$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1, \quad R=1$$

Section 11.9

1. Find a power series representation for the function and determine the radius of convergence.

a.) $f(x) = \dfrac{1}{1-5x}$ $= \sum_{n=0}^{\infty} (5x)^n$, $|5x|<1$, $|x| < \frac{1}{5}$

$$= \boxed{\sum_{n=0}^{\infty} 5^n x^n} \qquad \boxed{R = \frac{1}{5}}$$

b.) $f(x) = \dfrac{3}{2+x}$ $= \dfrac{3}{2(1+\frac{x}{2})}$

$$= \frac{3}{2}\left(\frac{1}{1-(-\frac{x}{2})}\right)$$

$$= \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n, \quad \left|-\frac{x}{2}\right|<1$$

$$|x|<2$$

$$= \frac{3}{2}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n} \qquad \boxed{R=2}$$

$$= \boxed{\sum_{n=0}^{\infty} 3\,\frac{(-1)^n x^n}{2^{n+1}}}$$

c.) $f(x) = \dfrac{x^4}{8-x^3}$ $= \dfrac{x^4}{8(1-\frac{x^3}{8})}$

$$= \frac{x^4}{8}\sum_{n=0}^{\infty}\left(\frac{x^3}{8}\right)^n, \quad \left|\frac{x^3}{8}\right|<1$$

$$|x^3|<8$$

$$= \frac{x^4}{8}\sum_{n=0}^{\infty}\frac{x^{3n}}{8^n} \qquad |x|<\sqrt[3]{8}=2$$

$$\boxed{R=2}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{x^{3n+4}}{8^{n+1}}}$$

c_n = coefficients

Theorem If $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ... + c_n x^n + ...$ has a radius of convergence R. Then

$10x^4$

$10 \quad 4x^3$

a.) $f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + ... + nc_n x^{n-1} + ...$ has a radius of convergence R.

$\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1}$

$$f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1} = \sum_{n=0}^{\infty} c_{n+1}(n+1)x^n$$

b.) $\int f(x)\,dx = C + c_0 x + \frac{c_1 x^2}{2} + \frac{c_2 x^3}{3} ... + \frac{c_n x^{n+1}}{n+1} + ...$ has a radius of convergence R.

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$$

Note: When **differentiating** a power series, the starting index **will** change if the first term is constant. When **integrating** a power series, the starting index does **not** change under any circumstances.

2. If $f(x) = \sum_{n=0}^{\infty} \frac{n^4 (8x)^n}{n!}$, find $f'(x)$ and $\int f(x)\,dx$.

c_n

$$f(x) = \sum_{n=0}^{\infty} \frac{n^4 8^n}{n!} x^n$$

$$\boxed{\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{n^4 8^n}{n!} \frac{x^{n+1}}{n+1}}$$

$$f'(x) = \sum_{\boxed{n=1}}^{\infty} \frac{n^4 8^n}{n!} n x^{n-1}$$

$\longrightarrow$ re-index!

$$= \sum_{n=0}^{\infty} \frac{(n+1)^4 8^{n+1}}{(n+1)!} (n+1) x^n \overset{\text{or}}{=} \sum_{n=0}^{\infty} \frac{(n+1)^4 8^{n+1} x^n}{n!}$$

since $(n+1)! = (n+1)n!$

3. Find a power series representation for the function and determine the radius of convergence.

a.) $\dfrac{1}{(5+2x)^2}$

$\dfrac{\#}{(\quad)^2}$ integrate first

$$\int \frac{1}{(5+2x)^2}\,dx = -\frac{1}{2}\,\frac{1}{5+2x}$$

$u = 5+2x$

$du = 2\,dx$

$\frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2}\left(\frac{1}{u}\right)$

$$= -\frac{1}{2}\,\frac{1}{5\left(1+\frac{2x}{5}\right)}$$

$$= \frac{-1}{2\cdot 5}\,\frac{1}{1-\left(-\frac{2x}{5}\right)}$$

$\left|-\frac{2x}{5}\right| < 1$

$|x| < \frac{5}{2}$

$\boxed{R = \frac{5}{2}}$

$$= -\frac{1}{10}\sum_{n=0}^{\infty}\left(-\frac{2x}{5}\right)^n$$

$$= -\frac{1}{2\cdot 5}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n x^n}{5^n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}\,2^{n-1}\,x^n}{5^{n+1}}$$

undo integral!

$$\frac{1}{(5+2x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\,2^{n-1}\,x^n}{5^{n+1}}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,2^{n-1}}{5^{n+1}}\,n\,x^{n-1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+2}\,2^{n}}{5^{n+2}}(n+1)\,x^n$$

note: $(-1)^{n+2} = (-1)^n$

b.) Use part a.) to find a power series representation for the function $\displaystyle\int \frac{x^2}{(5+2x)^2}\,dx$

$$x^2\left(\frac{1}{(5+2x)^2}\right) = x^2\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+2}\,2^n}{5^{n+2}}(n+1)\,x^n\right)$$

$$\int \frac{x^2}{(5+2x)^2}\,dx = \int \sum_{n=0}^{\infty}\frac{(-1)^{n+2}\,2^n}{5^{n+2}}(n+1)\,x^{n+2}\,dx$$

$$= \boxed{C + \sum_{n=0}^{\infty}\frac{(-1)^{n+2}\,2^n}{5^{n+2}}(n+1)\,\frac{x^{n+3}}{n+3}}$$

c.) $\ln(2-x^2)$ $\ln(a+x^n)$ Take derivative first

$$\frac{d}{dx}\ln(2-x^2)=\frac{-2x}{2-x^2}=\frac{-\cancel{2}x}{\cancel{2}(1-\frac{x^2}{2})}=-x\left(\frac{1}{1-\frac{x^2}{2}}\right)$$

$R=\sqrt{2}$

$$=-x\sum_{n=0}^{\infty}\left(\frac{x^2}{2}\right)^n$$

$$=-x\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n}$$

integrate both sides!

$$=-\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^n}$$

$$\ln(2-x^2)=\int-\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2^n}\,dx$$

$$=\overset{\ln 2}{\cancel{C}}-\sum_{n=0}^{\infty}\frac{x^{2n+2}}{2^n(2n+2)}$$

let $x=0$

$\ln 2=C-\Sigma(0)$ $C=\ln 2$

d.) Use part c.) to find a power series representation for the function $x\ln(2-x^2)$

$$x\ln(2-x^2)=x\left(\ln 2-\sum_{n=0}^{\infty}\frac{x^{2n+2}}{2^n(2n+2)}\right)$$

$$=\underline{x\ln 2}-\sum_{n=0}^{\infty}\frac{x^{2n+3}}{2^n(2n+2)}$$

e.) Use part d.) to find a power series representation for the function $\int x\ln(2-x^2)\,dx$

$$\int x\ln(2-x^2)\,dx=\frac{x^2}{2}\ln 2-\sum_{n=0}^{\infty}\frac{x^{2n+4}}{2^n(2n+2)(2n+4)}+C$$

f.) $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{2n+1}$

g.) Use part f.) to find a series representation for the function $\int_0^{1/2} \arctan\left(\frac{x}{3}\right)\, dx$

$$\int_0^{\frac{1}{2}} \arctan\left(\frac{x}{3}\right) dx = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{3}\right)^{2n+1}}{2n+1}\, dx$$

$$= \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{3^{2n+1}(2n+1)}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{3^{2n+1}(2n+1)(2n+2)} \Bigg|_0^{\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+2}}{3^{2n+1}(2n+1)(2n+2)}$$

Section 11.10

$\sum_{n=0}^{\infty} c_n(x-a)^n$

The **Taylor Series** for $f(x)$ about $x = a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$c_n = \frac{f^n(a)}{n!}$

4. Find the Taylor Series for f centered at $a = -1$ if

$\rightarrow f^{(n)}(-1) = \frac{(5)^n n!}{2^n(n+1)}$.

$$f(x) = \sum_{n=0}^{\infty} \frac{\boxed{f^n(-1)}}{n!}(x+1)^n$$

$$= \sum_{n=0}^{\infty} \frac{5^n n!}{2^n(n+1)n!}(x+1)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{5^n}{2^n(n+1)}(x+1)^n$$

5. Find $f^{(50)}(2)$ if $f(x) = \sum_{n=0}^{\infty} \frac{10^{n+1}(x-2)^n}{(n+5)!}$, that is the 50^{th} derivative of f at $x = 2$.

Taylor series centered at $x=2$

$c_n = \frac{f^n(2)}{n!}$

$c_n = \frac{10^{n+1}}{(n+5)!}$

equal

$n=50$

$$\frac{f^n(2)}{n!} = \frac{10^{n+1}}{(n+5)!}$$

$$\frac{f^{50}(2)}{50!} = \frac{10^{51}}{55!} \rightarrow \boxed{f^{50}(2) = \frac{10^{51}}{55!}(50!)}$$

6. Find the Taylor Series centered at $a = 4$ if $f(x) = \dfrac{1}{x^2}$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$\frac{1}{x^2} = \sum_{n=0}^{\infty} \frac{f^{n}(4)}{n!}(x-4)^n \longrightarrow f'(x) = \frac{-2}{x^3} = -2x^{-3}$$

$$f''(x) = \frac{3\cdot 2}{x^4}$$

$$f^{n}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}}$$

$$f'''(x) = \frac{-4\cdot 3\cdot 2}{x^5}$$

$$f^{n}(4) = \frac{(-1)^n (n+1)!}{4^{n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{4^{n+2}\, n!} (x-4)^n$$

7. Find the Maclaurin Series for $f(x) = e^{-x^2}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

8. Find the Maclaurin Series for
$$f(x) = x^2 \sin\left(\frac{x^4}{4}\right).$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$x^2 \sin \frac{x^4}{4} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^4}{4}\right)^{2n+1}}{(2n+1)!} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{4^{2n+1}(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+6}}{4^{2n+1}(2n+1)!}$$

9. Evaluate the indefinite integral as an infinite series
$$\int \cos(x^{10})\, dx.$$

$$(x^{10})^{2n} = x^{20n}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\int \cos(x^{10})\, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{20n}}{(2n)!}\, dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{20n+1}}{(2n)!\,(20n+1)}$$

10. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n(\pi)^{2n}}{4^{2n}(2n)!}$. $= \sum_{n=0}^{\infty} \frac{(-1)^n\left(\frac{\pi}{4}\right)^{2n}}{(2n)!}$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$= \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

what about $\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} \cdot 5}{(2n)!} = 5\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n)!}$

$= \boxed{5\cos 5}$

11. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{3n}}{n!}$. $= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (x^3)^n}{n!}$

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

$= \sum_{n=0}^{\infty} \frac{(-2x^3)^n}{n!}$

$= e^{-2x^3}$

Find the sum of $\sum_{n=1}^{\infty} \frac{4^n}{n!}$

Know: $\sum_{n=0}^{\infty} \frac{4^n}{n!} = e^4$

$\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$

$= a_0 + a_1 + \sum_{n=2}^{\infty} a_n$

$\underbrace{\sum_{n=0}^{\infty} \frac{4^n}{n!}}_{= e^4} = 1 + \sum_{n=1}^{\infty} \frac{4^n}{n!}$

$e^4 = 1 + \sum_{n=1}^{\infty} \frac{4^n}{n!} \rightarrow \boxed{e^4 - 1 = \sum_{n=1}^{\infty} \frac{4^n}{n!}}$