

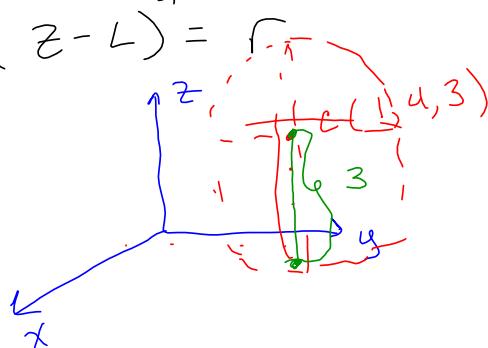
1. Find the equation of the sphere with center $(1, 4, 3)$ that touches the xy plane.

equation of a sphere with center $C(h, k, l)$ and radius r is

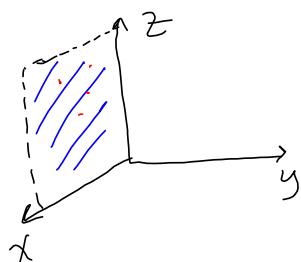
$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

$$C(1, 4, 3) \quad r = 3$$

$$(x-1)^2 + (y-4)^2 + (z-3)^2 = 9$$



2. Does the sphere $x^2 + y^2 + z^2 + 4x - 2y - 8z = 5$ intersect the xz plane? If so, what is the intersection?



if it touches the xz plane,
 $y=0$ must satisfy the
equation of the sphere.

$$y=0$$

$$x^2 + 0^2 + z^2 + 4x - 2(0) - 8z = 5$$

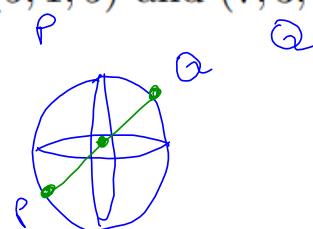
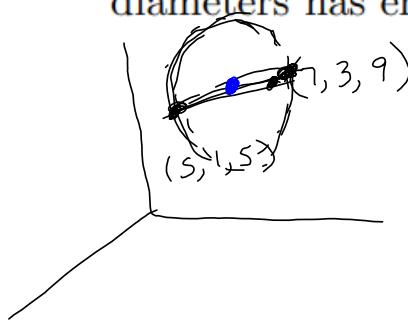
$$x^2 + z^2 + 4x - 8z = 5$$

$$x^2 + 4x + 4 + z^2 - 8z + 16 = 5 + 4 + 16$$

$$(x+2)^2 + (z-4)^2 = 25$$

yes! circle!

3. Find the equation of the sphere if one of their diameters has endpoints $(5, 1, 5)$ and $(7, 3, 9)$.



$C = \text{midpoint of } PQ$

$$C = \left(\frac{5+7}{2}, \frac{1+3}{2}, \frac{5+9}{2} \right)$$

$$P(5, 1, 5)$$

$$C(6, 2, 7)$$

$$\boxed{C = (6, 2, 7)}$$

$$r = |CP| = \sqrt{(-1)^2 + (-1)^2 + (-2)^2}$$

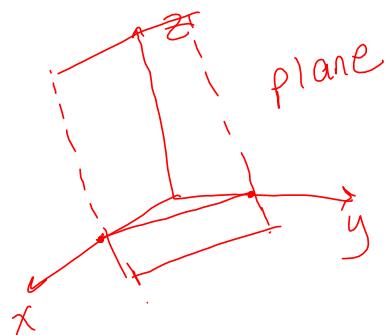
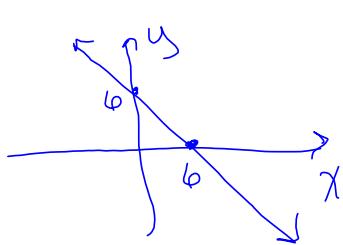
equation:

$$(x-6)^2 + (y-2)^2 + (z-7)^2 = r^2$$

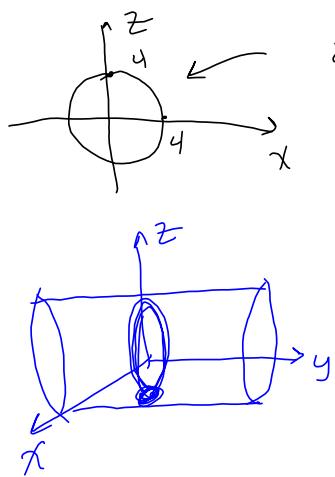
$$\boxed{r = \sqrt{4}}$$

4. What does $y = 6 - x$ represent in \mathbb{R}^3 ?

A degree one equation in x, y, z is a plane.

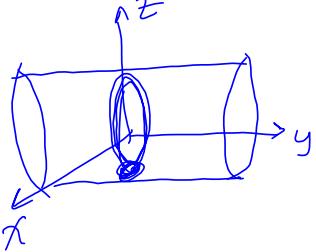


5. What does $x^2 + z^2 = 16$ represent in \mathbb{R}^3 ?



2-d circle of radius 4

cylinder centered
along y-axis radius 4.



6. Given $\mathbf{a} = \langle -7, 1, 2 \rangle$ and $\mathbf{b} = \langle 5, -1, 1 \rangle$, find a unit vector in the direction of $\mathbf{a} + 2\mathbf{b}$.

$$\vec{a} + 2\vec{b} = \langle -7, 1, 2 \rangle + 2 \langle 5, -1, 1 \rangle$$

$$= \langle -7, 1, 2 \rangle + \langle 10, -2, 2 \rangle$$

$$\overrightarrow{a+2b} = \langle 3, -1, 4 \rangle$$

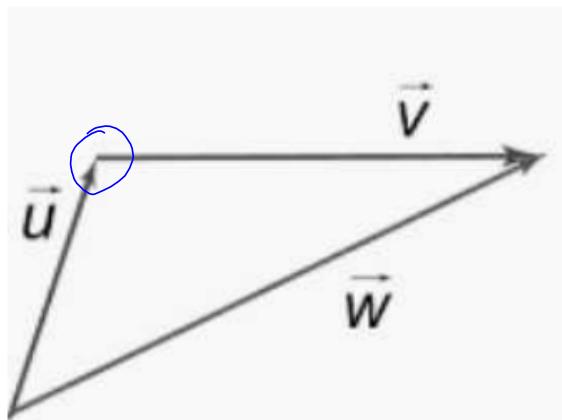
$$\overrightarrow{u} = \frac{\overrightarrow{a+2b}}{\| \overrightarrow{a+2b} \|} = \frac{\langle 3, -1, 4 \rangle}{\| \langle 3, -1, 4 \rangle \|}$$

$$= \frac{\langle 3, -1, 4 \rangle}{\sqrt{9 + 1 + 16}}$$

$$= \frac{\langle 3, -1, 4 \rangle}{\sqrt{26}}$$

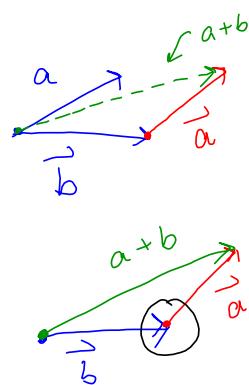
$$= \boxed{\left\langle \frac{3}{\sqrt{26}}, \frac{-1}{\sqrt{26}}, \frac{4}{\sqrt{26}} \right\rangle}$$

7. For the picture seen below, write \mathbf{v} in terms of \mathbf{u} and \mathbf{w} .



$$\mathbf{u} + \mathbf{v} = \mathbf{w}$$

$$\mathbf{v} = \mathbf{w} - \mathbf{u}$$



8. Compute $\mathbf{a} \cdot \mathbf{b}$ if

a.) $\mathbf{a} = \langle 4, 5, -1 \rangle$ and $\mathbf{b} = \langle 2, 1, 3 \rangle$.

$$\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\langle 4, 5, -1 \rangle \cdot \langle 2, 1, 3 \rangle = 8 + 5 + -3 \\ = 10$$

b.) $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and $\theta = 120^\circ$.

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos \theta \\ &= (2)(5) \cos 120^\circ \\ &= (2)(5)(-\frac{1}{2}) = \boxed{-5} \end{aligned}$$

~~$\approx 60^\circ$~~ $120^\circ = 180^\circ - 60^\circ$
 $\cos 60^\circ = \frac{1}{2}$
 $\cos 120^\circ = -\frac{1}{2}$

c.) $|\mathbf{a}| = 6$, $|\mathbf{b}| = 4$ and \mathbf{a} is perpendicular to \mathbf{b} .

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos \theta \\ &= (6)(4) \cos 90^\circ \\ &= 0 \end{aligned}$$

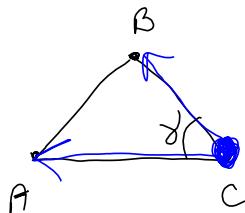
d.) $|\mathbf{a}| = 6$, $|\mathbf{b}| = 4$ and \mathbf{a} is parallel to \mathbf{b} .

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos 0^\circ \quad (0^\circ \text{ angle}) \quad \theta = 0^\circ \quad \rightarrow \\ \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos 180^\circ \quad \theta = 180^\circ \quad \leftarrow \rightarrow \end{aligned}$$

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos(180^\circ) \\ \mathbf{a} \cdot \mathbf{b} &= -24 \quad (180^\circ \text{ angle}) \end{aligned}$$

$$\boxed{\mathbf{a} \cdot \mathbf{b} = \pm 24}$$

9. The points $A(0, -1, 6)$, $B(2, 1, -3)$ and $C(5, 4, 2)$ form a triangle. Find $\angle C$.



Know: $a \cdot b = |a||b|\cos\theta$

$$\boxed{\frac{a \cdot b}{|a||b|} = \cos\theta}$$

\vec{CA} = "terminal - initial"
 $\vec{CA} = \langle -5, -5, 4 \rangle$, $\vec{CB} = \langle -3, -3, -5 \rangle$

$$\cos\gamma = \frac{\langle -5, -5, 4 \rangle \cdot \langle -3, -3, -5 \rangle}{\sqrt{25+25+16} \sqrt{9+9+25}}$$

$$= \frac{15 + 15 - 20}{\sqrt{160} \sqrt{43}}$$

$$\gamma = \arccos \left(\frac{10}{\sqrt{160} \sqrt{43}} \right)$$

γ = degree mode rounded to nearest degree.

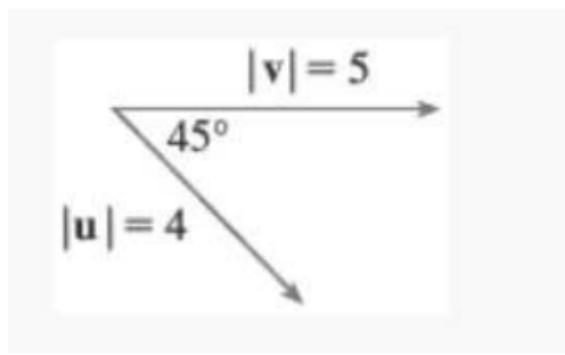
10. Find the cross product of $\langle 1, 1, 3 \rangle$ and $\langle -2, -1, -5 \rangle$.

$$\begin{aligned} \langle 1, 1, 3 \rangle \times \langle -2, -1, -5 \rangle &= \begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ -2 & -1 & -5 \end{vmatrix} \\ &= \langle -5 - (-3), -(-5 - (-6)), -1 - (-2) \rangle \\ &= \langle -2, -1, 1 \rangle \end{aligned}$$

$a \times b$ is perpendicular to both
 \overrightarrow{a} and \overrightarrow{b}

$$b \times a = -(a \times b)$$

11. Find $|\mathbf{u} \times \mathbf{v}|$ and determine if $\mathbf{u} \times \mathbf{v}$ points in or out of the page.



Theorem

$$\text{recall } \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$$

$$= (4)(5)\sin 45^\circ$$

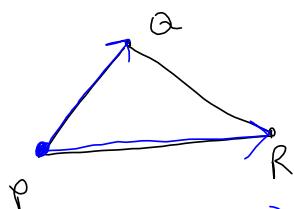
$$= (20) \frac{\sqrt{2}}{2} = 10\sqrt{2}$$

$\underline{\mathbf{u}} \times \underline{\mathbf{v}}$ points out of page

fingers of right hand along \mathbf{u}
so they curl thru θ towards \mathbf{v} .

12. Find a vector that is orthogonal to the plane that passes through the points $P(1, 0, 1)$, $Q(2, 3, 4)$ and $R(2, 1, 1)$.

Three points determine a plane



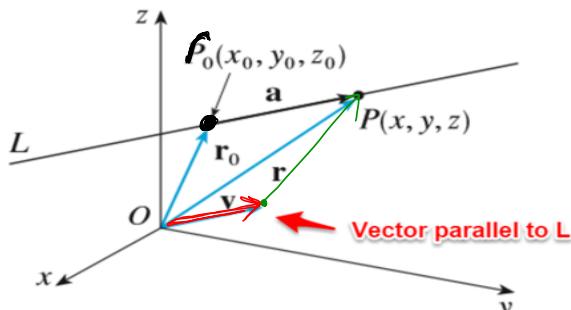
$\vec{PR} \times \vec{PQ}$ will be perpendicular to the plane

$$\vec{PR} = \langle 1, 1, 0 \rangle, \quad \vec{PQ} = \langle 1, 3, 3 \rangle$$

$$\vec{PR} \times \vec{PQ} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 3 & 3 \end{vmatrix} = \langle 3, -3, 2 \rangle$$

13. Find a vector equation of the line that passes through the point $(2, -5, 1)$ and is parallel to the vector $\langle 8, 10, -7 \rangle$.

recall in 2-dimension



scalar t such that $\mathbf{a} = t\mathbf{v}$

$$\text{Vector equation of line } L \quad \boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$

$$y = mx + b$$

m = slope

b = y -intercept

in space, the equation of a line L requires

- ① Point on line \mathbf{r}_0
- ② Any vector \mathbf{v} parallel to L

the vector equation of L
is $\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$

13. Find a vector equation of the line that passes through the point $(2, -5, 1)$ and is parallel to the vector $\langle 8, 10, -7 \rangle$.

$$\mathbf{r}_0 = (2, -5, 1)$$

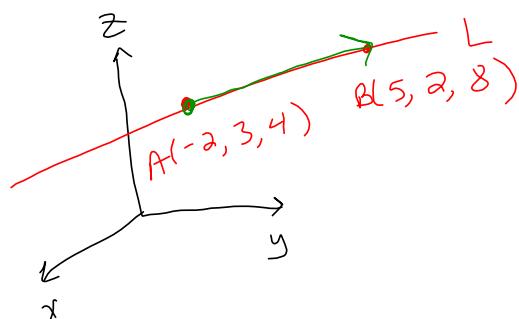
$$\mathbf{v} = \langle 8, 10, -7 \rangle$$

$$\begin{aligned} \mathbf{r}(t) &= \overrightarrow{\mathbf{r}_0} + t \overrightarrow{\mathbf{v}} \\ &= \langle 2, -5, 1 \rangle + t \langle 8, 10, -7 \rangle \\ &= \langle 2, -5, 1 \rangle + \langle 8t, 10t, -7t \rangle \end{aligned}$$

vector equation
of line

$$\mathbf{r}(t) = \langle 2 + 8t, -5 + 10t, 1 - 7t \rangle$$

14. Find parametric equations and a symmetric equations for the line passing through the points $(-2, 3, 4)$ and $(5, 2, 8)$.



r_0 = any point on L

$$r_0 = (-2, 3, 4) \text{ or } (5, 2, 8)$$

$$r_0 = (-2, 3, 4)$$

Equation of line

$$\begin{aligned} r &= \vec{r}_0 + t \vec{v} \\ &= \langle -2, 3, 4 \rangle + t \langle 7, -1, 4 \rangle \\ &= \langle -2 + 7t, 3 - t, 4 + 4t \rangle \end{aligned}$$

v = any vector parallel to L

$$\begin{array}{l} v = \overrightarrow{AB} \text{ or } \overrightarrow{BA} \\ v = \langle 7, -1, 4 \rangle \end{array}$$

parametric equations:

$$\begin{aligned} x &= -2 + 7t & t &= \frac{x+2}{7} \\ y &= 3 - t & t &= \frac{y-3}{-1} \\ z &= 4 + 4t & t &= \frac{z-4}{4} \end{aligned}$$

symmetric equations

$$\frac{x+2}{7} = \frac{y-3}{-1} = \frac{z-4}{4}$$

15. Do the lines $\frac{x-1}{2} = y = \frac{z-1}{4}$ and

$$x = \frac{y+2}{2} = \frac{z+2}{3}$$
 intersect?

If so, what is the point of intersection?

$$L_1: \frac{x-1}{2} = y = \frac{z-1}{4}$$

let $t = \text{parameter}$

$$\begin{aligned}\frac{x-1}{2} &= t \rightarrow x = 2t + 1 \\ y &= t \\ \frac{z-1}{4} &= t \rightarrow z = 4t + 1\end{aligned}$$

 L_1

$$L_2: x = \frac{y+2}{2} = \frac{z+2}{3}$$

they intersect if there
is a solution to

$$\textcircled{1} \quad 2t+1 = v$$

$$\textcircled{2} \quad t = 2v - 2$$

$$\textcircled{3} \quad 4t+1 = 3v-2$$

let $v = \text{parameter}$

$$\begin{aligned}x &= v \\ \frac{y+2}{2} &= v \\ \frac{z+2}{3} &= v\end{aligned}$$

 L_2

Look at $\textcircled{1}$ and $\textcircled{2}$

$$\begin{aligned}2t+1 &= v \\ t &= 2v - 2\end{aligned}$$

$$t = 2(2t+1) - 2$$

$$t = 4t$$

$$3t = 0$$

$$t = 0$$

see if $t=0$ and $v=1$
satisfies equation $\textcircled{3}$

$$\textcircled{3} \quad 4t+1 = 3v-2$$

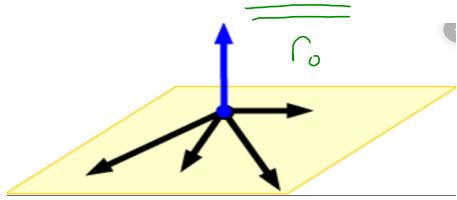
$$4(0)+1 \stackrel{?}{=} 3(1)-2$$

$$\begin{aligned}1 &= 1 \\ \text{true!} &\end{aligned}$$

intersection point
is $(1, 0, 1)$

$$v = 1$$

16. Find an equation of the plane passing through the point $(3, 4, 5)$ and perpendicular to $\langle -1, 2, 5 \rangle$.



plane is determined by
any point $r_0(x_0, y_0, z_0)$
and any vector \vec{n}
perpendicular to the plane.

equation of the plane is $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$$r = \langle x, y, z \rangle$$

$$n = \langle -1, 2, 5 \rangle$$

$$\vec{r}_0 = \langle 3, 4, 5 \rangle$$

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\langle -1, 2, 5 \rangle \cdot \langle x-3, y-4, z-5 \rangle = 0$$

$$-1(x-3) + 2(y-4) + 5(z-5) = 0$$

17. Find the equation of the plane that passes through the points $P(1, 0, 1)$, $Q(2, 3, 4)$ and $R(2, 1, 1)$.

point on plane $\boxed{r_0(1, 0, 1)}$

vector perpendicular to the plane

earlier in problem #12, we computed

$$\overrightarrow{PR} \times \overrightarrow{PQ} = \langle 3, -3, 2 \rangle$$

$$\boxed{\vec{n} = \langle 3, -3, 2 \rangle}$$

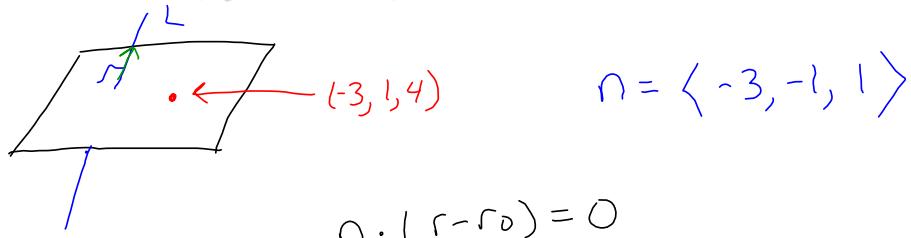
plane: $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$$\langle 3, -3, 2 \rangle \cdot \langle x-1, y-0, z-1 \rangle = 0$$

$$3(x-1) - 3(y-0) + 2(z-1) = 0$$

18. Find an equation of the plane passing through the point $(-3, 1, 4)$ and is perpendicular to the line

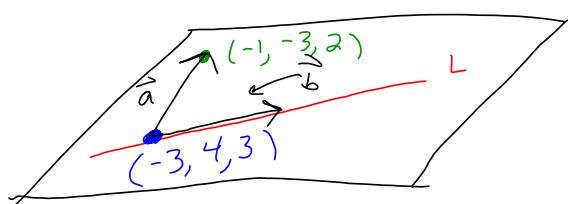
$x = 2 - 3t, y = 3 - t, z = t.$ \leftarrow direction vector is $\langle -3, -1, 1 \rangle$



$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

$$\langle -3, -1, 1 \rangle \cdot \langle x + 3, y - 1, z - 4 \rangle = 0$$

19. Find an equation of the plane passing through the point $(-1, -3, 2)$ that contains the line $x = -1 - 2t$, $y = 4t$, $z = 2 + t$.



$$\vec{a} = \langle 2, -3, -1 \rangle$$

$$\vec{b} = \langle -2, 4, 1 \rangle$$

to find a plane

$$P_0(-1, -3, 2)$$

vector perpendicular to
the plane

$$\vec{n} = \langle 2, -3, -1 \rangle \times \langle -2, 4, 1 \rangle$$

20. Consider the planes $z = x + y$ and $2x - 5y - z = 1$.

a.) Find the angle between the planes.

b.) Find the line of intersection of the planes.

a.) angle between two planes is the angle between their normal vectors.

$$n_1 = \langle 1, 1, -1 \rangle \quad \cos \theta = \frac{\langle 1, 1, -1 \rangle \cdot \langle 2, -5, -1 \rangle}{\sqrt{3} \sqrt{30}}$$

$$n_2 = \langle 2, -5, -1 \rangle$$

$$\theta = \arccos \left(\frac{-2}{\sqrt{3} \sqrt{30}} \right)$$

b) line of intersection, we need:

① a point

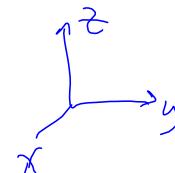
② a vector parallel to the line

$$z = x + y, \quad 2x - 5y - z = 1$$

$$y=0 \text{ in both equations}$$

$$y=0 \quad \begin{aligned} z &= x+y \rightarrow z=x \\ 2x - 5y - z &= 1 \rightarrow 2x - z = 1 \\ 2x - x &= 1 \end{aligned}$$

$$r_0 (1, 0, 1)$$



$$x = 1$$

$$z = 1$$

$$v = n_1 \times n_2$$

$$= \langle 1, 1, -1 \rangle \times \langle 2, -5, -1 \rangle$$

$$= \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & -5 & -1 \end{vmatrix}$$

$$v = \langle -6, -1, -7 \rangle$$

Line

$$r_0 + t v$$

$$\boxed{\langle 1, 0, 1 \rangle + t \langle -6, -1, -7 \rangle}$$

