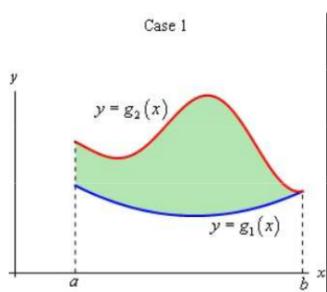


we will begin shortly!

$$\iint_D f(x, y) dA \rightarrow dy dx$$

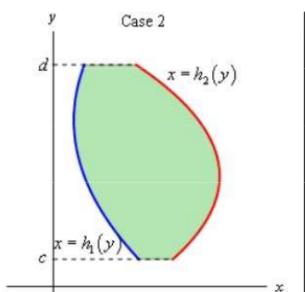
Type I: A plane region D is said to be of type I if it lies between the graphs of two continuous functions of x , that is $D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$.



If f is continuous on a type I region $D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

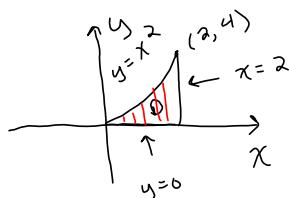
Type II: A plane region D is said to be of type II if it lies between the graphs of two continuous functions of y , that is $D = \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$.



If f is continuous on a type II region $D = \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$, then

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

1. Sketch the region of integration and evaluate $\iint_D xe^y dA$ where D is the region bounded by $y = 0$, $y = x^2$ and $x = 2$.



Type 1: $0 \leq x \leq 2$ "du dx"
 $0 \leq y \leq x^2$

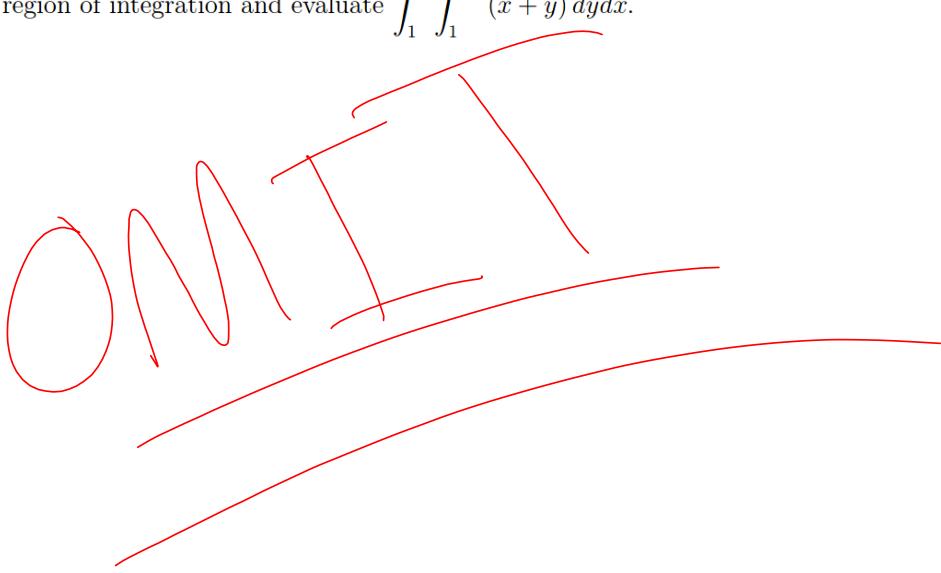
$$\iint_D xe^y dA = \int_0^2 \left(\int_0^{x^2} xe^y dy \right) dx$$

$$= \int_0^2 \left[xe^y \Big|_{y=0}^{y=x^2} \right] dx$$

$$\begin{aligned} & \int xe^x dx \\ & u = x^2 \quad du = 2x dx \\ & \frac{1}{2} \int e^u du = \frac{1}{2} e^u \\ & = \int_0^2 \left(xe^{x^2} - x e^0 \right) dx \\ & = \int_0^2 \left(\underline{xe^{x^2}} - x \right) dx \\ & = \left(\frac{1}{2} e^{x^2} - \frac{x^2}{2} \right) \Big|_0^2 \end{aligned}$$

$$\begin{aligned} & = \frac{1}{2} e^4 - 2 - \left(\frac{1}{2} \right) \\ & = \boxed{\frac{1}{2} e^4 - \frac{5}{2}} \end{aligned}$$

2. Sketch the region of integration and evaluate $\int_1^4 \int_1^{\sqrt{x}} (x + y) dy dx$.



3. Sketch the region of integration and evaluate $\int_0^1 \int_0^y (3 + x^2 y) dx dy$

$$0 \leq x \leq y \rightarrow x=0 \quad x=y \quad x=0 \rightarrow x=y$$

$$0 \leq y \leq 1$$

$$\int_0^1 \left(\int_0^y (3 + x^2 y) dx \right) dy$$

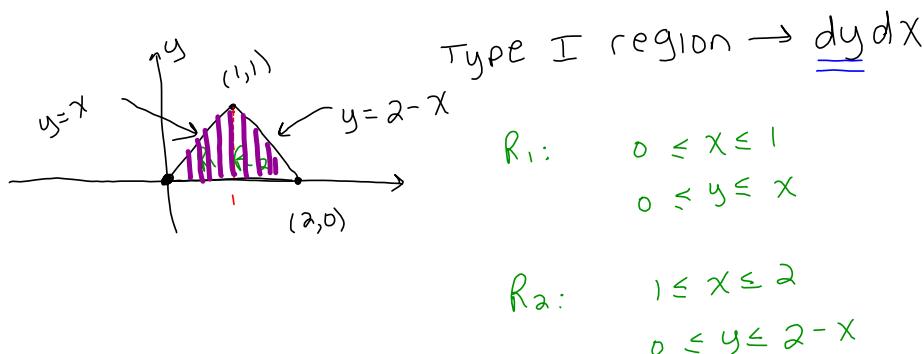
$$\int_0^1 \left(\left(3x + \frac{x^3}{3} y^3 \right) \Big|_{x=0}^{x=y} \right) dy$$

$$\int_0^1 \left(3y + \frac{1}{3} y^4 \right) dy$$

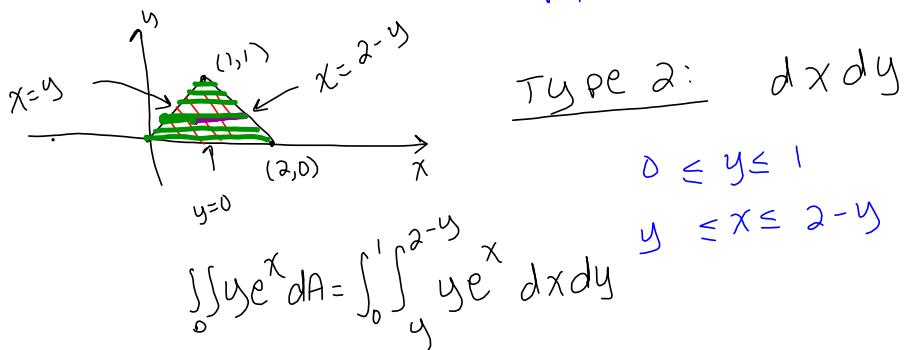
$$\frac{3y^2}{2} + \frac{1}{15} y^5 \Big|_0^1 = \frac{3}{2} + \frac{1}{15}$$

$$= \boxed{\frac{47}{30}}$$

4. Set up but do not evaluate both a type I and type II integral for $\iint_D ye^x dA$, where D is the triangular region with vertices $(0, 0)$, $(1, 1)$ and $(2, 0)$.



$$\iint_D ye^x dA = \underbrace{\int_0^1 \int_0^x ye^x dy dx}_{R_1} + \int_1^2 \int_0^{2-x} ye^x dy dx$$

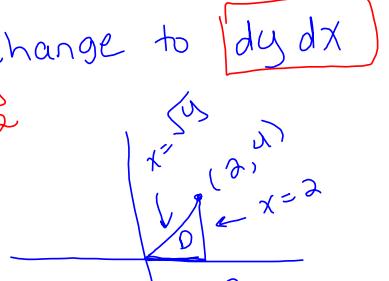


5. Sketch the region of integration and change the order of integration.

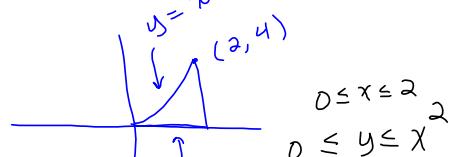
$$(i) \int_0^4 \int_{\sqrt{y}}^2 f(x, y) dx dy$$

\star $\boxed{\sqrt{y} \leq x \leq 2}$ \star

$\boxed{0 \leq y \leq 4}$



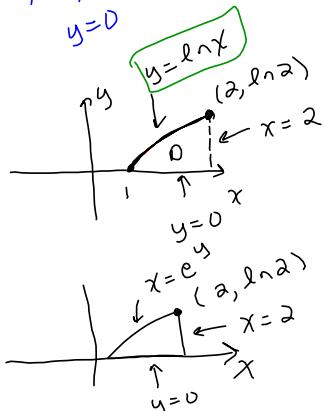
$$\iint_0^2 f(x, y) dy dx$$



$$(ii) \int_1^2 \int_0^{\ln x} f(x, y) dy dx \rightarrow \boxed{dx dy}$$

$$0 \leq y \leq \ln x$$

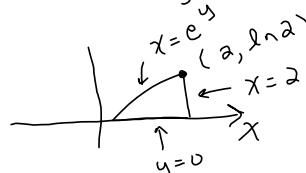
$$1 \leq x \leq 2$$



$$\int_0^{\ln 2} \int_{e^y}^2 f(x, y) dx dy$$

$$0 \leq y \leq \ln 2$$

$$e^y \leq x \leq 2$$



6. Set up but do not evaluate a double integral that gives the volume of the solid under the surface $z = xy$ and above the triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$

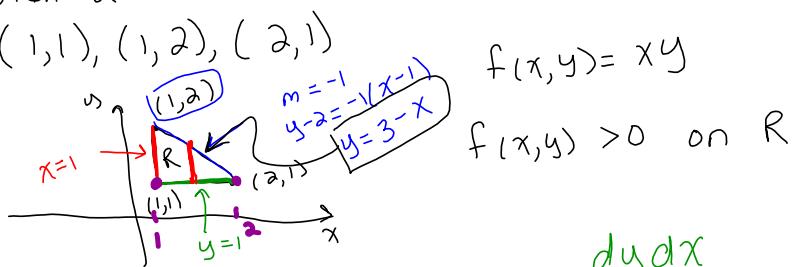
Recall: $f(x, y) \geq 0$ on R
 $(R \text{ in } xy \text{ plane})$

$$V = \iint_R f(x, y) dA$$



R : triangular region with vertices

$$(1, 1), (1, 2), (2, 1)$$



$$f(x, y) = xy$$

$$f(x, y) > 0 \text{ on } R$$

$$dy dx$$

$$\text{Volume} = \iint_R xy dA \quad 1 \leq x \leq 2 \\ 1 \leq y \leq 3-x$$

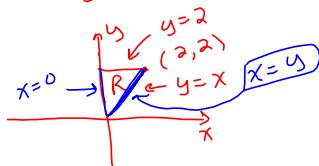
$$= \int_1^2 \int_1^{3-x} xy dy dx$$

7. Evaluate $\int_0^2 \int_x^2 e^{-y^2} dy dx$

reverse order
of integration!

$$\begin{cases} x \leq y \leq 2 \\ 0 \leq x \leq 2 \end{cases}$$

$$\begin{array}{l} y=x \\ y=2 \end{array}$$



$dx dy$

$$\int_0^2 \int_0^y e^{-y^2} dx dy$$

$$\begin{array}{l} 0 \leq y \leq 2 \\ 0 \leq x \leq y \end{array}$$

$$\int_0^2 \int_0^{x^y} e^{-y^2} (x) \Big|_{x=0}^{x=y} dy$$

$$\begin{array}{l} u = -y \\ du = -dy \end{array} \quad \begin{array}{l} y=2, u=-4 \\ y=0, u=0 \end{array}$$

$$\int_0^2 e^{-y^2} y dy$$

$$du = -2y dy$$

$$-\frac{1}{2} \int_0^{-4} e^u du$$

$$\begin{aligned} \frac{1}{2} \int_{-4}^0 e^u du &= \frac{1}{2} e^u \Big|_{-4}^0 \\ &= \left(\frac{1}{2} (1 - e^{-4}) \right) \end{aligned}$$

8. Evaluate $\int_0^2 \int_{y^2}^4 \sqrt{x} \sin x \, dx \, dy$

$$\int_0^2 \int_{y^2}^4 \sqrt{x} \sin x \, dx \, dy$$

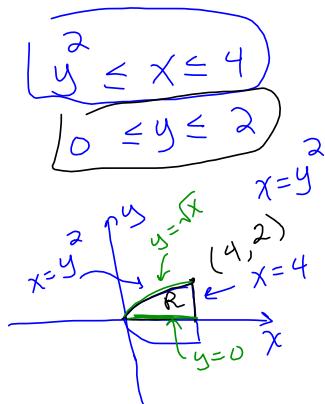
convert
to $dy \, dx$

$$\int_0^4 \left(\int_0^{\sqrt{x}} \sqrt{x} \sin x \, dy \right) dx$$

$$\int_0^4 \left[(\sqrt{x} \sin x) y \Big|_{y=0}^{y=\sqrt{x}} \right] dx$$

$$0 \leq x \leq 4$$

$$0 \leq y \leq \sqrt{x}$$



Parts!

$$\int_0^4 (\sqrt{x} \sin x) \sqrt{x} \, dx$$

$$\int_0^4 x \sin x \, dx$$

$$\int u \, dv = uv - \int v \, du$$

| \cancel{u} | \cancel{dv} |
|--------------|---------------|
| x | $\sin x$ |
| 1 | $-\cos x$ |
| 0 | $-\sin x$ |

$$u = x \quad dv = \sin x \, dx$$

$$du = dx \quad v = -\cos x$$

$$-x \cos x + \int \cos x \, dx$$

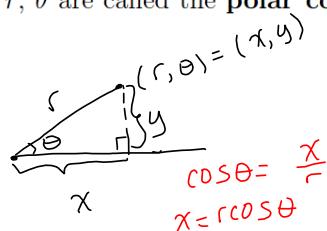
$$-x \cos x + \sin x$$

$$\int_0^4 x \sin x \, dx = (-x \cos x + \sin x) \Big|_0^4$$

$$= [-4 \cos 4 + \sin 4]$$

Recall: If $P(x, y)$ is a point in the xy -plane, we can represent the point P in polar form: Let r be the distance from O to P and let θ be the angle between the polar axis and the line OP . Then the point P is represented by the ordered pair (r, θ) , and r, θ are called the **polar coordinates** of P . Furthermore,

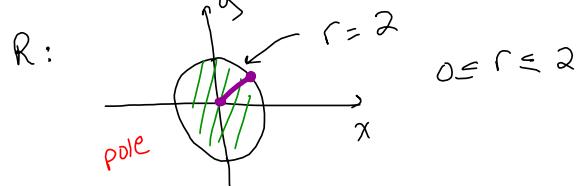
- a.) $x = r \cos(\theta), y = r \sin(\theta)$
- b.) $\tan(\theta) = \frac{y}{x}$, thus $\theta = \arctan\left(\frac{y}{x}\right)$
- c.) $x^2 + y^2 = r^2$



Change to Polar Coordinates in a Double Integral: If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\underbrace{\iint_R f(x, y) dA}_{R = \text{circle}} = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

9. Evaluate $\iint_R (x+2) dA$, where R is the region bounded by the circle $x^2 + y^2 = 4$.



Define R in polar!

r = distance from "pole"
to the polar curve

$$0 \leq r \leq 2$$

$$0 \leq \theta \leq 2\pi$$

integrand is $x+2 = r \cos \theta + 2$

$$\iint_R (x+2) dA = \int_0^{2\pi} \int_0^2 (r \cos \theta + 2) r dr d\theta \quad dA = r dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 (r^2 \cos \theta + 2r) dr d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^3}{3} \cos \theta + r^2 \right) \Big|_{r=0}^{r=2} d\theta$$

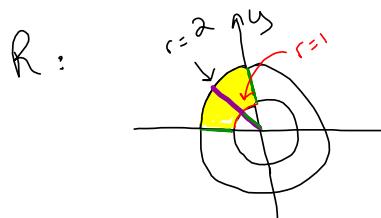
$$= \int_0^{2\pi} \left(\frac{8}{3} \cos \theta + 4 \right) d\theta$$

$$= \left(\frac{8}{3} \sin \theta + 4\theta \right) \Big|_0^{2\pi}$$

$$= \boxed{8\pi}$$

10. Evaluate $\iint_R 4y \, dA$, where R is the region in the second quadrant bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

and $y=0, \pi=0$



Polar!

$$1 \leq r \leq 2$$

$$\frac{\pi}{2} \leq \theta \leq \pi$$

$$\text{integrand } 4y = 4r \sin \theta$$

$$dA = r \, dr \, d\theta$$

Fubini!!

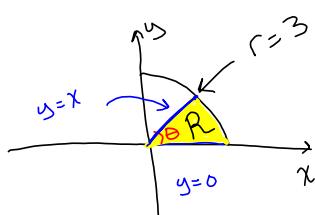
$$\int_{\frac{\pi}{2}}^{\pi} \int_1^2 (4r \sin \theta) r \, dr \, d\theta$$

$$\left(\int_{\frac{\pi}{2}}^{\pi} \sin \theta \, d\theta \right) \left(\int_1^2 4r^2 \, dr \right)$$

$$(-\cos \theta) \left[\int_{\frac{\pi}{2}}^{\pi} \frac{4}{3} r^3 \Big|_1^2 \right]$$

$$[-(-1)] \left[\frac{32}{3} \right] = \boxed{\frac{32}{3}}$$

11. Evaluate $\iint_R 3x^2 dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 9$ and the lines $y = 0$ and $y = x$.



define R in polar

$$0 \leq r \leq 3$$

$$0 \leq \theta \leq \frac{\pi}{4}$$

$$\text{integrand } 3x^2 = 3r^2 \cos^2 \theta$$

$$dA = r dr d\theta$$

$$\int_0^{\frac{\pi}{4}} \int_0^3 3r^2 \cos^2 \theta \, r \, dr \, d\theta$$

Fubini:

$$\int_0^{\frac{\pi}{4}} \cos^2 \theta \, d\theta \int_0^3 3r^3 \, dr$$

recall: $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

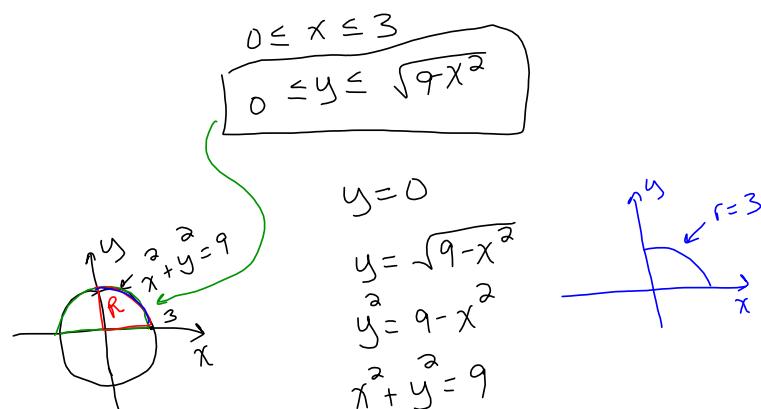
$$\int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 2\theta) \, d\theta \int_0^3 3r^3 \, dr$$

$$\frac{1}{2} \left(\theta + \frac{1}{2} \sin(2\theta) \right) \Big|_0^{\frac{\pi}{4}} \quad \frac{3r^4}{4} \Big|_0^3$$

$$\frac{1}{2} \left(\frac{\pi}{4} + \frac{1}{2} \sin \frac{\pi}{2} \right) \left(\frac{3^5}{4} \right)$$

$$\frac{1}{2} \left(\frac{\pi}{4} + \frac{1}{2} \right) \left(\frac{3^5}{4} \right)$$

12. Change $\int_0^3 \int_0^{\sqrt{9-x^2}} x^2 dy dx$ to a polar double integral. Do not evaluate



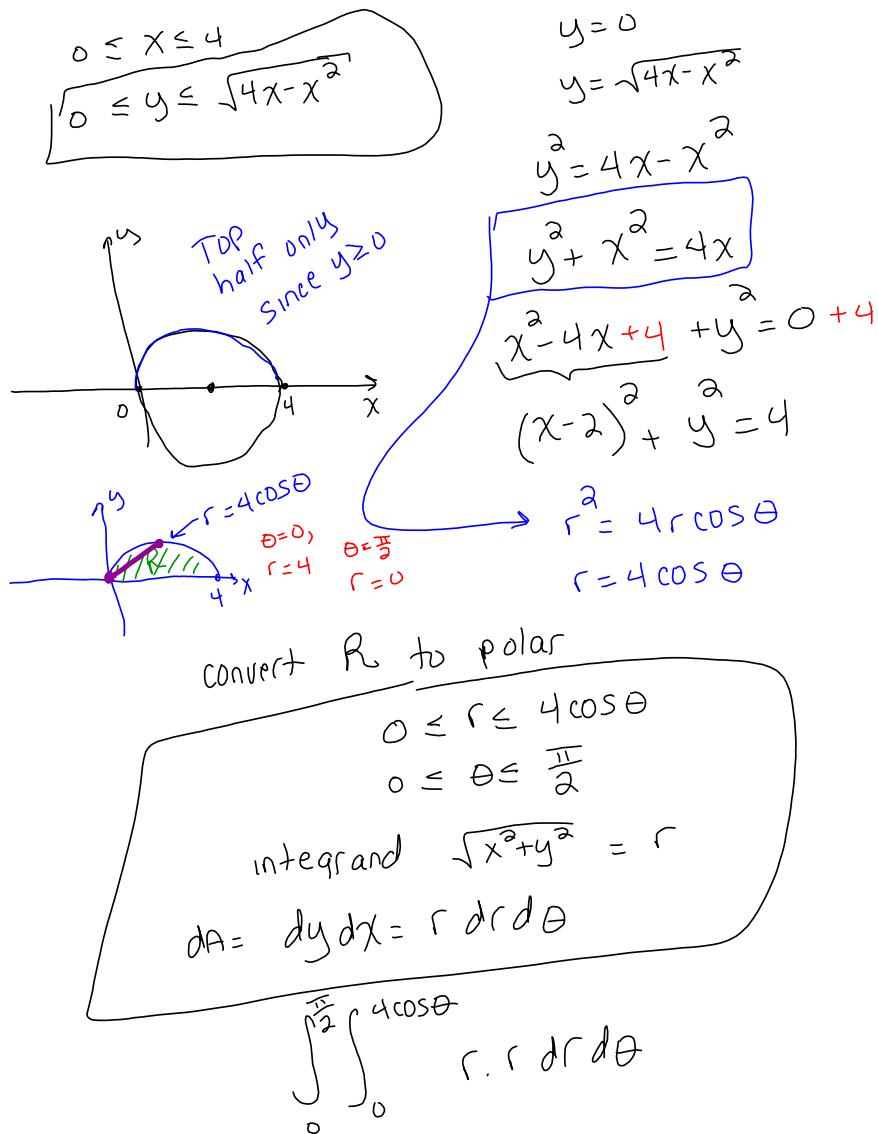
define R in polar :

$$\int_0^{\frac{\pi}{2}} \int_0^3 r^2 \cos^2 \theta r dr d\theta$$

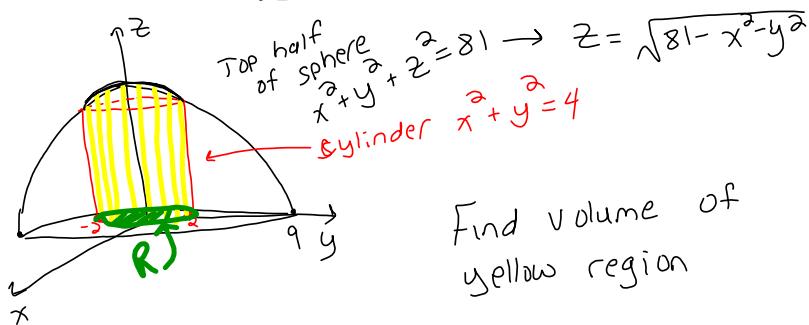
$0 \leq r \leq 3$
 $0 \leq \theta \leq \frac{\pi}{2}$

integrand: $x^2 = r^2 \cos^2 \theta$
 $dA = r dr d\theta$

13. Change $\int_0^4 \int_0^{\sqrt{4x-x^2}} \sqrt{x^2 + y^2} dy dx$ to a polar double integral. Do not evaluate



14. Set up but do not evaluate a double integral that gives the volume of the solid that lies above the xy -plane, below the sphere $x^2 + y^2 + z^2 = 81$ and inside the cylinder $x^2 + y^2 = 4$



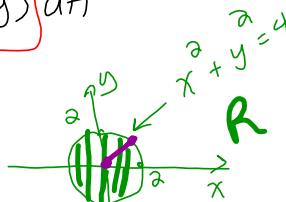
Find volume of
yellow region

If $f(x,y) \geq 0$,

Volume under $z = f(x,y)$ above the
region R in the xy plane

$$\text{is } V = \iint_R f(x,y) dA$$

For us, R is



$$V = \iint_R \sqrt{81 - x^2 - y^2} dA$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{81 - r^2} r dr d\theta$$

R in polar is:

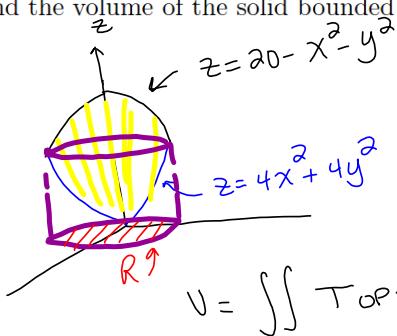
$$0 \leq r \leq 2$$

$$0 \leq \theta \leq 2\pi$$

$$\text{integrand } \sqrt{81 - r^2}$$

$$dA = r dr d\theta$$

15. Find the volume of the solid bounded by the paraboloids $z = 20 - x^2 - y^2$ and $z = 4x^2 + 4y^2$



Intersection:

$$20 - x^2 - y^2 = 4x^2 + 4y^2$$

$$20 = 5x^2 + 5y^2$$

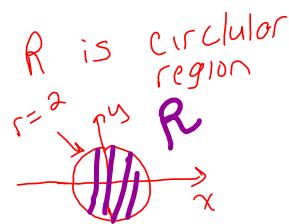
$$x^2 + y^2 = 4$$

$$V = \iint_R \text{Top} - \text{Bottom} \, dA$$

$$= \iint_R 20 - x^2 - y^2 - (4x^2 + 4y^2) \, dA$$

$$\iint_R (20 - 5x^2 - 5y^2) \, dA$$

$$20 - 5(x^2 + y^2) = 20 - 5r^2$$



convert to polar

$$0 \leq r \leq 2$$

$$0 \leq \theta \leq 2\pi$$

$$\text{integrand} = 20 - 5r^2$$

$$dA = r \, dr \, d\theta$$

$$V = \int_0^{2\pi} \int_0^2 (20 - 5r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 (20r - 5r^3) \, dr$$

$$= \theta \Big|_0^{2\pi} \left(10r^2 - \frac{5r^4}{4} \right) \Big|_0^2$$