

Spring 2020 Math 251

Week in Review 3

courtesy: Amy Austin

(covering sections 14.5, 14.6)

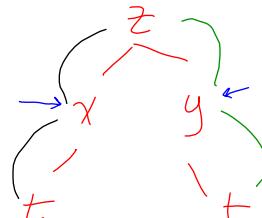
1. If $y = x^4$ and $x = \tan t$, find $\frac{dy}{dt}$.

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \frac{dx}{dt} \\ &= [4x^3] (\sec^2 t)\end{aligned}$$

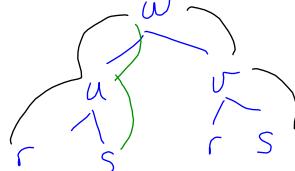
y
|
 x
|
 t

2. If $\underline{z = \ln(9x - 6y)}$, $x = \cos(e^t)$, $y = \sin^3(4t)$, find $\frac{dz}{dt}$.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= \frac{9}{9x-6y} (-\sin(e^t)e^t) + \frac{-6}{9x-6y} (3\sin^2(4t)\cos(4t)) \\ &= \frac{9}{9x-6y} (-e^t \sin(e^t)) - \frac{18\sin^2(4t)(\cos(4t))(4)}{9x-6y}\end{aligned}$$



3. If $w = u^2 + 2uv$, $u = r \ln s$, $v = 2r + s$, find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.



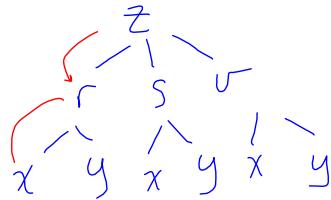
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial r} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial r}$$

$$\frac{\partial w}{\partial r} = (2u + 2v)(\ln s) + (2u)(2)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial s}$$

$$= (2u + 2v)\left(\frac{1}{s}\right) + (2u)(1)$$

4. If $z = r^3 + s + v^2$, $r = xe^y$, $s = ye^x$, $v = x^2y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial z}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = (3r^2)(e^y) + (1)(ye^x) + (2v)(2x)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial z}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = (3r^2)(xe^y) + (1)(e^x) + (2v)(x^2)$$

5. If $z = x^4 + xy^3$, $x = uv^3 + w^4$, $y = u + ve^w$, find

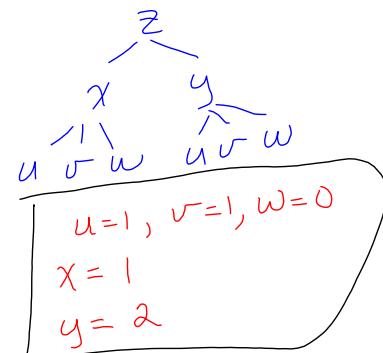
find $\frac{\partial z}{\partial u}$ when $u = 1$, $v = 1$, $w = 0$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$= (4x^3 + y^3)(v^3) + (3xy^2)(1)$$

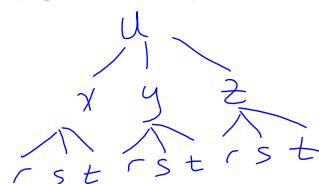
$$\frac{\partial z}{\partial u} = (4(1)^3 + 2^3)(1^3) + 3(1)(4)$$

$$= 4 + 8 + 12 = \boxed{24}$$



6. If $u = x^4y + y^2z^3$, $x = 2rs + 3st$, $y = rs^2t$,

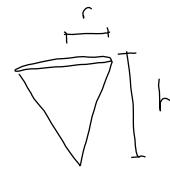
$z = r^2e^{2t}$, find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}$$

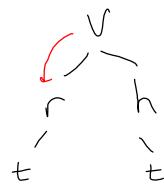
$$= (4x^3y)(2r + 3t) + (x^4 + 2y^2z^2)(2s^2t) + (3y^2z^2)(0)$$

7. The height and radius of a right circular cone are changing with respect to time. If the base radius of the cone is increasing at a rate of $\frac{1}{4}$ inches per minute while the height is decreasing at a rate $\frac{1}{10}$ inches per minute, [find the rate in which the volume if the cone is changing when the radius of the cone is 2 inches and the height of the cone is 1 inch.



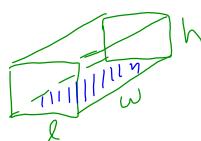
$$V(r, h) = \frac{1}{3} \pi r^2 h$$

given $\frac{dr}{dt} = \frac{1}{4}$ find $\frac{dV}{dt} \Big|_{r=2, h=1}$
 $\frac{dh}{dt} = -\frac{1}{10}$



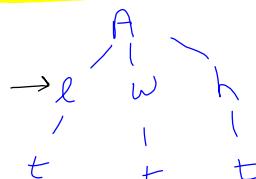
$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} \\ &= \frac{2}{3} \pi r h \frac{dr}{dt} + \frac{1}{3} \pi r^2 \frac{dh}{dt} \\ \frac{dV}{dt} &= \frac{2}{3} \pi (2)(1) \left(\frac{1}{4}\right) + \frac{1}{3} \pi (4) \left(-\frac{1}{10}\right) \\ \frac{dV}{dt} &= \frac{2\pi}{3} - \frac{4\pi}{30} \text{ in}^3/\text{min} \end{aligned}$$

8. The length l , width w and height h of a box change with time. [At a certain instant, the dimensions are $l = 1$ m, $w = 3$ m and $h = 2$ m, and l and w are increasing at rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that same instant, find the rate at which the surface area is changing.



these are not two's

$$A(l, w, h) = 2lw + 2wh + 2lh$$



when
 $l=1, w=3, h=2$
 $\frac{dh}{dt} = -3$
 $\frac{dw}{dt} = 2$
 $dl/dt = 2$

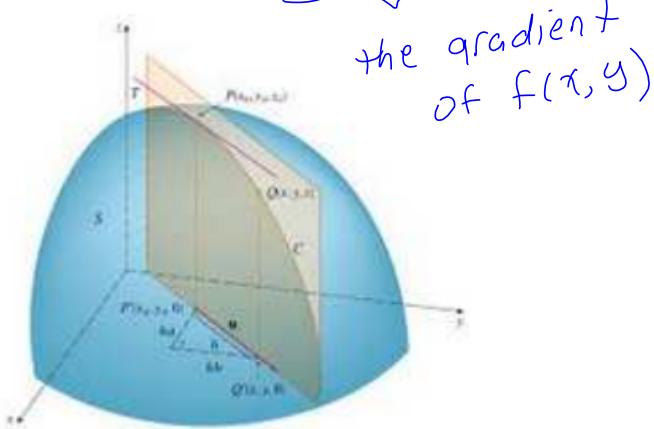
$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial l} \frac{dl}{dt} + \frac{\partial A}{\partial w} \frac{dw}{dt} + \frac{\partial A}{\partial h} \frac{dh}{dt} \\ &= (2w + 2h) \frac{dl}{dt} + (2l + 2h) \frac{dw}{dt} + (2w + 2l) \frac{dh}{dt} \\ &= [2(3) + 2(2)](2) + [(2(1) + 2(2))(2) + [2(3) + 2(1)](-3)] \end{aligned}$$

$$= 20 + 12 - 24$$

$$= 8 \text{ m}^2/\text{sec}$$

Definition: The **directional derivative** of

$z = f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is $D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \underbrace{\langle f_x(x, y), f_y(x, y) \rangle}_{\text{the gradient of } f(x, y)} \cdot \langle a, b \rangle$.



9. Given $f(x, y) = xy \sin x$, find the directional derivative at the point $\left(\frac{\pi}{2}, -1\right)$ in the direction

$$\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

$$f_x(x, y) = y \sin x + xy \cos x$$

$$f_y(x, y) = x \sin x$$

$$\begin{aligned} D_{\mathbf{u}} f\left(\frac{\pi}{2}, -1\right) &= \left\langle f_x\left(\frac{\pi}{2}, -1\right), f_y\left(\frac{\pi}{2}, -1\right) \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= \left\langle (-1)(1) + 0, \frac{\pi}{2} \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= \left\langle -1, \frac{\pi}{2} \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \boxed{\left[\frac{-3}{5} + \frac{4\pi}{10} \right]} \end{aligned}$$

10. Given $f(x, y) = x^3y^2$, find the directional derivative at the point $(-1, 2)$ in the direction $4\mathbf{i} - 3\mathbf{j}$.

$$\vec{u} = \frac{\langle 4, -3 \rangle}{\sqrt{16+9}} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$

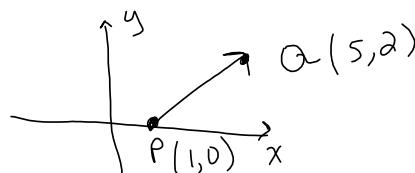
$$D_u f(-1, 2) = \langle f_x(-1, 2), f_y(-1, 2) \rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$

$$f_x(x, y) = 3x^2y^2 \quad f_x(-1, 2) = 12$$

$$f_y(x, y) = 2x^3y \quad f_y(-1, 2) = -4$$

$$D_u f(-1, 2) = \langle 12, -4 \rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle \\ = \frac{48}{5} + \frac{12}{5} = \frac{60}{5}$$

11. If $f(x, y) = x^2e^{xy}$, find the rate of change of f at the point $(1, 0)$ in the direction of the point $P(1, 0)$ to the point $Q(5, 2)$.



$$\overrightarrow{PQ} = \langle 4, 2 \rangle \quad \vec{u} = \frac{\langle 4, 2 \rangle}{\sqrt{20}} = \frac{\langle 4, 2 \rangle}{2\sqrt{5}} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$\vec{u} = \frac{\langle 4, 2 \rangle}{\sqrt{16+4}}$$

$$D_u f(1, 0) = \langle f_x(1, 0), f_y(1, 0) \rangle \cdot \vec{u}$$

$$f_x(x, y) = 2x e^{xy} + x^2 y e^{xy} \quad f_x(1, 0) = 2$$

$$f_y(x, y) = x^2 x e^{xy} = x^3 e^{xy} \quad f_y(1, 0) = 1$$

$$D_u f(1, 0) = \langle 2, 1 \rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = \frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}} \\ = 5/\sqrt{5}$$

max rate of change of $f(x,y)$ at the point (x_0, y_0) occurs in the direction of

$$\nabla f = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$$

12. Find the gradient of $f(x, y) = x^2 + y^3 - 4xy$ at the point $(1, -1)$.

$$\nabla f = \langle f_x(1, -1), f_y(1, -1) \rangle$$

$$f_x = 2x - 4y$$

$$f_x(1, -1) = 6$$

$$f_y = 3y^2 - 4x$$

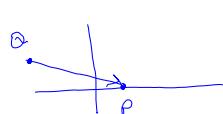
$$f_y(1, -1) = -1$$

13. If $f(x, y) = x^2 e^{-2y}$, $P(2, 0)$, $Q(-3, 1)$.

- a.) Find the directional derivative at Q in the direction of P .

- b.) Find a vector in the direction in which f increases most rapidly at P , and find the rate of change of f in that direction.

a) $D_u f(-3, 1) = \langle f_x(-3, 1), f_y(-3, 1) \rangle \cdot \vec{u}$



$$\vec{u} = \frac{\vec{QP}}{|\vec{QP}|} = \left\langle \frac{5}{\sqrt{26}}, \frac{-1}{\sqrt{26}} \right\rangle$$

$$f(x, y) = x^2 e^{-2y}$$

$$f_x = 2x e^{-2y}$$

$$f_y = -2x^2 e^{-2y}$$

$$\begin{aligned} f_x(-3, 1) &= -6e^{-2} \\ f_y(-3, 1) &= -18e^{-2} \end{aligned}$$

$$D_u f(-3, 1) = \langle -6e^{-2}, -18e^{-2} \rangle \cdot \left\langle \frac{5}{\sqrt{26}}, \frac{-1}{\sqrt{26}} \right\rangle$$

$$= \left[\frac{-30e^{-2}}{\sqrt{26}} + \frac{18e^{-2}}{\sqrt{26}} \right]$$

If $f(x, y) = x^2 e^{-2y}$, $P(2, 0)$, $Q(-3, 1)$.

- b.) Find a vector in the direction in which f increases most rapidly at P , and find the rate of change of f in that direction.

max rate of change occurs in the direction of the gradient.

$$\nabla f(2, 0) = \langle f_x(2, 0), f_y(2, 0) \rangle$$

$$f_x = 2x e^{-2y}$$

$$= \langle 4, -8 \rangle$$

$$f_y = -2x^2 e^{-2y}$$

The maximum rate of change at the point P is $|\nabla f(2, 0)|$

$$|\nabla f| = |\langle 4, -8 \rangle|$$

$$= \sqrt{16 + 64}$$

$$= \sqrt{80}$$

14. Find the maximum rate of change of

$f(x, y) = \sin^2(3x + 2y)$ at the point $\left(\frac{\pi}{6}, -\frac{\pi}{8}\right)$ and the direction in which it occurs.

direction of max rate of change is ∇f $\rightarrow_{3x+2y} \frac{\pi}{6} - \frac{\pi}{8} = \frac{\pi}{4}$

$$\nabla f = \left\langle f_x\left(\frac{\pi}{6}, -\frac{\pi}{8}\right), f_y\left(\frac{\pi}{6}, -\frac{\pi}{8}\right) \right\rangle$$

$$f_x = 2 \sin(3x+2y) \cos(3x+2y)(3)$$

$$\begin{aligned} f_x\left(\frac{\pi}{6}, -\frac{\pi}{8}\right) &= 2 \sin \frac{\pi}{4} \cos \frac{\pi}{4}(3) \\ &= 2 \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2}(3) \\ &= 3 \end{aligned}$$

$$\begin{aligned} f_y &= 2 \sin(3x+2y) \cos(3x+2y)(2) \\ f_y\left(\frac{\pi}{6}, -\frac{\pi}{8}\right) &= 2 \sin \frac{\pi}{4} \cos \frac{\pi}{4}(2) \\ &= 2 \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2}(2) \\ &= 2 \end{aligned}$$

$$\boxed{\nabla f = \langle 3, 2 \rangle} \quad \text{← direction of max rate of change}$$

$$\text{max rate of change } |\langle 3, 2 \rangle| = \sqrt{9+4} = \sqrt{13}$$

15. Find the equation of the tangent plane to the surface $f(x, y) = x^2 + y^2 - 4xy$ at the point $(1, 2)$.

$$\boxed{z - z_0 = f_x(1, 2)(x-1) + f_y(1, 2)(y-2)}$$

$$z_0 = f(1, 2) = 1 + 4 - 4(2) = 5 - 8 = -3$$

$$z + 3 = f_x(1, 2)(x-1) + f_y(1, 2)(y-2)$$

$$f_x = 2x - 4y, \text{ so } f_x(1, 2) = 2 - 8 = -6$$

$$f_y = 2y - 4x, \text{ so } f_y(1, 2) = 4 - 4 = 0$$

$$\boxed{z + 3 = -6(x-1)}$$

16. Find the equation of the tangent plane to the surface $\underbrace{xyz^3 = 8}$ at the point $(2, 2, 1)$.

A level surface to $w = xyz^3$

$$\vec{n} = \nabla F$$

$$F(x, y, z) = xyz^3$$

$$\vec{n} = \nabla F = \langle F_x, F_y, F_z \rangle = \langle yz^3, xz^3, 3xyz^2 \rangle$$

$$\text{at } (2, 2, 1), \quad \vec{n} = \nabla F(2, 2, 1) = \langle 2, 2, 12 \rangle$$

recall: A plane has equation $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$$\langle 2, 2, 12 \rangle \cdot \langle x-2, y-2, z-1 \rangle = 0$$

17. Find the equation of the tangent plane to the surface $xy + yz + zx = 5$ at the point $(1, 2, 1)$.

level surface to $F(x, y, z) = xy + yz + zx$

$$\vec{n} = \nabla F$$

$$\vec{n} = \langle y+z, x+z, y+x \rangle$$

$$\text{at } (1, 2, 1), \quad \vec{n} = \langle 3, 2, 3 \rangle$$

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \rightarrow \langle 3, 2, 3 \rangle \cdot \langle x-1, y-2, z-1 \rangle = 0$$