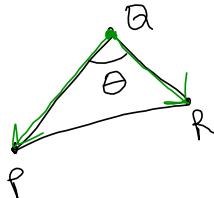


1. Consider the triangle formed by the points  $P = (2, 3, 0)$ ,  $Q = (4, -1, -1)$  and  $R = (2, 0, 2)$ .

(a) Find the angle at vertex  $\underline{Q}$ .



$$\overrightarrow{QP} = \langle -2, 4, 1 \rangle$$

$$\overrightarrow{QR} = \langle -2, 1, 3 \rangle$$

$$\cos \theta = \frac{\langle -2, 4, 1 \rangle \cdot \langle -2, 1, 3 \rangle}{\sqrt{4+16+1} \sqrt{4+1+9}}$$

$$\cos \theta = \frac{4 + 4 + 3}{\sqrt{21} \sqrt{14}} \quad \theta = \arccos\left(\frac{11}{\sqrt{21} \sqrt{14}}\right)$$

$$\theta \approx 50^\circ$$

(b) Find an equation of the plane containing  $P, Q$ , and  $R$ .

$\vec{n}$  normal vector perpendicular to plane

$r_0$  any point on plane  $r_0 = R = (2, 0, 2)$

$$\vec{n} = \overrightarrow{QP} \times \overrightarrow{QR} = \langle -2, 4, 1 \rangle \times \langle -2, 1, 3 \rangle$$

$$= \begin{vmatrix} i & j & k \\ -2 & 4 & 1 \\ -2 & 1 & 3 \end{vmatrix}$$

$\vec{n} = \langle 11, 4, 6 \rangle$

 $r_0 = (2, 0, 2)$

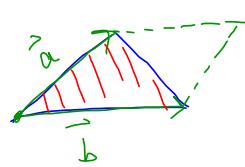
equation of plane is  $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$   $\vec{r} = \langle x, y, z \rangle$

$$\langle 11, 4, 6 \rangle \cdot \langle x-2, y-0, z-2 \rangle = 0$$

$$11x - 22 + 4y + 6z - 12 = 0$$

$11x + 4y + 6z = 34$

(c) Find the area of the triangle formed by the points  $P, Q$ , and  $R$ .



$$\text{A parallelogram} = |\vec{a} \times \vec{b}|$$

$$A_{\Delta PAR} = \frac{1}{2} \left| \langle 11, 4, 6 \rangle \right|$$

$$= \frac{1}{2} \sqrt{121 + 16 + 36}$$

2. Consider the plane  $P_1$  given by the equation  $2x - y + 3z = 7$  and the plane  $P_2$  given by the equation  $3x + y + 2z = 3$ .

(a) Are  $P_1$  and  $P_2$  parallel? Justify your answer.

A normal vector to the plane  $ax + by + cz = d$  is

$$\vec{n} = \langle a, b, c \rangle$$

$$\vec{n}_1 = \langle 2, -1, 3 \rangle$$

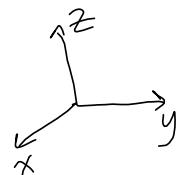
$$\vec{n}_2 = \langle 3, 1, 2 \rangle$$

$\vec{n}_1 + \vec{n}_2$  are not scalar multiples.  
or show  $\vec{n}_1 \times \vec{n}_2 \neq \langle 0, 0, 0 \rangle$

(b) Find a point  $(x_0, y_0, z_0)$  that lies on both planes.

$$\begin{cases} P_1: 2x - y + 3z = 7 \\ P_2: 3x + y + 2z = 3 \end{cases}$$

let  $z = 0$      $2x - y = 7$   
 $3x + y = 3$



$$2x - 3 + 3x = 7$$

$$5x = 10$$

$$x = 2$$

$$y = 3 - 3x$$

$$y = -3$$

A point on both planes is  
 $(2, -3, 0)$

(c) Find a parametric equation for the line where the two planes intersect.

The line of intersection of the two planes is:

$$\text{line in space is } \vec{r}(t) = \vec{r}_0 + t \vec{v}$$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2$$

$$\vec{r}(t) = \langle 2, -3, 0 \rangle + t \langle -5, 5, 5 \rangle$$

$$\begin{aligned} x &= 2 - 5t \\ y &= -3 + 5t \\ z &= 5t \end{aligned}$$

$$= \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$\vec{v} = \langle -5, 5, 5 \rangle$$

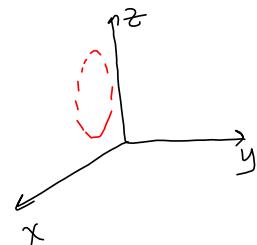
3. What is the equation for a sphere centered at  $(3, 4, 12)$  of radius 6. Does it intersect the  $xz$  plane? If so, what is the intersection?

$$(x-3)^2 + (y-4)^2 + (z-12)^2 = 36$$

on  $xz$  plane set  $y=0$

$$(x-3)^2 + 16 + (z-12)^2 = 36$$

$$(x-3)^2 + (z-12)^2 = 20$$



Intersects the  
 $xz$  plane in a  
circle

4. Let  $\mathbf{a} = \langle 1, 2, -2 \rangle$  and  $\mathbf{b} = \langle 2, -1, 2 \rangle$ . Find  $\underline{\underline{\text{Proj}_{\mathbf{b}} \mathbf{a}}}$ .

vector projection of  $\vec{a}$  onto  $\vec{b}$

$$\text{Proj}_{\mathbf{b}} \mathbf{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

$$= \frac{\langle 1, 2, -2 \rangle \cdot \langle 2, -1, 2 \rangle}{(\sqrt{4+1+4})^2} \langle 2, -1, 2 \rangle$$

$$= \frac{2 - 2 - 4}{9} \langle 2, -1, 2 \rangle$$

$$= -\frac{4}{9} \langle 2, -1, 2 \rangle$$

5. Let  $\mathbf{a} = \langle -2, 2, 1 \rangle$ . Find a vector  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  so that  $\text{Comp}_{\mathbf{a}} \mathbf{b} = -4$ .

$\uparrow$   
scalar projection of  $\vec{b}$  onto  $\vec{a}$

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$$

$$\begin{aligned} \text{comp}_{\mathbf{a}} \mathbf{b} &= \frac{\langle -2, 2, 1 \rangle \cdot \langle b_1, b_2, b_3 \rangle}{\sqrt{4+4+1}} \\ &= \frac{-2b_1 + 2b_2 + b_3}{3} \end{aligned}$$

$$\text{comp}_{\mathbf{a}} \mathbf{b} = -4$$

$$\frac{-2b_1 + 2b_2 + b_3}{3} = -4$$

$\langle 0, 0, -12 \rangle$   
infinitely many answers!

$$-2b_1 + 2b_2 + b_3 = -12$$

$$\begin{array}{l} \text{choose } \begin{cases} b_1 = 0 \\ b_2 = 0 \end{cases} \\ b_3 = -12 \end{array}$$

6. Let  $\mathbf{r}(t) = \langle t^2, \frac{t-1}{t^2-1}, \frac{\sin t}{t} \rangle$ .

(a) Find the domain of  $\mathbf{r}(t)$ .

$$t \neq \pm 1$$

$$t \neq 0$$



$$(-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$$

(b) Find  $\lim_{t \rightarrow 1} \mathbf{r}(t)$ .

$$\lim_{t \rightarrow 1} \left\langle t^2, \frac{t-1}{(t+1)(t-1)}, \frac{\sin t}{t} \right\rangle = \boxed{\left\langle 1, \frac{1}{2}, \frac{\sin(1)}{1} \right\rangle}$$

7. Let  $\mathbf{r}(t) = (\cos(t^2), \sin(t^2), t^2)$ .

(a) Find  $\mathbf{T}(\sqrt{\pi})$ , the unit tangent vector at  $t = \sqrt{\pi}$ .

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

$$\mathbf{r}'(t) = \langle -2t \sin(t^2), 2t \cos(t^2), 2t \rangle$$

$$\mathbf{T}(\sqrt{\pi}) = \frac{\mathbf{r}'(\sqrt{\pi})}{|\mathbf{r}'(\sqrt{\pi})|}$$

$$\mathbf{r}'(\sqrt{\pi}) = \langle 0, 2\sqrt{\pi}(-1), 2\sqrt{\pi} \rangle$$

$$\mathbf{r}'(\sqrt{\pi}) = \langle 0, -2\sqrt{\pi}, 2\sqrt{\pi} \rangle$$

$$\mathbf{T}(\sqrt{\pi}) = \frac{\langle 0, -2\sqrt{\pi}, 2\sqrt{\pi} \rangle}{2\sqrt{2}\sqrt{\pi}}$$

$$\boxed{\mathbf{T}(\sqrt{\pi}) = \langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle}$$

$$|\mathbf{r}'(\sqrt{\pi})| = \sqrt{4\pi + 4\pi}$$

$$= \sqrt{8\pi}$$

$$= 2\sqrt{2\pi}$$

(b) Find  $\mathbf{a}(t)$ , the acceleration vector, at time  $t$ .

$$\mathbf{r}'(t) = \langle -2\sin(t^2) - 4t^2 \cos(t^2), 2t\cos(t^2) - 4t\sin(t^2), 2 \rangle$$

$$\text{note: } \mathbf{r}''(\sqrt{\pi}) = \langle -4\pi(-1) - 2, 2 \rangle = \langle 4\pi, -2, 2 \rangle$$

which is used in part d)

(c) Find the length of the curve from  $(1, 0, 0)$  to  $(1, 0, 2\pi)$ .

length of  $\overrightarrow{r(t)}$  from  $t=a$  to  $t=b$  is

$$L = \int_a^b |\mathbf{r}'(t)| dt$$

$$\boxed{\mathbf{r}(t) = \langle \cos(t^2), \sin(t^2), t^2 \rangle}$$

$a = \text{value of } t$   
that yields  $(1, 0, 0)$

$$t=0$$

$$\mathbf{r}(0) = \langle 1, 0, 0 \rangle \checkmark$$

$b = \text{value of } t$   
that yields  $(1, 0, 2\pi)$

$$b = \sqrt{2\pi} \quad \text{set } z = 2\pi$$

$$t^2 = 2\pi$$

$$t = \sqrt{2\pi}$$

$$\mathbf{r}'(t) = \langle -2t \sin(t^2), 2t \cos(t^2), 2t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{4t^2 \sin^2 t^2 + 4t^2 \cos^2 t^2 + 4t^2}$$

$$= \sqrt{4t^2 (\sin^2 t^2 + \cos^2 t^2) + 4t^2}$$

$$= \sqrt{4t^2 + 4t^2}$$

$$= \sqrt{8t^2}$$

$$|\mathbf{r}'(t)| = \sqrt{8t^2}$$

$$L = \int_0^{\sqrt{2\pi}} \sqrt{8t^2} dt$$

$$= \sqrt{8} \left[ \frac{t^2}{2} \right]_0^{\sqrt{2\pi}}$$

$$= \frac{\sqrt{8}(2\pi)}{2}$$

$$\boxed{L = \sqrt{8}\pi \text{ or } 2\sqrt{2}\pi}$$

(d) Find the curvature of the curve traced out by  $\mathbf{r}(t)$  when  $t = \sqrt{\pi}$ .

Formula for curvature of a curve

$$K = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{|\langle 0, -2\sqrt{\pi}, 2\sqrt{\pi} \rangle \times \langle 4\pi, -2, 2 \rangle|}{|\langle 0, -2\sqrt{\pi}, 2\sqrt{\pi} \rangle|^3}$$

$$\mathbf{r}'(\sqrt{\pi}) = \langle 0, -2\sqrt{\pi}, 2\sqrt{\pi} \rangle$$

$$\mathbf{r}''(\sqrt{\pi}) = \langle 4\pi, -2, 2 \rangle$$

$$\mathbf{r}'(\sqrt{\pi}) \times \mathbf{r}''(\sqrt{\pi}) = \begin{vmatrix} i & j & k \\ 0 & -2\sqrt{\pi} & 2\sqrt{\pi} \\ 4\pi & -2 & 2 \end{vmatrix} = \langle 0, 8\pi\sqrt{\pi}, 8\pi\sqrt{\pi} \rangle$$

$$K = \frac{\sqrt{64\pi^3 + 64\pi^3}}{(\sqrt{8\pi})^3} = \frac{8\pi\sqrt{\pi}}{8\sqrt{8}\sqrt{\pi}} = \frac{\pi}{\sqrt{8}}$$

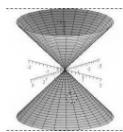
Worked quickly!  
double check

8. Match the equation with the images that follow.

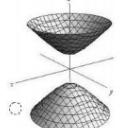
$$\text{Equation 1: } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$\text{Equation 2: } -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

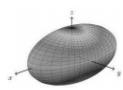
$$\text{Equation 3: } \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$



equation 3



equation 2



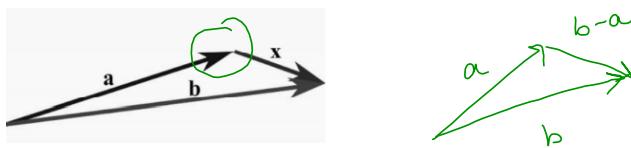
equation 1

sum of three squares = positive number  
ellipsoid

$x^2 - y^2 + z^2 = 1$       1 sheet hyperboloid

$x^2 - y^2 - z^2 = 1$       2 sheet hyperboloid

9. Use the figure below to answer the questions that follow.



(a) Write  $x$  in terms of  $a$  and  $b$ .

$$a + x = b$$

$$x = b - a$$

(b) If the angle between  $a$  and  $b$  is  $60^\circ$ ,  $|a| = 7$ , and  $|b| = 6$ , find  $a \cdot b$ .

$$\begin{aligned} a \cdot b &= |a||b|\cos\theta \\ &= (7)(6)\cos 60^\circ \\ &= \boxed{21} \end{aligned}$$

(c) If the angle between  $a$  and  $b$  is  $60^\circ$ ,  $|a| = 7$ , and  $|b| = 6$ , find  $|a \times b|$  and determine whether  $a \times b$  is directed into or out of the page.

$$\begin{aligned} |a \times b| &= |a||b|\sin\theta \\ &= (7)(6)\sin 60^\circ \\ &= 21\sqrt{3} \end{aligned}$$

$a \times b$   
directed into the  
page

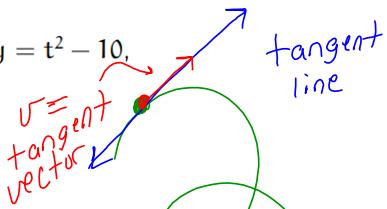
10. Find parametric equations for the tangent line to the curve  $x = 4\sqrt{t}$ ,  $y = t^2 - 10$ ,  $z = \frac{4}{t}$  at  $(8, 6, 1)$ .

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

derivative

$$\vec{r}_0 = (8, 6, 1)$$

$$\vec{v} =$$



$$\boxed{\vec{r}(t) = \langle 4\sqrt{t}, t^2 - 10, \frac{4}{t} \rangle}$$

$$\vec{r}'(t) = \left\langle \frac{2}{\sqrt{t}}, 2t, -\frac{4}{t^2} \right\rangle$$

$$\vec{v} = \vec{r}'(4) = \left\langle 1, 8, -\frac{1}{4} \right\rangle$$

what value of  
t yields the  
point  $(8, 6, 1)$

$$t = 4$$

$$\vec{r}(t) = \langle 8, 6, 1 \rangle + t \left\langle 1, 8, -\frac{1}{4} \right\rangle$$

$$\boxed{\begin{aligned} x &= 8 + t \\ y &= 6 + 8t \\ z &= 1 - \frac{1}{4}t \end{aligned}}$$

11. If  $\mathbf{r}'(t) = \langle t, e^t, te^{3t} \rangle$  and  $\underline{\mathbf{r}(0) = \langle 1, 3, 2 \rangle}$ , find  $\mathbf{r}(t)$ .

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt$$

$$\mathbf{r}(t) = \left\langle \frac{t^2}{2} + C_1, e^t + C_2, \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + C_3 \right\rangle$$

$$\mathbf{r}(0) = \langle C_1, 1+C_2, -\frac{1}{9} + C_3 \rangle = \langle 1, 3, 2 \rangle$$

$$C_1 = 1, 1 + C_2 = 3, -\frac{1}{9} + C_3 = 2$$

$$C_2 = 2 \quad C_3 = 2 + \frac{1}{9} = \frac{19}{9}$$

$$\begin{aligned} \int te^{3t} dt & \text{ parts} \\ u = t & \quad dv = e^{3t} dt \\ du = dt & \quad v = \frac{1}{3}e^{3t} \end{aligned}$$

$$\begin{aligned} \int te^{3t} dt &= uv - \int v du \\ &= \frac{1}{3}te^{3t} - \int \frac{1}{3}e^{3t} dt \\ &= \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} \end{aligned}$$

12. Find  $\int_0^1 \left( \frac{4t}{t^2+1} \mathbf{j} - \frac{1}{1+t^2} \mathbf{k} \right) dt$ .

$= \left( 2 \ln(t^2+1) \Big|_0^1 \right) \vec{\mathbf{j}} - \left( \arctan(t) \Big|_0^1 \right) \vec{\mathbf{k}}$   
 $= \boxed{(2 \ln 2) \vec{\mathbf{j}} - \frac{\pi}{4} \vec{\mathbf{k}}}$

u-sub  
 $u = t^2 + 1$   
 $du = 2t dt$

$\frac{4}{2} \int \frac{du}{u}$

$2 \ln|u|$   
 $2 \ln|t^2 + 1|$

13. Given the curves  $\mathbf{r}_1(t) = \langle 3t, t^2, t^3 \rangle$  and  $\mathbf{r}_2(v) = \langle \sin v, \sin(2v), 6v \rangle$  intersect at the origin, find the angle of intersection.

$$\mathbf{r}_1(0) = \langle 0, 0, 0 \rangle$$

$$\mathbf{r}_2(0) = \langle 0, 0, 0 \rangle$$

Angle of intersection is  
the angle between their  
tangent vectors at the intersection  
point.

$$\mathbf{r}'_1(t) = \langle 3, 2t, 3t^2 \rangle$$

$$\mathbf{r}'_1(0) = \langle 3, 0, 0 \rangle$$

$$\mathbf{r}'_2(v) = \langle \cos v, 2\cos(2v), 6 \rangle$$

$$\mathbf{r}'_2(0) = \langle 1, 2, 6 \rangle$$

$$\cos \theta = \frac{\langle 3, 0, 0 \rangle \cdot \langle 1, 2, 6 \rangle}{\sqrt{9} \sqrt{1+4+36}}$$

$$\theta = \arccos\left(\frac{3}{3\sqrt{41}}\right)$$

$$\theta = \arccos\left(\frac{1}{\sqrt{41}}\right)$$

14. Find parametric equations for the line that passes through  $(2, -1, 5)$  and is

(a) parallel to the line  $\frac{x+1}{3} = \frac{y-6}{4} = z$ .

$$\begin{aligned}\frac{x+1}{3} &= t \\ x &= 3t - 1 \\ \frac{y-6}{4} &= t \\ y &= 4t + 6 \\ z &= t\end{aligned}$$

direction vector of this line is

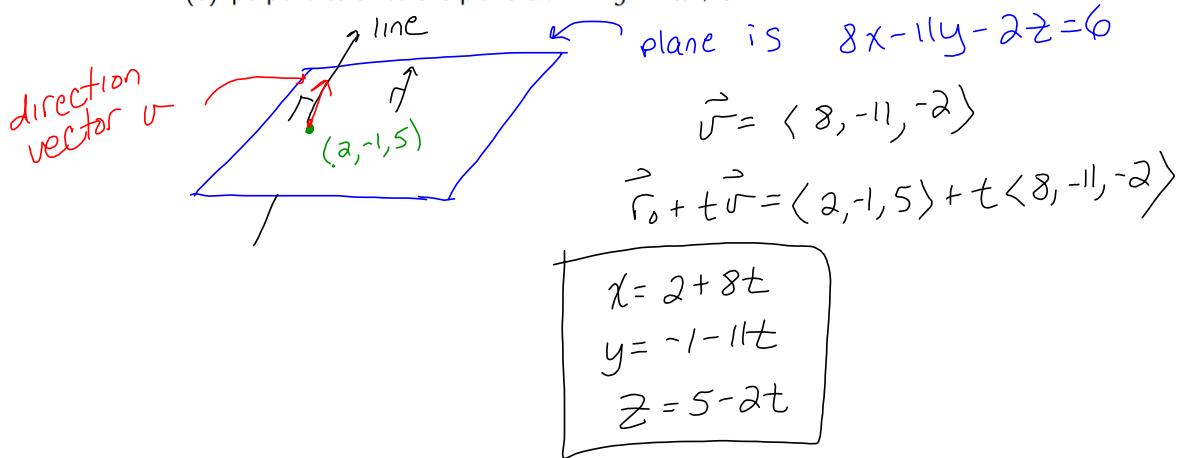
$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \quad \vec{v} = \langle 3, 4, 1 \rangle$$

Parallel lines have parallel direction vectors.

$$\vec{r}_0 + t\vec{v} = \langle 2, -1, 5 \rangle + t\langle 3, 4, 1 \rangle$$

$$\boxed{\begin{aligned}x &= 2+3t \\ y &= -1+4t \\ z &= 5+t\end{aligned}}$$

(b) perpendicular to the plane  $8x - 11y = 2z + 6$ .



15. Consider the line that passes through the points  $(4, 3, -1)$  and  $(5, 3, 5)$ . Where does this line intersect the three coordinate planes, and if it does not intersect all of the three coordinate planes, explain why not.

equation of line

$$\vec{r}_0 + t \vec{v}$$

$$\vec{r}_0 = (4, 3, -1)$$

$$\vec{v} = \langle 1, 0, 6 \rangle$$

$$\begin{aligned} r(t) &= \vec{r}_0 + t \vec{v} \\ &= \langle 4, 3, -1 \rangle + t \langle 1, 0, 6 \rangle \end{aligned}$$

*intersects the xy plane when  $z=0$*

$$\begin{cases} x = 4 + t \\ y = 3 \\ z = -1 + 6t \end{cases}$$

$$x = 4 + \frac{1}{6} = \frac{25}{6}$$

$$y = 3$$

$$\boxed{\text{point: } \left( \frac{25}{6}, 3, 0 \right)}$$

*xy plane intersection*

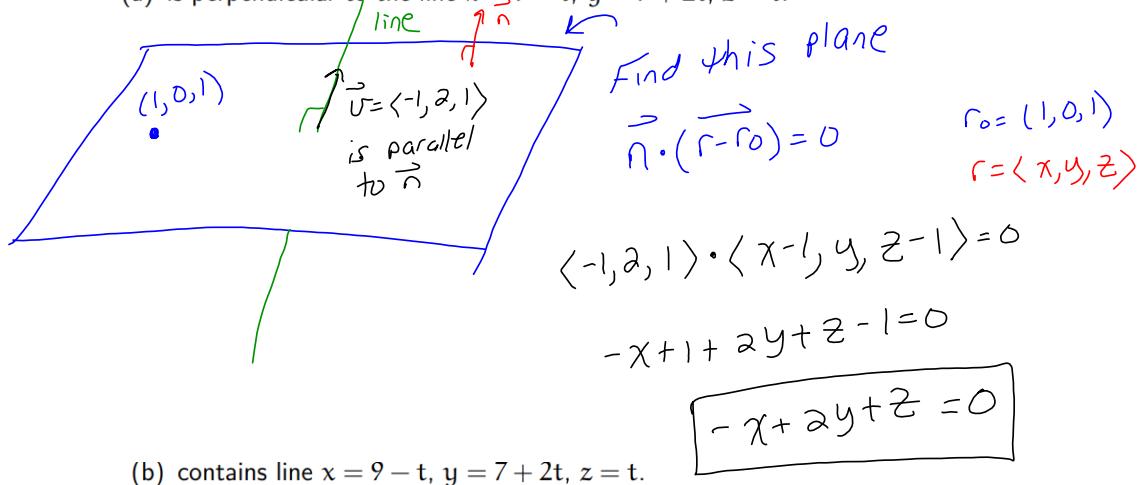
$$\begin{cases} x = 0 \\ t = -4 \\ y = 3 \\ z = -1 - 24 = -25 \end{cases}$$

*xz plane  $y=0 \rightarrow y=3$  is never zero!*

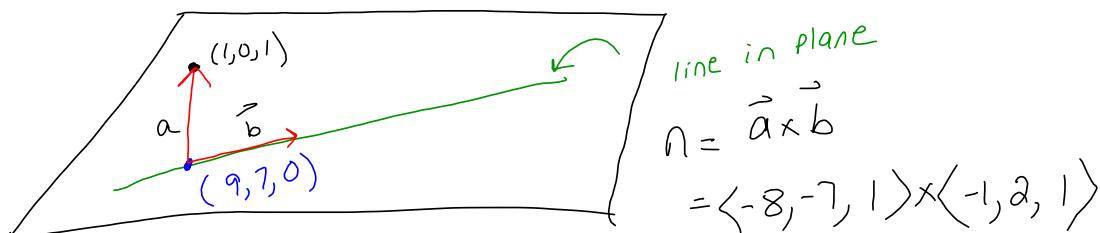
*no intersection*

16. Find the equation of the plane that passes through the point  $(1, 0, 1)$  and

(a) is perpendicular to the line  $x = 9 - t, y = 7 + 2t, z = t$ .



(b) contains line  $x = 9 - t, y = 7 + 2t, z = t$ .



$$\vec{n} = \begin{vmatrix} i & j & k \\ -8 & -7 & 1 \\ -1 & 2 & 1 \end{vmatrix} = \langle -9, 7, -23 \rangle$$

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\langle -9, 7, -23 \rangle \cdot \langle x-1, y, z-1 \rangle = 0$$

17. Describe the space curve  $x = \sin t$ ,  $y = 3$ ,  $z = \cos t$ .

$$\sin^2 t + \cos^2 t = 1$$

$$x^2 + z^2 = 1$$

