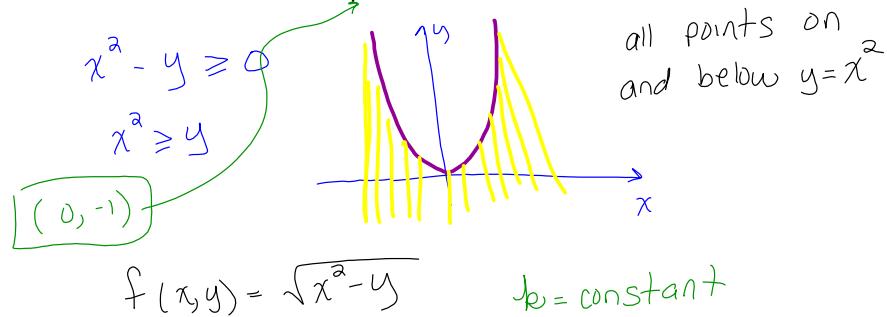


1. Sketch the domain of $f(x, y) = \sqrt{x^2 - y}$ and describe the level curves.



all points on
and below $y = x^2$

$$k = \sqrt{x^2 - y}$$

$$k^2 = x^2 - y \rightarrow y = x^2 - k^2$$

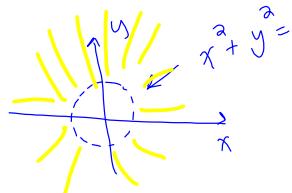
level curves is a family of parabolas

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{l} k=0 \\ k=1 \end{array} \rightarrow y = x^2 - 1$$

2. Sketch the domain of $f(x, y) = \ln(y^2 + x^2 - 1)$ and describe the level curves.

$$y^2 + x^2 - 1 > 0$$

$$y^2 + x^2 > 1$$



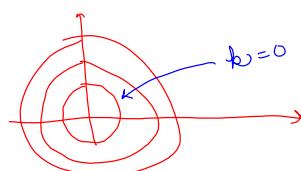
domain
all points
outside the
 $x^2 + y^2 = 1$,
excluding the
circle

Level curves set $f(x, y) = k$

$$k = \ln(y^2 + x^2 - 1)$$

$$e^k = y^2 + x^2 - 1 \rightarrow e^k + 1 = y^2 + x^2$$

Family of circles!



3. What are the level surfaces to the equation $f(x, y, z) = x + y + z$?

$$k = x + y + z \quad \text{Family of planes!}$$

4. For $f(x, y) = \sin(x^2 + y^2)$, find all first and second partial derivatives.

$$\frac{\partial f}{\partial x} = f_x(x, y) = 2x \cos(x^2 + y^2)$$

$$f_y(x, y) = 2y \cos(x^2 + y^2)$$

second partial:
 $f_{xx}, f_{yy}, f_{xy} = f_{yx}$

$$f_{xx} = \frac{\partial f}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(2x \cos(x^2 + y^2) \right)$$

product rule +
chain rule

$$= 2 \cos(x^2 + y^2) + (2x)(-2x \sin(x^2 + y^2))$$

Do others on your own!

5. Find an equation for the tangent plane to the surface $\underline{z = 2x^2 + y^2}$ at the point $(1, 1)$.

solved for z

$$x_0 = 1, y_0 = 1, z_0 = 3$$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 3 = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$$

$$f_x = \frac{\partial z}{\partial x} = 4x, \text{ so } f_x(1, 1) = 4$$

$$f_y = \frac{\partial z}{\partial y} = 2y, \text{ so } f_y(1, 1) = 2$$

$$z - 3 = 4(x - 1) + 2(y - 1)$$

6. Find the tangent plane to the surface $2xy + 3yz + 7xz = -9$ at the point $(1, 2, -1)$.

not explicitly solve for z

$$w = f(x, y, z) = \underline{2xy + 3yz + 7xz}$$

$$\vec{n} = \nabla f(1, 2, -1) \quad \nabla f = \langle f_x, f_y, f_z \rangle$$

$$\vec{r}_0 = (1, 2, -1)$$

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\nabla f = \langle 2y + 7z, 2x + 3z, 3y + 7x \rangle$$

$$\nabla f(1, 2, -1) = \langle 4 - 7, 2 - 3, 6 + 7 \rangle$$

$$\langle -3, -1, 13 \rangle \cdot \langle x-1, y-2, z+1 \rangle = 0$$

$$\vec{n} = \langle -3, -1, 13 \rangle$$

$$-3(x-1) - (y-2) + 13(z+1) = 0$$

7. If $z = x^3y^2$, find the differential, \underline{dz} , and explain what it measures.

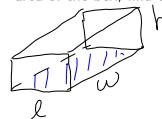
$$z = f(x, y), \text{ then } dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$z = x^3y^2$$

$$dz = 3x^2y^2 dx + 2yx^3 dy$$

dz measures the error in z
when x & y are approximate values

8. Consider a rectangular box with length l , width w and height h . If A is the surface area of the box, find the differential, \underline{dA} .



$$A(l, w, h) = 2wl + 2wh + 2lh$$

$$dA = A_l dl + A_w dw + A_h dh$$

$$dA = (2w + 2h)dl + (2l + 2h)dw + (2w + 2l)dh$$

$$dl = \text{"error in } l\text{"} \quad dh = \text{error in } h$$

$$dw = \text{error in } w$$

9. The height of a cone was measured to be 3 cm with a maximum error of 0.1 cm and the radius of the cone is measured to be 2 cm with a maximum error of 0.2 cm. Use differentials to estimate the maximum error in the calculated volume of the cone. $\leftarrow dV = ?$

$$V(r, h) = \frac{1}{3}\pi r^2 h$$

$$\begin{aligned} h &= 3, |dh| \leq \frac{1}{10} \\ r &= 2, |dr| \leq \frac{2}{10} \end{aligned}$$

$$dV = V_r dr + V_h dh$$

$$dV = \frac{\pi}{3}(ar)hdr + \frac{1}{3}\pi r^2 dh$$

$$\text{max error in } V \approx dV$$

$$dV = \frac{\pi}{3}(4)(3)\frac{2}{10} + \frac{1}{3}\pi(4)\frac{1}{10}$$

$$= \frac{24\pi}{30} + \frac{4\pi}{30}$$

$$dV = \frac{28\pi}{30} \text{ cm}^3$$

10. Find the equation of the tangent plane to the surface $f(x, y) = xe^{xy}$ at the point $(1, 0, 1)$ and use this plane to approximate $f(1.1, -0.2)$.

① Find tangent plane at $(1, 0, 1)$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 1 = f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

$$f_x = e^{xy} + xy^2 e^{xy} \quad f_x(1, 0) = 1$$

$$f_y = (x) x e^{xy} \quad f_y(1, 0) = 1$$

$$z - 1 = x - 1 + y \rightarrow z = x + y$$

$$f(x, y) \approx z = x + y \quad \text{near } (1, 0, 1)$$

$(1.1, -0.2)$ near $(1, 0)$

$$f(1.1, -0.2) \approx 1.1 - 0.2 = 0.9$$

11. Use a linear approximation to estimate $((2.1)^2 + (0.1)^3)^3$

↳ using tangent plane

$$f(x, y) = (x^2 + y^3)^3 \quad \text{Find tangent plane at } (2, 0, (2^2 + 0^3)^3)$$

$$z - 64 = f_x(2, 0)(x - 2) + f_y(2, 0)(y - 0) \quad = (2, 0, 64)$$

$$z = 64 + 192(x - 2)$$

$$f(2.1, 0.1) \approx 64 + 192(0.1) = 83.2$$

12. Use differentials to approximate $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

$$w = f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \quad x = 3 \\ f(3, 2, 6) = \sqrt{9+4+36} = 7 \quad y = 2 \\ z = 6$$

$$dw = w_x dx + w_y dy + w_z dz \quad dx = .02$$

$$dy = -0.03$$

$$dz = -0.01$$

$$dw = \frac{x}{\sqrt{x^2 + y^2 + z^2}} dx + \frac{y}{\sqrt{x^2 + y^2 + z^2}} dy + \frac{z}{\sqrt{x^2 + y^2 + z^2}} dz$$

$$dw = \frac{3}{7} \cancel{\frac{2}{100}} + \frac{2}{7} \cancel{\left(-\frac{3}{100}\right)} + \frac{6}{7} \left(-\frac{1}{100}\right)$$

$$dw = \frac{-6}{700} \quad \text{error}$$

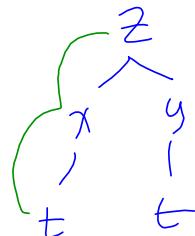
$$f(3.02, 1.97, 5.99) \approx f(3, 2, 6) + dw$$

$$\approx 7 - \frac{6}{700} = 6.94$$

$$\text{Actual } f(3.02, 1.97, 5.99) = 6.99$$

13. If $z = e^{x^2+y^2}$, $x = e^t$. $y = \cos t$, find $\frac{dz}{dt}$.

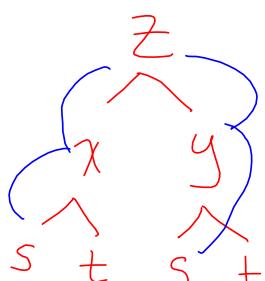
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$



$$= (2xe^{x^2+y^2})(e^t) + 2ye^{x^2+y^2}(-\sin t)$$

14. For $z = xy$, $x = \cos(st^2)$, $y = \underline{\sin e^t}$, find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$



$$= (y)(-\sin(st^2)(t^2)) + (x)(0)$$

15. The radius and height of a circular cylinder change with time. When the height is 1 meter and the radius is 2 meters, the radius is increasing at a rate of 4 meters per second and the height is decreasing at a rate of 2 meters per second. At that same instant, find the rate at which the volume is changing.

$$V(r, h) = \pi r^2 h$$

when $h=1, \frac{dh}{dt} = -2$
 $r=2, \frac{dr}{dt} = 4$

Find $\frac{dV}{dt}$ at this same instant

$$\frac{dV}{dt} = \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} = 2\pi rh \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

$$\frac{dV}{dt} = 2\pi(2)(1)(4) + \pi(4)^2(-2)$$

$$\frac{dV}{dt} = 16\pi - 8\pi$$

$$\frac{dV}{dt} = 8\pi \text{ m}^3/\text{sec}$$

16. Let $f(x, y) = \sqrt{xy}$. Find the directional derivative of f at the point P(4, 1) in the direction from P to Q(6, 2).

$$D_u f(a, b) = \nabla f(a, b) \cdot \vec{u}$$

$$D_u f(4, 1) \cdot \vec{u}$$

\vec{u} = unit vector

$$\nabla f(4, 1)$$

$$\nabla f = \left\langle \frac{y}{2\sqrt{xy}}, \frac{x}{2\sqrt{xy}} \right\rangle$$

$$\nabla f(4, 1) = \left\langle \frac{1}{2\sqrt{4}}, \frac{4}{2\sqrt{4}} \right\rangle$$

$$\nabla f(4, 1) = \left\langle \frac{1}{4}, 1 \right\rangle$$

$$\vec{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$\vec{u} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$D_u f(4, 1) = \nabla f(4, 1) \cdot \vec{u}$$

$$= \left\langle \frac{1}{4}, 1 \right\rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$= \frac{1}{2\sqrt{5}} + \frac{1}{\sqrt{5}}$$

17. Let $f(x, y) = \sqrt{xy}$. What is the direction of the largest rate of change at the point $P(4, 1)$?

direction of maximum rate of change is
the gradient vector.

} in problem 16, we found $\nabla f(4, 1) = \left\langle \frac{1}{4}, 1 \right\rangle$

18. Let $f(x, y) = e^{x+y}$. What is the maximum rate of change at the point $P(-1, 1)$?

max rate of change is $|\nabla f(-1, 1)|$

$$\nabla f = \left\langle e^{x+y}, e^{x+y} \right\rangle$$

$$\nabla f(-1, 1) = \left\langle 1, 1 \right\rangle$$

$$\text{max ROC} = |\langle 1, 1 \rangle|$$

$$= \boxed{\sqrt{2}}$$

19. Find an equation for the tangent plane and normal line to the surface $x^2 - 5y^2 + z^2 = 4$ at the point $(7, 3, 0)$.

$$\vec{n} = \nabla f, \quad f(x, y, z) = x^2 - 5y^2 + z^2 \quad \text{level surface}$$

$$\nabla f = \langle 2x, -10y, 2z \rangle$$

$$\nabla f(7, 3, 0) = \langle 14, -30, 0 \rangle$$

Plane $\langle 14, -30, 0 \rangle \cdot \langle x-7, y-3, z \rangle = 0$

$$\boxed{14(x-7) - 30(y-3) = 0}$$

Normal line at $(7, 3, 0)$ $\vec{r}(t) = \vec{r}_0 + t \vec{v}$

$$\vec{v} = \vec{n}$$

$$\vec{v} = \langle 14, -30, 0 \rangle$$

$$f(t) = \langle 7, 3, 0 \rangle + t \langle 14, -30, 0 \rangle$$

$$\boxed{\begin{aligned} x &= 7 + 14t \\ y &= 3 - 30t \\ z &= 0 \end{aligned}}$$

20. For the $f(x, y) = 2x^3 - xy^2 + 5x^2 + y^2 + 5$, find all local minima, maxima, and saddle points.

$$\text{CP : } \begin{array}{l} f_x = 6x^2 - y^2 + 10x \\ f_y = -2xy + 2y \\ f_{xx} = 12x + 10 \\ f_{yy} = -2x + 2 \\ f_{xy} = -2y \end{array}$$

$\begin{cases} 6x^2 - y^2 + 10x = 0 \\ -2xy + 2y = 0 \end{cases} \rightarrow -2y(x-1) = 0$

$y=0, x=1$ (plug into $f_x = 0$)

$y=0: 6x^2 + 10x = 0 \quad x=0, x = -\frac{5}{3}$

$x=1: 6 - y^2 + 10 = 0 \quad (0,0), (-\frac{5}{3}, 0)$

$16 = y^2 \quad (1, 4), (1, -4)$

$((y = \pm 4))$

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

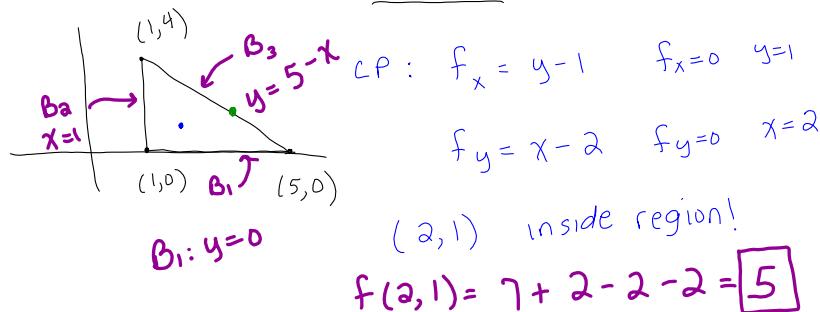
$$D > 0: \begin{cases} f_{xx} > 0 & \text{local min} \\ f_{xx} < 0 & \text{local max} \end{cases}$$

$D < 0$ saddle point

$D = 0$ no conclusion

CP	$D = (12x+10)(-2x+2) - (-2y)^2$	$f_{xx} = 12x+10$	Conclusion
$(0,0)$	$D = 20 > 0$	$f_{xx} = 10 > 0$	min
$(-\frac{5}{3}, 0)$	$D < 0$		saddle
$(1, 4)$	$D < 0$		saddle
$(1, -4)$	$D < 0$		saddle

21. Find the absolute maximum and minimum values of $f(x, y) = 7 + xy - x - 2y$ over the closed triangular region with vertices $(1, 0)$, $(5, 0)$, $(1, 4)$.



$$f(x, y) = 7 + xy - x - 2y$$

$B_1: y=0 \quad f(x, 0) = 7 - x, \quad 1 \leq x \leq 5$

$f'(x, 0) = -1$ no CP. Only need to evaluate f at endpoints.

$x=1: f(1, 0) = \boxed{6}$

$x=5: f(5, 0) = \boxed{2}$

$B_2: x=1 \quad f(1, y) = 7 + y - 1 - 2y$

$= -y + 6, \quad 0 \leq y \leq 4$

$f'(1, y) = -1, \quad \text{no CP} \quad \text{evaluate } f \text{ at endpoints}$

$f(x, y) = 7 + xy - x - 2y$

$B_3: y = 5 - x$

$y=0: f(1, 0) = \boxed{6}$

$y=4: f(1, 4) = \boxed{2}$

$$f(x, 5-x) = 7 + x(5-x) - x - 2(5-x)$$

$$= 7 + 5x - x^2 - x - 10 + 2x$$

$$f(x, 5-x) = -x^2 + 6x - 3, \quad 1 \leq x \leq 5$$

$f'(x, 5-x) = -2x + 6 \quad \text{cn: } x=3 \quad \checkmark$

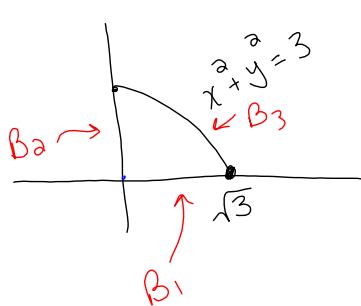
$x=3: f(3, 2) = -9 + 18 - 3 = \boxed{6}$

$x=1: f(1, 4) = -1 + 6 - 3 = \boxed{2}$

$x=5: f(5, 0) = -25 + 30 - 3 = \boxed{2}$

Abs max = 6, Abs min = 2

22. Find the absolute maximum and minimum values of $f(x, y) = xy^2 + 2$ over the region
 $D = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$.



$$\text{Cn: } f_x = y^2 \quad \text{Cn: } (0,0)$$

$$f_y = 2xy$$

$$f(0,0) = \boxed{2}$$

$$B_1: y=0 : f(x,0) = \boxed{2}, \quad 0 \leq x \leq \sqrt{3}$$

constant!

$$B_2: x=0 \quad f(0,y) = 2 \quad 0 \leq y \leq \sqrt{3}$$

constant!

$$B_3: x^2 + y^2 = 3 \quad f(x,y) = xy^2 + 2$$

$$y = \sqrt{3 - x^2}$$

$$f(x, \sqrt{3-x^2}) = x(3-x^2) + 2, \quad 0 \leq x \leq \sqrt{3}$$

$$= 3x - x^3 + 2$$

$$f(x, \sqrt{3-x^2}) = 3 - 3x^2$$

$$3 - 3x^2 = 0 \rightarrow x = 1$$

~~x = -1~~

$\boxed{x=1} : f(1, \sqrt{2}) = 3 - 1 + 2 = \boxed{4}$

$$x=0 : f(0, \sqrt{3}) = \boxed{2}$$

$$x=\sqrt{3} : f(\sqrt{3}, 0) = 3\sqrt{3} - (\sqrt{3})^3 + 2$$

$$= 3\sqrt{3} - 3\sqrt{3} + 2$$

$$= \boxed{2}$$

Abs max = 4
 min = 2

23. Use the method of Lagrange to find the maximum and minimum values of

$$f(x, y) = 6x + 6y \text{ subject to the constraint } \underline{x^2 + y^2 = 18}.$$

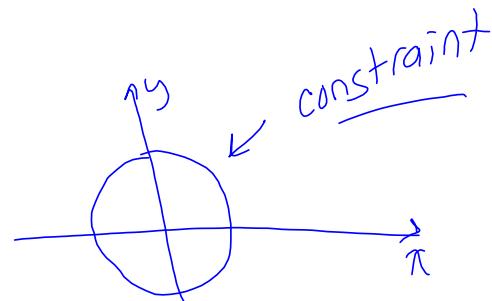
$$\nabla f = \lambda \nabla g$$

$$g(x, y) = x^2 + y^2$$

$$\langle 6, 6 \rangle = \lambda \langle 2x, 2y \rangle$$

$$\begin{aligned} 6 &= 2\lambda x \\ 6 &= 2\lambda y \\ x^2 + y^2 &= 18 \end{aligned}$$

SOLVE
SYSTEM!



λ cannot be zero
divide by λ

$$6 = 2\lambda x \rightarrow$$

$$x = \frac{6}{2\lambda} = \frac{3}{\lambda}$$

$$6 = 2\lambda y \rightarrow$$

$$y = \frac{6}{2\lambda} = \frac{3}{\lambda}$$

$$\frac{9}{\lambda^2} + \frac{9}{\lambda^2} = 18 \rightarrow \frac{18}{\lambda^2} = 18$$

$$1 = \lambda^2 \rightarrow \boxed{\lambda = \pm 1}$$

$$\lambda = 1 \left\langle \begin{array}{l} x = 3 \\ y = 3 \end{array} \right.$$

$$f(3, 3) =$$

$$\lambda = -1 \left\langle \begin{array}{l} x = -3 \\ y = -3 \end{array} \right.$$

$$f(-3, -3) =$$

$$f(x, y) = 6x + 6y$$

$$f(3, 3) = 18 + 18 = 36$$

$$f(-3, -3) = -18 - 18 = -36$$

$$\boxed{\begin{array}{l} \text{Abs max} = 36 \\ \text{Abs min} = -36 \end{array}}$$

24. Use the method of Lagrange to find the maximum and minimum values of
 $f(x, y) = y^2 - x^2$ subject to the constraint $\frac{1}{4}x^2 + y^2 = 25$.

25. Use the method of Lagrange to find the extreme values of
 $f(x, y, z) = xyz$ subject to the constraint $2x + y + 3z = 60$.