

Spring 2020 Math 251

Week in Review 4
courtesy: Amy Austin
(covering sections 15.1)

Partial Integration and Iterated Integrals: Suppose $z = f(x, y)$ is a function of two variables that is integrable over the rectangle $R = [a, b] \times [c, d]$. This means $a \leq x \leq b$ and $c \leq y \leq d$.

(a) We use the notation $\int_a^b f(x, y) dx$ to mean that y is held fixed and $f(x, y)$ is integrated with respect to x from $x = a$ to $x = b$. This is called the **partial integration of $f(x, y)$ with respect to x** .

(b) We use the notation $\int_c^d f(x, y) dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This is called the **partial integration of $f(x, y)$ with respect to y** .

(c) An **iterated integral** is an integral of the form $\int_c^d \int_a^b f(x, y) dx dy$ or $\int_a^b \int_c^d f(x, y) dy dx$.

$$1. \text{ Find } \int_0^{\pi/4} x \sin(3y) dy \quad \text{hold } x \text{ constant}$$

$$\begin{aligned} & u = 3y \quad u = 0, u = \frac{3\pi}{4} \\ & du = 3 \rightarrow \frac{du}{3} = dy \\ & \int_0^{\frac{3\pi}{4}} x \sin u du \\ & -\frac{1}{3} \left(x \cos u \right) \Big|_{u=0}^{u=\frac{3\pi}{4}} \\ & = -\frac{x}{3} \left[\cos \frac{3\pi}{4} - \cos 0 \right] \\ & = -\frac{x}{3} \left(-\frac{\sqrt{2}}{2} - 1 \right) \end{aligned}$$

$$2. \text{ Find } \int_1^e \frac{y \ln(x)}{x} dx = \int_1^e y \left(\frac{\ln x}{x} \right) dx$$

$$\begin{aligned} & u = \ln x \quad x = e, u = \ln e = 1 \\ & du = \frac{1}{x} dx \quad x = 1, u = \ln 1 = 0 \\ & du = \frac{dx}{x} \\ & = \int_0^1 y u du \\ & = y \frac{u^2}{2} \Big|_0^1 \\ & = \frac{y}{2} (1 - 0) = \boxed{\frac{y}{2}} \end{aligned}$$

$$3. \text{ Evaluate } \int_0^2 \int_0^3 (xy + x + y) dy dx \text{ and } \int_0^3 \int_0^2 (xy + x + y) dx dy$$

$$\begin{aligned} & \int_0^2 \left[xy + \frac{x^2}{2} + y^2 \right] \Big|_{y=0}^{y=3} dx \\ & \quad \text{dx} \rightarrow \left(\frac{9}{4}x^2 + \frac{3}{2}x^2 + \frac{9}{2}x \right) \Big|_0^2 \\ & \quad 9 + 6 + 9 = \boxed{24} \end{aligned}$$

$$\begin{aligned} & \int_0^3 \left[\frac{9x}{2} + 3x + \frac{9}{2} \right] dx \\ & \quad \text{dy} \rightarrow \int_0^3 (4y + 2) dy \\ & \quad = (2y^2 + 2y) \Big|_0^3 = \boxed{24} \end{aligned}$$

Fubini's Theorem: If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$(a) \iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

(b) In the case where $f(x, y) = g(x)h(y)$, then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d g(x)h(y) dy dx = \int_a^b g(x) dx \int_c^d h(y) dy$$

$$\begin{aligned} & \int_1^2 \int_3^4 xy dx dy = \left(\int_1^2 y dy \right) \left(\int_3^4 x dx \right) \\ & = \left(\frac{y^2}{2} \Big|_1^2 \right) \left(\frac{x^2}{2} \Big|_3^4 \right) \\ & = \left(\frac{1}{2}(4-1) \right) \left(\frac{1}{2}(16-9) \right) \end{aligned}$$

$$\begin{aligned}
 4. \int_{-3}^3 \left[\int_0^{\pi/2} (y + y^2 \cos x) dx dy \right] &= \int_{-3}^3 \left[yx + y^2 \sin x \right] \Big|_{x=0}^{x=\frac{\pi}{2}} dy \\
 &= \int_{-3}^3 \left[\frac{\pi}{2}y + y^2 \sin \frac{\pi}{2} - 0 \right] dy \\
 &= \left(\frac{\pi}{2} \frac{y^2}{2} + \frac{y^3}{3} \right) \Big|_{-3}^3 \\
 &= \frac{9\pi}{4} + 9 - \left(\frac{9\pi}{4} - 9 \right) = \boxed{18}
 \end{aligned}$$

5. Find $\iint_R \frac{x}{y^2} dA$, where $R = [0, 4] \times [1, 2]$

$$0 \leq x \leq 4$$

$$1 \leq y \leq 2$$

$$\begin{aligned}
 \frac{x}{y^2} &= x \left(\frac{1}{y^2} \right) \quad \underline{\text{Fubini}} \Big|_0 \\
 \iint_R \frac{x}{y^2} dA &\rightarrow \begin{array}{l} dx dy \\ \text{or} \\ dy dx \end{array} \quad \left(\int_0^4 x dx \right) \left(\int_1^2 \frac{1}{y^2} dy \right) \\
 &\quad \left(\frac{x^2}{2} \Big|_0^4 \right) \left(-\frac{1}{y} \Big|_1^2 \right) \\
 &\quad (8) \left(-\frac{1}{2} + 1 \right) = \boxed{4}
 \end{aligned}$$

6. Find $\int_0^2 \left[\int_0^1 \frac{(2x+3y)^3}{3y} dx dy \right]$

$$u = 2x + 3y \quad \begin{array}{l} x=1, u=2+3y \\ x=0, u=3y \end{array}$$

$$\begin{aligned}
 \frac{du}{dx} &= 2 \\
 \frac{du}{2} &= dx
 \end{aligned}$$

$$\int_0^2 \left[\frac{1}{2} \int_{3y}^{2+3y} u^3 du \right] dy = \frac{1}{2} \int_0^2 \left[\frac{u^4}{4} \Big|_{u=3y}^{u=2+3y} \right] dy$$

$$\int (2+3y)^4 dy \quad u^* = 2+3y$$

$$= \frac{1}{8} \int_0^2 \left[(2+3y)^4 - 81y^4 \right] dy$$

$$\frac{1}{3} \int (u^*)^4 du^* = \frac{1}{3} \frac{(u^*)^5}{5}$$

$$= \frac{1}{15} \left[(2+3y)^5 - 81y^5 \right] \Big|_0^2$$

$$= \frac{1}{15} (2+3y)^5$$

$$= \frac{1}{8} \left[\frac{8}{15} - \frac{81(3^2)}{5} \right]$$

7. Find $\iint_R e^{2x+y} dA$, where $R = [0, \ln 2] \times [0, \ln 3]$

$$\begin{aligned} e^{2x+y} &= \underline{e^{2x}} \cdot \underline{e^y} \quad \text{Fubini} \\ \iint_R e^{2x+y} dA &= \iint_R e^{2x} e^y dA \\ \int e^{2x} dx &= \frac{1}{2} e^{2x} \quad \text{ln}x \\ e^{2x} &= x \quad \text{ln}x \\ k \ln x &= \ln x \end{aligned}$$

$$\begin{aligned} &= \int_0^{\ln 2} e^{2x} dx \int_0^{\ln 3} e^y dy \\ &= \left(\frac{1}{2} e^{2x} \Big|_0^{\ln 2} \right) \left(e^y \Big|_0^{\ln 3} \right) \\ &= \frac{1}{2} \left(e^{2\ln 2} - 1 \right) \left(e^{\ln 3} - 1 \right) \\ &= \boxed{\frac{1}{2}(3)(2)} \end{aligned}$$

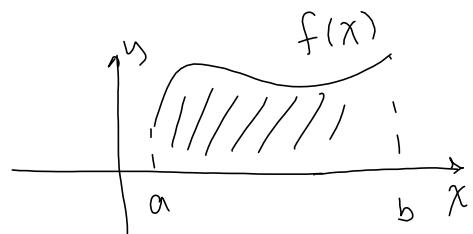
8. Find $\iint_R (y \cos(xy)) dA$, where $R = [0, 2] \times [0, \pi]$

$$\begin{aligned} \textcircled{1} \quad \iint_R y \cos(xy) dy dx &\quad \textcircled{2} \quad \iint_R y \cos(xy) dx dy \\ \text{Parts!} \uparrow & \\ \textcircled{2} \quad \int_0^\pi \left[\int_0^2 y \cos(xy) dx \right] dy & \quad u = xy \quad \begin{array}{l} x=2, u=2y \\ x=0, u=0 \end{array} \\ du & \quad \frac{du}{dx} = y \rightarrow du = y dx \\ \int_0^\pi \left[\int_0^{2y} \cos(u) du \right] dy & \quad \int_0^\pi \left[\sin(2y) - \sin(0) \right] dy \\ \left. \sin u \right|_{u=0}^{u=2y} dy & \quad -\frac{1}{2} \cos(2y) \Big|_0^\pi \\ & \quad -\frac{1}{2} [\cos(2\pi) - \cos(0)] \\ & = \boxed{0} \end{aligned}$$

9. Find $\iint_R x \sec^2 y dA$, where $R = \{(x, y) | 0 \leq x \leq 2, 1 \leq y \leq \frac{\pi}{4}\}$

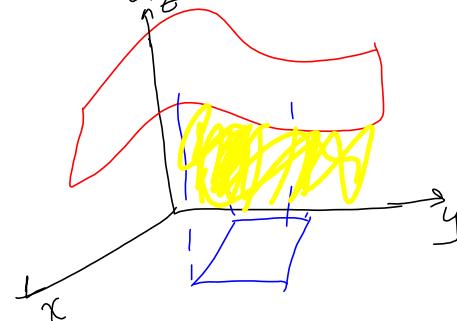
$$\begin{aligned} \text{Fubini} \quad & \left(\int_0^2 x dx \right) \left(\int_1^{\frac{\pi}{4}} \sec^2 y dy \right) \\ & \left(\frac{x^2}{2} \Big|_0^2 \right) \left(\tan y \Big|_1^{\frac{\pi}{4}} \right) \\ & (2) \left[\tan \frac{\pi}{4} - \tan(1) \right] = 2(1 - \tan(1)) \end{aligned}$$

10. Find the volume of the solid S that is bounded by the paraboloid $x^2 + y^2 + z = 16$, $z = 0$, $0 \leq x \leq 4$, $0 \leq y \leq 4$.



$$A = \int_a^b f(x) dx$$

If $f(x, y) \geq 0$ on \underline{R}
region in xy plane

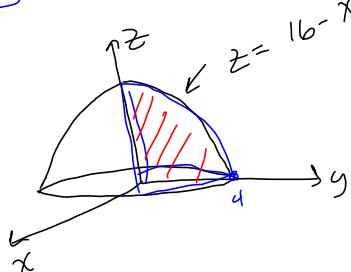


volume under $f(x, y)$ above the xy -plane

over the region R is

$$V = \iint_R f(x, y) dA$$

10. Find the volume of the solid S that is bounded by the paraboloid $x^2 + y^2 + z = 16$, $z = 0$, $0 \leq x \leq 4$, $0 \leq y \leq 4$.



$\underbrace{z = 16 - x^2 - y^2}_{xy \text{ plane}}$

$x\text{-int: } 0 = 16 - x^2$

$x = 4$

$$V = \iint_R (16 - x^2 - y^2) dA$$

$$= \int_0^4 \int_0^4 (16 - x^2 - y^2) dx dy$$

$$= \int_0^4 \left[16x - \frac{x^3}{3} - xy^2 \right] \Big|_{x=0}^{x=4} dy$$

$$= \int_0^4 \left[64 - \frac{64}{3} - 4y^2 \right] dy$$

$$= \left(64y - \frac{64}{3}y - \frac{4y^3}{3} \right) \Big|_0^4 = \frac{256}{3}$$

