16.2 Line integral \( \int_C \)

1. not conservative: path dependent
   
   \[ C: r(t), \quad a \leq t \leq b \]
   
   \[ \int_C f(x,y,z) \, ds = \int_a^b f(r(t)) |r'(t)| \, dt \]
   
   \[ \int_C P \, dx + Q \, dy = \int_C (x + 2y) \, dx + 8y^2 \, dy \]
   
   \[ r(t) = (\sin t, \cos t), \quad 0 \leq t \leq \frac{\pi}{2} \]
   
   \[ x = \sin t, \quad dx = \cos t \, dt \]
   
   \[ y = \cos t, \quad dy = -\sin t \, dt \]
   
   \[ = \int_0^{\frac{\pi}{2}} (\sin t + 2 \cos t) \cos t \, dt + 8 \cos t (-\sin t) \, dt \]

2. conservative path independent only endpoints matter!
   
   \[ \int_C F \, dr = \int_C \nabla f \, dr, \quad f = \text{potential} \]
   
   \[ C: r(t), \quad a \leq t \leq b \]
   
   \[ = f(r(b)) - f(r(a)) \]

Green's Theorem: \( \int_C P \, dx + Q \, dy = \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \)

F: \( \langle P, Q \rangle \)

conservative:

\[ \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \]

D: region \( C \) encloses

C: planar curve

Recall: work done in moving a particle along a curve

\[ W = \int_C F \, dr \]

\[ C: r(t), \quad a \leq t \leq b \]

\[ = \int_a^b F(r(t)) \cdot r'(t) \, dt \]

What is \( C \) closed?

- If \( F \) is conservative, \( W = 0 \)
- If \( F \) is not conservative, Green's theorem can apply

\[ F = \langle e^x \cos y, (x+y) \rangle, \quad C \text{ is the boundary of the region bounded} \]

by \( y = x^2, \quad y = 1 \)

if \( F \) is not conservative \( \rightarrow \) is the curve closed? yes!!

Green's theorem \( \int_C F \cdot dr = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \)

D: region bounded by \( C \).

\[ = \iint_D (\cos x + e^x \sin y) \, dx \, dy \]

\[ = \int_0^1 \int_0^1 (\cos x + e^x \sin y) \, dy \, dx \]

\[ x \leq y \leq 1 \]

\[ 0 \leq x \leq 1 \]
Surface integrals

\[ \iint_S f(x, y, z) \, ds = \iint_D f(r(u, v)) |r_u \times r_v| \, dA \]

\[ D = \text{parametric domain} \]

\[ \iint_S F \cdot ds = \iint_D F(r(u, v)) \cdot (r_u \times r_v) \, dA \]

Stokes' Theorem: \( C = \text{closed space curve} \)

\[ \int_C F \cdot dr = \iint_S \text{curl} F \cdot ds, \quad S = \text{surface } C \text{ bounds} \]

\( S \) parameterized by

\[ r(u, v) = \iint_D \text{curl} F(r(u, v)) \cdot (r_u \times r_v) \, dA \]

\[ \iint_S \text{curl} F \cdot ds = \int_C F \cdot dr \]

Divergence Theorem:

Flux = \[ \iint_S F \cdot ds = \iiint_E \text{div} (F) \, dV \]

\( E = \text{volume enclosed by } S \)
1. Evaluate $\int_C (\sin x + \cos y)ds$, where $C$ is the line segment going from the point $(0,0)$ to the point $(3\pi, 4\pi)$.

(a) $\frac{2}{3\pi}$  
(b) $\frac{10}{3}$  
(c) $\frac{-10}{3}$  
(d) $30\pi^2$  
(e) $\frac{-2}{3\pi}$  

$\mathbf{\overrightarrow{r}}_0 = (0,0)$  
$\mathbf{\overrightarrow{r}}' = (3\pi, 4\pi)$  
$\mathbf{\overrightarrow{r}}_0 + t\mathbf{\overrightarrow{v}} = (0,0) + t (3\pi, 4\pi)$  
$\mathbf{\overrightarrow{r}}(t) = (3\pi t, 4\pi t)$, $0 \leq t \leq 1$

$\int_0^1 (\sin (3\pi t) + \cos (4\pi t)) \sqrt{9\pi^2 + 16\pi^2} \, dt$

2. Find the surface area of the part of the plane $z = x - y - 4$ that lies inside the cylinder $x^2 + y^2 = 9$.

(a) $9\sqrt{2}\pi$  
(b) $6\sqrt{3}\pi$  
(c) $\sqrt{19}\pi$  
(d) $9\sqrt{3}\pi$  
(e) $2\sqrt{19}\pi$

$S$ parameterized by $\mathbf{r}(u,v)$

$A(S) = \iint_D \left| \mathbf{r}_u \times \mathbf{r}_v \right| dA$

$D$: domain of $\mathbf{r}(u,v)$.

Parameterize the plane $z = x - y - 4$

$z = 4 + x + y$

$\mathbf{\overrightarrow{r}}(x,y) = (x, y, 4 + x + y)$

$\mathbf{\overrightarrow{r}}_x \times \mathbf{\overrightarrow{r}}_y = \begin{pmatrix} -z_x & -z_y & 1 \\ -x_y & -y_y & 1 \\ -x_x & -y_x & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$

$\iint_D \left| \mathbf{\overrightarrow{r}}_x \times \mathbf{\overrightarrow{r}}_y \right| dA = \iint_D \sqrt{3} dA$

$D$: $x^2 + y^2 \leq 9$
3. Which of the following vector functions \( \mathbf{F} \) has the vector field shown below?

\[ \mathbf{F}(1,1) = \langle -2, 8 \rangle \]

\( \mathbf{F}(1,1) = \langle 2, -8 \rangle \)

(a) \( \mathbf{F} = (-2x, -8) \)

(b) \( \mathbf{F} = (2x, -8) \)

(c) \( \mathbf{F} = (2x, 8) \)

(d) \( \mathbf{F} = (-2x, 8) \)

(e) None of these

4. Using the Divergence Theorem, find the flux of \( \mathbf{F} = \langle e^x + \cos y, y^3 + \sin(xz), z^3 + 2e^{-z} \rangle \) across the positively oriented surface \( S \), where \( S \) is the part of the cylinder \( y^2 + z^2 = 1 \) that lies between the planes \( x = 0 \) and \( x = 3 \).

\[
\text{flux} = \iint_S \mathbf{F} \cdot d\mathbf{S}
\]

\[
= \iiint_E \text{div} \mathbf{F} \, d\mathbf{V}
\]

\[
= \iiint_E (0 + 3y^2 + 3z^2) \, dV
\]

\[
= \iint_D \left[ \int_0^3 (3y^2 + 3z^2) \, dx \right] \, dA
\]

\[
= \iint_D \left[ (3y^2 + 3z^2)x \big|_{x=0}^{x=3} \right] \, dA
\]

\[
= \int_0^3 \int_0^1 3(y^2 + z^2) \, r \, dr \, d\theta
\]
5. Find the work done by the force field \( \mathbf{F} = (x^2 + y^2, xy) \) in moving a particle along the curve \( C: \mathbf{r}(t) = (t^2, t^3) \), \( 0 \leq t \leq 1 \).

(a) \( \frac{20}{21} \)  
(b) \( \frac{52}{21} \)  
(c) \( \frac{23}{24} \)  
(d) \( \frac{22}{56} \)  
(e) \( \frac{167}{68} \)

\[ w = \int_C \mathbf{F} \cdot d\mathbf{r} \quad \frac{2 \theta}{2x} \neq \frac{2 \rho}{2y} \]

not conservative

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)), \mathbf{r}'(t) \, dt \]

\[ = \int_0^1 \langle t^4 + t^3, t^5 \rangle \cdot \langle 2t, 3t^2 \rangle \, dt \]

6. Find \( \int_C y^2 \, dx + x \, dy \) where \( C \) is the triangle with vertices \((0, 0), (1, 1)\) and \((0, 1)\). Assume positive (counterclockwise) orientation.

(a) \( \frac{7}{6} \)  
(b) \( \frac{1}{6} \)  
(c) \( -\frac{7}{6} \)  
(d) 0  
(e) \( -\frac{1}{6} \)

\[ \int_C y^2 \, dx + x \, dy = \iint_D (1 - 2y) \, dA \]

\[ = \int_0^1 \int_0^y (1 - 2y) \, dx \, dy \]
7. Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F} = (2xy^3z^4, 3x^2y^2z^4, 4x^2y^2z^5) \) and \( C \) is any path from the point \((0,0,0)\) to the point \((1,1,1)\).

(a) 2
(b) 1
(c) -2
(d) 0
(e) -1

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{f} \cdot d\mathbf{r} \]

Find potential function \( f \)

\[
\begin{align*}
\int 2xy^2z^4 \, dx &= \frac{2}{3}x^2y^2z^4 \\
\int 3x^2y^2z^4 \, dy &= \frac{3}{3}x^2y^4z^4 \\
\int 4x^3y^2z^3 \, dz &= x^3y^2z^4 \\
\end{align*}
\]

\[
f(x, y, z) = x^3y^2z^4
\]

\[ f(1,1,1) - f(0,0,0) = 1
\]

8. Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F} = (e^z, xy) \) and \( C: r(t) = (t^2, t^3), \ 0 \leq t \leq 1 \).

(a) \( \frac{11}{22} \) - \( \frac{1}{2e} \)
(b) \( \frac{17}{22} \)
(c) \( \frac{11}{8} \)
(d) \( \frac{11}{8} - \frac{e}{2} \)
(e) \( \frac{11}{8} - \frac{1}{e} \)
9. Which of the following integrals is the correct setup in order to evaluate\( \iint_S z\,dS \) where \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 1 \) that lies between the planes \( z = 0 \) and \( z = \frac{1}{2} \)? Note: If we parameterize the sphere \( x^2 + y^2 + z^2 = \rho^2 \) by \( r(\theta, \phi) = (\rho \sin(\theta) \cos(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi)) \), then \( r_\theta \times r_\phi = \rho^2 \sin(\phi) \).

(a) \( \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \sin^2(\phi) \cos(\theta) \cos(\phi) \,d\phi \,d\theta \)

(b) \( \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \sin^2(\phi) \cos(\theta) \cos(\phi) \,d\phi \,d\theta \)

(c) \( \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \sin^2(\phi) \cos(\theta) \cos(\phi) \,d\phi \,d\theta \)

(d) \( \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \sin^2(\phi) \cos(\theta) \cos(\phi) \,d\phi \,d\theta \)

(e) \( \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \sin^2(\phi) \cos(\theta) \cos(\phi) \,d\phi \,d\theta \)

10. Which of the following integrals is a result of using Stokes' Theorem to find \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = (-y^2, z, x^2) \) and \( C \) is the curve of intersection of \( z = 9 - x^2 - y^2 \) in the \( xy \) plane. Orient \( C \) to be counterclockwise when looking from above.

(a) \( \int_0^{2\pi} \int_0^1 \mathbf{F} \cdot d\mathbf{S} \)

(b) \( \int_0^{2\pi} \int_0^1 \mathbf{F} \cdot d\mathbf{S} \)

(c) \( \int_0^{2\pi} \int_0^1 \mathbf{F} \cdot d\mathbf{S} \)

(d) \( \int_0^{2\pi} \int_0^1 \mathbf{F} \cdot d\mathbf{S} \)

(e) \( \int_0^{2\pi} \int_0^1 \mathbf{F} \cdot d\mathbf{S} \)

11. Given \( \mathbf{F} = (x + y, 2xy + y^3) \), find \( f(-1,2) \) where \( f \) is the potential function of \( \mathbf{F} \).

(a) \( -\frac{2}{3} \)

(b) \( -\frac{5}{6} \)

(c) \( \frac{2}{3} \)

(d) \( 2 \)

(e) \( \frac{23}{6} \)
Part II: Work out. For problems (13-15), you are being asked for the set up only. Be sure your limits of integration are defined and matched with the appropriate differential.

12. (7 pts) Let $f$ be a scalar field and let $\mathbf{F}$ be a vector field. Determine whether each expression is meaningful or meaningless (circle one). If the expression is meaningful, circle whether the expression is a vector or a scalar.

a.) $\text{curl} \, \mathbf{F}$ meaningful (vector or scalar) meaningless
b.) $\text{grad} \, f$ meaningful (vector or scalar) meaningless
c.) $\text{grad} \, (\text{div} \, F)$ meaningful (vector or scalar) meaningless
d.) $\text{grad} \, (f \times (\text{curl} \, F))$ meaningful (vector or scalar) meaningless
e.) $\text{curl} \, f$ meaningful (vector or scalar) meaningless
f.) $\text{div} \, (\text{grad} \, f)$ meaningful (vector or scalar) meaningless

curl operates on a vector
gradient operates on a scalar functions
divergence operates on vectors
13. (10 pts) Consider \( \int_C (\sin(xy)dx + (x + \ln(y + 1))dy \), where \( C \) is the triangle with vertices \((0,0), (2,1)\) and \((3,0)\).

Using Green's Theorem, set up the resulting double integral. Do not evaluate the integral.
14. (10 pts) Set up but do not evaluate the surface integral obtained by using Stokes' Theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \),

where \( \mathbf{F} = \langle y^2, z^3, x^2 \rangle \) and where \( C \) is the curve of intersection of the plane \( x + z = 1 \) and the cylinder \( x^2 + y^2 = 4 \), oriented counterclockwise when viewed from above.
15. (8 pts) Set up but do not evaluate \( \iiint_S xy \, dS \) where \( S \) is the part of the paraboloid \( z = x^2 + y^2 \) between the planes \( z = 1 \) and \( z = 4 \).
16. (10 pts) Using the Divergence Theorem, find the flux of \( \mathbf{F} = \langle z \arctan(y^2), x^3 \ln(x^2 + 1), z \rangle \) where \( S \) is the surface enclosed by the paraboloid \( x^2 + y^2 + z = 2 \) and the plane \( z = 1 \). Simplify your answer.
Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$.

$\mathbf{F}(x, y, z) = x^2 \sin(z)i + y^2j + xzk,$

$S$ is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the $xy$-plane, oriented upward.

$$
\int_C \mathbf{F} \cdot d\mathbf{r}
$$

Parameterize $C$:

\begin{align*}
\mathbf{r}(	heta) &= \langle \cos \theta, \sin \theta, 0 \rangle \\
\mathbf{r'}(\theta) &= \langle -\sin \theta, \cos \theta, 0 \rangle
\end{align*}

This curve is a boundary curve for $S$.

$$
\int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta)) \cdot \mathbf{r'}(\theta) d\theta
$$

$$
= \int_0^{2\pi} \langle 0, \sin^2 \theta \cos \theta \sin \theta, 0 \rangle \cdot \langle -\sin \theta, \cos \theta, 0 \rangle d\theta
$$

$$
= \int_0^{2\pi} \sin^3 \theta \cos \theta \sin \theta d\theta = 0
$$