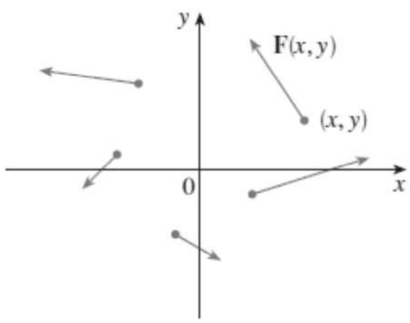
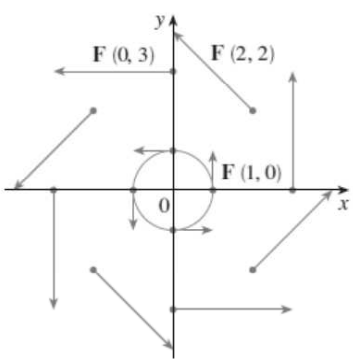
Spring 2020 Math 251

Week in Review 6
courtesy: Amy Austin
(covering sections 16.1-16.3)

Definition: A **vector field** in two dimension is a function **F** that assigns to each point (x, y) in $D \subset \mathbb{R}^2$ a two dimensional vector, $\mathbf{F}(x, y)$. In two dimension, the vector field lies entirely in the xy plane. A few vector fields in $\mathbb{R}^2$:

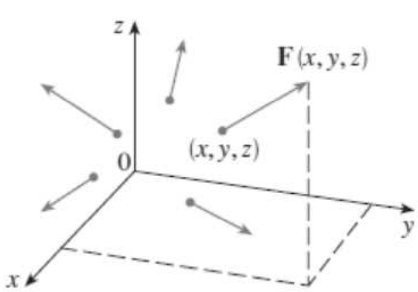




Definition: A **vector field** in three dimension is a function **F** that assigns to each point (x, y, z) in $D \subset \mathbb{R}^3$ a three dimensional vector, $\mathbf{F}(x, y, z)$.

In three dimension, the vector field is in space.

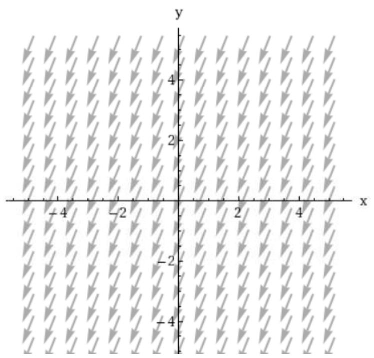
A vector field in $\mathbb{R}^3$:



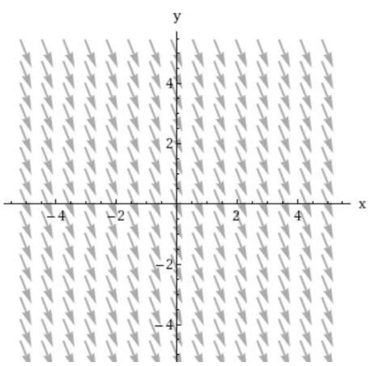
In order to match **F** with it's vector field, choose a several points, (x, y), in each quadrant, and look at the *direction* of $\mathbf{F}(x, y)$. Often times, it is a process of elimination.

1. Which of the following is the vector field for $\mathbf{F}(x, y) = \langle 2x, -7 \rangle$?

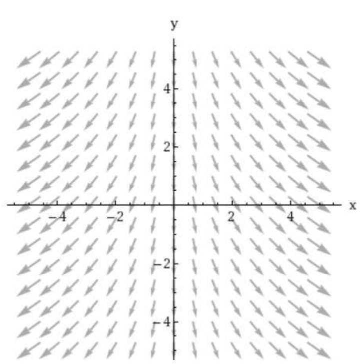
   a.)



   b.)

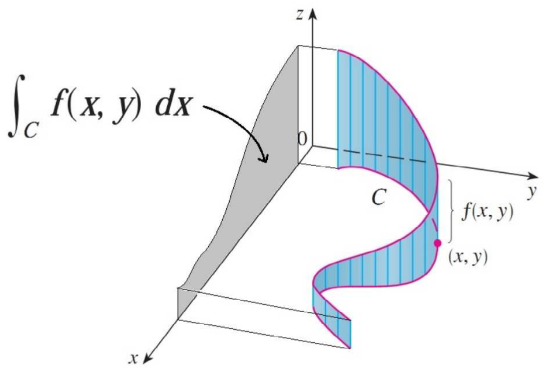


   c.)



Definition: If f is defined on a smooth curve C defined as $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then the **line integral of f along C** is

$$\int_C f(x,y)ds = \int_a^b (f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b (f(x(t),y(t))|\mathbf{r}'(t)|\, dt$$



In order to find a line integral along a curce C, we must first parameteterize the curve. Sometimes, the parameterization will be given explicitly, other times you must parameterize the curve.

2. Evaluate $\int_C (2x + y)ds$, where C is defined as $\mathbf{r}(t) = \langle 2 + t, 3 - t \rangle$, $0 \le t \le 1$.

3. Set up but do not evaluate $\int_C (2x + x^2y)ds$, where C is the arc of the curve $y = x^2$ from $(1, 1)$ to $(2, 4)$ **using two different parameterizations**.

4. Evaluate $\int_C (x^2 + y)\ ds$ where C consists of the line segment from the point $(1, 4)$ to $(3, -1)$.

5. Evaluate $\int_C (x + y)ds$, where C is the top half of the circle $x^2 + y^2 = 4$, oriented counterclockwise.

6. Set up but do not evaluate $\int_C (2 + x^2y)ds$, where C is the arc of the curve $x = y^2$ from $(1, -1)$ to $(4, 2)$ and then along the line segment from the point $(4, 2)$ to the point $(3, 7)$.

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ for $a \le t \le b$. The **line integral of f along C with respect to x** is $\int_C f(x,y)dx = \int_a^b (f(x(t),y(t))\ x'(t)\ dt$.

The **line integral of f along C with respect to y** is $\int_C f(x,y)dy = \int_a^b (f(x(t),y(t))\ y'(t)\ dt$

7. Evaluate $\int_C y\,dx + x^2 dy$, where C is decribed by $\mathbf{r}(t) = \langle 3e^t, e^{2t} \rangle$, $0 \le t \le 1$.

8. Evaluate $\int_C x\,dx + y\,dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.

9. Evaluate $\int_C (x+y)dz + (y-x)dy + z\,dx$ where C: $x = t^4$, $y = t^3$, $z = t^2$, $0 \le t \le 1$.

   Line Integrals over vector fields: Suppose now are moving a particle along a curve C through a vector (force) field, $\mathbf{F}$. We define the **line integral of F along C** to be $\int_c \mathbf{F} \cdot d\mathbf{r} = \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$

10. Find $\int_c \mathbf{F} \cdot d\mathbf{r}$, C: $\mathbf{r}(\mathbf{t}) = \langle t, t^2, t^4 \rangle$, $0 \le t \le 1$, and $\mathbf{F}(x, y, z) = \langle x, z^2, -4y \rangle$.

11. Find the work done by the force field $\mathbf{F}(x, y) = \langle x^2, xy \rangle$ in moving an object counterclockwise around the right half of the circle $x^2 + y^2 = 9$.

12. Suppose we are moving a particle from the point $(0, 0)$ to the point $(2, 4)$ in a force field $\mathbf{F}(x, y) = \langle y^2, x \rangle$. Find $\int_c \mathbf{F} \cdot d\mathbf{r}$ where:

    a.) The particle travels along the line segment from $(0, 0)$ to $(2, 4)$.

    b.) The particle travels along the curve $y = x^2$ from $(0, 0)$ to $(2, 4)$.

    Definition: If $\mathbf{F}$ is a continuous vector field, we say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if and only if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 with the same starting and ending points. In other words, the line integral is the same **no matter what path** you travel on as long as the endpoints are the same.

    Recall from chapter 14: The **gradient** of a function $f(x, y)$ is $\nabla f = \langle f_x(x, y), f_y(x, y) \rangle$. **Thus we can now think of the gradient as being a vector field**.

13. Find the gradient of $f(x, y) = \sqrt{x^2 + y^2}$.

    Definition: A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar funtion f, that is there exists a function f so that $\mathbf{F} = \nabla f$. We call f the **potential function**.

Recall the Fundamental Theorem of Calculus tells us that $\int\limits_a^b f'(x)dx = f(b) - f(a)$.

Since $\nabla f = \langle f_x, f_y \rangle$, we can think of the potential function, f, as some sort of antiderivative of ∇f. Hence

$$\int \mathbf{F} \cdot d\mathbf{r} = \int \nabla f \cdot d\mathbf{r}$$

Fundamental Theorem for Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let $\mathbf{F}$ be a conservative vector field. Let f be a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C. Then

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Note: **Line integrals of conservative vectors fields are independent of path** because in a **conservative** vector field, the line integral is computed by only using the **endpoints** (not the PATH). Therefore, if we are in a **conservative** vector field, the line integral along a curve C in that vector field will be the same **no matter what curve we travel across** that connects the endpoints together. **WHICH MEANS WE DON'T EVEN NEED TO PARAMETERIZE THE CURVE!**

14. Let $f(x,y) = 3x + x^2y - yx^2$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \nabla f$ and C is the curve given by $\mathbf{r}(t) = \langle 2t, t^2 \rangle$, $1 \leq t \leq 2$.

   Question: How do we determine if a vector field is conservative, and if so, how do we find the potential function? The 'test for conservative' we use depends on whether $\mathbf{F}$ is in $\mathbb{R}^2$ or $\mathbb{R}^3$.

   Theorem: $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = P\mathbf{i} + Q\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, if and only if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$. Note: This above criteria to determine if a vector field is conservative works only for $\mathbb{R}^2$.

15. Is $\mathbf{F}(x,y) = \langle 3x^2 - 4y, 4y^2 - 2x \rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

16. Is $\mathbf{F}(x,y) = \langle x + y, x - 2 \rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

17. Given $\mathbf{F}(x,y) = \langle 2xy^3, 3x^2y^2 \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \langle t^3 + 2t^2 - t, 3t^4 - t^2 \rangle$, $0 \leq t \leq 2$.

18. Let $\mathbf{F}(x, y) = \langle 3 + 2xy^2, 2x^2y \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the arc of the hyperbola $y = \frac{1}{x}$ from $(1, 1)$ to $\left(4, \frac{1}{4}\right)$.

19. Given $\mathbf{F}(x, y) = \langle 3x^2 - 4y, 4y^2 - 2x \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by

    $\mathbf{r}(t) = \langle t^2, t^2 + t - 2 \rangle$, $0 \leq t \leq 1$.

20. Given that $\mathbf{F} = \langle 4xe^z, \cos(y), 2x^2e^z \rangle$ is conservative and $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $0 \leq t \leq \frac{\pi}{2}$. Note: We had to tell you $\mathbf{F}$ is conservative since we have not yet learned the testing criteria for conservativness in $\mathbb{R}^3$