

c.) Determine the minimum number of terms we need to add in order to find the sum with error less than $\frac{1}{120}$.

how large is n so $|R_n| < \frac{1}{120}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

solve

$\frac{1}{(n+1)^2} < \frac{1}{120}$

$120 < (n+1)^2$

$n=1$? $120 < 2^2$

$n=2$

$\vdots$

$\boxed{n=10}$

$n=10$ works because

$120 < (10+1)^2 = 121$