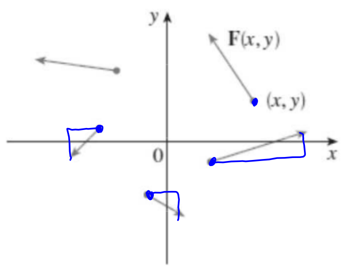
Spring 2020 Math 251

Week in Review 6

courtesy: Amy Austin

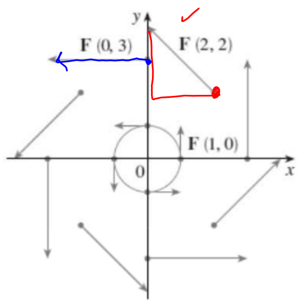
(covering sections 16.1-16.3)

Definition: A **vector field** in two dimension is a function **F** that assigns to each point (x, y) in $D \subset \mathbb{R}^2$ a two dimensional vector, $\mathbf{F}(x, y)$. In two dimension, the vector field lies entirely in the xy plane. A few vector fields in $\mathbb{R}^2$:



$$\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$$

$F(x,y) = \langle -y, x \rangle$



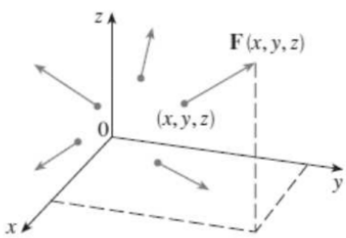
$F(2,2) = \langle -2, 2 \rangle$

$F(0,3) = \langle -3, 0 \rangle$

Definition: A **vector field** in three dimension is a function **F** that assigns to each point (x, y, z) in $D \subset \mathbb{R}^3$ a three dimensional vector, $\mathbf{F}(x, y, z)$.

In three dimension, the vector field is in space.

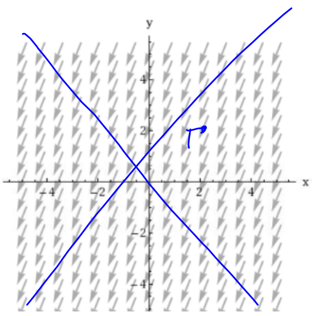
A vector field in $\mathbb{R}^3$:



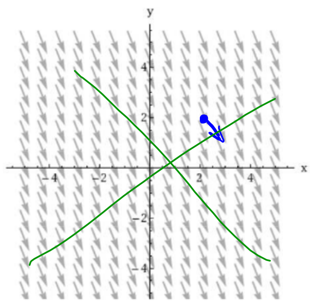
In order to match $\mathbf{F}$ with it's vector field, choose a several points, (x, y), in each quadrant, and look at the *direction* of $\mathbf{F}(x, y)$. Often times, it is a process of elimination.

1. Which of the following is the vector field for $\mathbf{F}(x, y) = \langle 2x, -7 \rangle$?

a.)

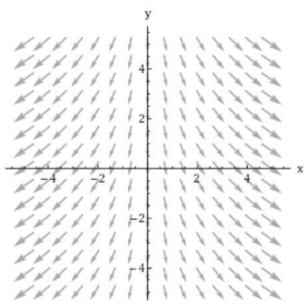


b.)



$F(2,2) = \langle 4, -7 \rangle$

c.)



$F\cdot(x,y) = \langle 2x, -7 \rangle$

$F(-2,2) = \langle -4, -7 \rangle$

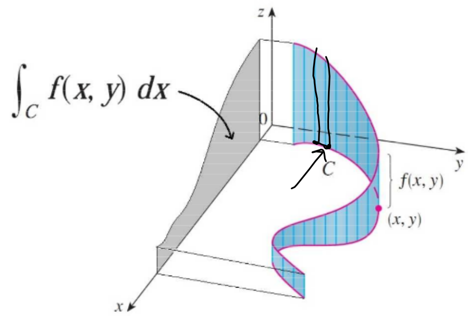
Definition: If f is defined on a smooth curve C defined as $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then the **line integral of f along C** is

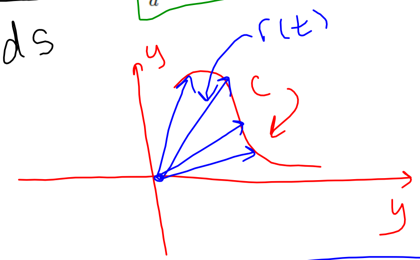
$f(x,y) =$ surface

$f(x,y)$ evaluated at $r(t)$

$$\int_C f(x,y)\,ds = \int_a^b (f(x(t),y(t))\underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}_{ds} = \boxed{\int_a^b (f(x(t),y(t))|\mathbf{r}'(t)|\,dt}$$

$ds =$ arc length differential





$$\boxed{\begin{gathered} r(t) = \langle x(t), y(t)\rangle \\ a \le t \le b \end{gathered}}$$

In order to find a line integral along a curce C, we must first parameteterize the curve. Sometimes, the parameterization will be given explicitly, other times you must parameterize the curve.

2. Evaluate $\int_C (2x + y)\,ds$, where C is defined as $\mathbf{r}(t) = \langle \overset{x}{2 + t}, \overset{y}{3 - t} \rangle$, $0 \le t \le 1$.

$$\int_C f(x,y)\,ds = \int_a^b f(r(t))\,|r'(t)|\,dt$$

$r'(t) = \langle 1, -1 \rangle$

$|r'(t)| = \sqrt{2}$

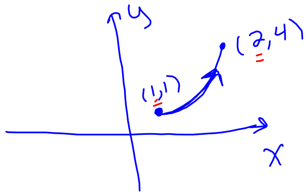
$$= \int_0^1 \underbrace{\big(2(2+t) + 3 - t\big)}_{f(r(t))} \underbrace{\sqrt{2}\,dt}_{|r'(t)|\,dt}$$

$$= \sqrt{2} \int_0^1 (4 + 2t + 3 - t)\,dt$$

$$= \sqrt{2} \int_0^1 (7 - t)\,dt = \ldots$$

3. Set up but do not evaluate $\int_C (2x + x^2 y)ds$, where C is the arc of the curve $\boxed{y = x^2}$ from $(1,1)$ to $(2,4)$ **using two different parameterizations**.

Parameterize $y = x^2$



① let $x = t$, $y = t^2$

$$\boxed{r(t) = \langle t, t^2 \rangle}, \quad 1 \le t \le 2$$

$r'(t) = \langle 1, 2t \rangle$, so $|r'(t)| = \sqrt{1 + 4t^2}$

$$\int_C (2x + x^2 y)\, ds = \int_1^2 \underbrace{(2t + t^2 t^2)}_{f(r(t))} \underbrace{\sqrt{1 + 4t^2}\, dt}_{|r'(t)|\, dt}$$

② $y = x^2$, from $(1,1)$ to $(2,4)$

$x = \sqrt{y}$

$y = t$, $x = \sqrt{t}$

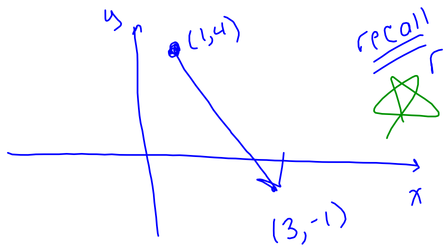
$$\boxed{r(t) = \langle \sqrt{t}, t \rangle}, \quad 1 \le t \le 4$$

$r'(t) = \left\langle \frac{1}{2\sqrt{t}}, 1 \right\rangle$

$$\int_C (2x + x^2 y)\, dS = \int_1^4 \underbrace{(2\sqrt{t} + t t)}_{f(r(t))} \underbrace{\sqrt{\frac{1}{4t} + 1}}_{r'(t)}\, dt$$

4. Evaluate $\int_C (x^2 + y)\, ds$ where C consists of the line segment from the point $(1, 4)$ to $(3, -1)$.

To parameterize a line



recall $r(t) = r_0 + tv$

$= \langle 1, 4 \rangle + t\langle 2, -5 \rangle, \quad 0 \le t \le 1$

$\boxed{r(t) = \langle 1 + 2t,\ 4 - 5t \rangle}$

starting at $(1,4)$

$1 + 2t = 1$

$\boxed{t = 0}$

ending at

$1 + 2t = 3$

$\boxed{t = 1}$

$r'(t) = \langle 2, -5 \rangle$

$|r'(t)| = \sqrt{4 + 25}$

$= \boxed{\sqrt{29}}$

$\int (x^2 + y)\, ds$

$\int_0^1 \left((1+2t)^2 + 4 - 5t\right)\sqrt{29}\, dt$

$\sqrt{29}\int_0^1 \left(1 + 4t + 4t^2 + 4 - 5t\right) = \sqrt{29}\int_0^1 \left(4t^2 - t + 5\right) dt$

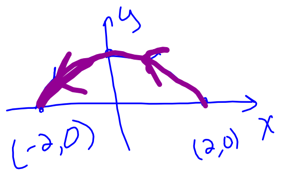
5. Evaluate $\int_C (x+y)\,ds$, where C is the top half of the circle $x^2+y^2=4$, oriented counterclockwise.

To parameterize a circle $x^2+y^2=r^2$

CCW $\quad x = r\cos\theta,\ y = r\sin\theta$

CW $\quad x = r\sin\theta,\ y = r\cos\theta$

$x^2+y^2=4 \qquad x = 2\cos\theta,\ y = 2\sin\theta$



$\theta = 0:\ (2,0)$

$\theta = \frac{\pi}{2}:\ (0,2)$

$\theta = \pi:\ (-2,0)$

$\boxed{r(\theta) = \langle 2\cos\theta, 2\sin\theta\rangle} \qquad 0 \le \theta \le \pi$

$r'(\theta) = \langle -2\sin\theta, 2\cos\theta\rangle$

$|r'(\theta)| = \sqrt{4\sin^2\theta + 4\cos^2\theta} = \sqrt{4} = 2$

$$\int_C (x+y)\,ds = \int_0^\pi \underbrace{(2\cos\theta + 2\sin\theta)}_{f(r(\theta))}\underbrace{2\,d\theta}_{|r'(\theta)|\,d\theta}$$

$$= 4\int_0^\pi (\cos\theta + \sin\theta)\,d\theta$$

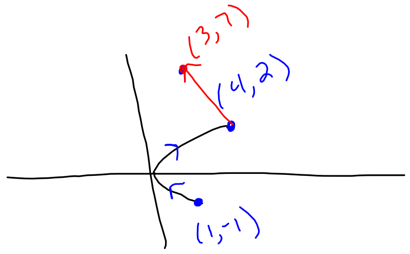
$$= 4\left(\sin\theta - \cos\theta\right)\Big|_0^\pi$$

$$= 4\left(-(-1) - (-1)\right)$$

$$= 4(2)$$

6. Set up but do not evaluate $\int_C (2 + x^2 y)ds$, where C is the arc of the curve $x = y^2$ from $(1, -1)$ to $(4, 2)$ and then along the line segment from the point $(4, 2)$ to the point $(3, 7)$.

C_1 $x = y^2$ from $(1,-1)$ to $(2,2)$



C_2: line segment from $\underbrace{(4,2)}_{r_0}$ to $(3,7)$

$$\int_C (2+x^2y)ds = \int_{C_1}(2+x^2y)ds + \int_{C_2}(2+x^2y)ds$$

C_1: $x = y^2$ from $(1,-1)$ to $(4,2)$

Parameterize $x = y^2$ $y = t$, $x = t^2$, $-1 \le t \le 2$

$$\boxed{\begin{array}{l} C_1 \quad r_1(t) = \langle t^2, t\rangle, \quad -1 \le t \le 2 \\ r_1'(t) = \langle 2t, 1\rangle, \quad \text{so } |r_1'(t)| = \sqrt{4t^2+1} \end{array}}$$

C_2: $r_2(t) = \vec{r}_0 + t\vec{v} = \langle 4,2\rangle + t\langle -1, 5\rangle$, $0 \le t \le 1$

$\boxed{r_2(t) = \langle 4-t, 2+5t\rangle}$, $r_1'(t) = \langle -1, 5\rangle$ $|r_1'(t)| = \sqrt{26}$

$$\int_C (2+x^2y)ds = \underbrace{\int_{-1}^{2}(2+t^4 t)\sqrt{4t^2+1}\,dt}_{\text{along } C_1} + \underbrace{\int_0^1 (2+(4-t)^2(2+5t))\sqrt{26}\,dt}_{\text{along } C_2}$$

Definition: Let C be a smooth curve defined by the parametric equations $x = x(t)$, $y = y(t)$ for $a \leq t \leq b$. The **line integral of f along** C **with respect to** x is $\int_C f(x,y)dx = \int_a^b (f(x(t), y(t))\ x'(t)\ dt$.

The **line integral of f along** C **with respect to** y is $\int_C f(x,y)dy = \int_a^b (f(x(t), y(t))\ y'(t)\ dt$

7. Evaluate $\int_C y dx + x^2 dy$, where C is decribed by $\mathbf{r}(t) = \langle 3e^t, e^{2t} \rangle$, $0 \leq t \leq 1$.

$x = 3e^t, \quad y = e^{2t}$

$dx = 3e^t dt, \quad dy = 2e^{2t} dt$

$$\int_0^1 (e^{2t})(3e^t dt) + (9e^{2t})(2e^{2t})\, dt$$

$$\int_0^1 \left(3e^{3t} + 18e^{4t}\right) dt = \left(3 \cdot \frac{1}{3} e^{3t} + 18 \frac{1}{4} e^{4t}\right)\Big|_0^1$$

$$= \left(e^{3t} + \frac{9}{2} e^{4t}\right)\Big|_0^1$$

$$= e^3 + \frac{9}{2} e^4 - \left(1 + \frac{9}{2}\right)$$

8. Evaluate $\int_C x dx + y dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(3, 1)$.

$y = t, \quad x = 4 - t^2, \quad -3 \leq t \leq 1$

$dy = dt, \quad dx = -2t\, dt$

$$\int_{-3}^1 (4 - t^2)(-2t\, dt) + (t)\, dt = \int_{-3}^1 \left(-8t + 2t^3 + t\right) dt = \cdots =$$

9. Evaluate $\int_C (x + y)dz + (y - x)dy + z dx$ where C: $x = t^4$, $y = t^3$, $z = t^2$, $0 \leq t \leq 1$.

$dx = 4t^3 dt, \quad dy = 3t^2 dt, \quad dz = 2t\, dt$

$$\int_0^1 (t^4 + t^3)(2t\, dt) + (t^3 - t^4)(3t^2 dt) + (t^2)\, 4t^3 dt$$

Before: $\int_c f(x,y)\,ds = \int_a^b f(r(t))\,|r'(t)|\,dt$

Line Integrals over vector fields: Suppose now are moving a particle along a curve C through a vector (force) field, $\mathbf{F}$. We define the **line integral of F along C** to be $\int_c \mathbf{F}\cdot d\mathbf{r} = \int_a^b (\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt$

dot product!

10. Find $\int_c \mathbf{F}\cdot d\mathbf{r}$, C: $\mathbf{r}(\mathbf{t}) = \langle t, t^2, t^4\rangle$, $0 \le t \le 1$, and $\mathbf{F}(x,y,z) = \langle x, z^2, -4y\rangle$.

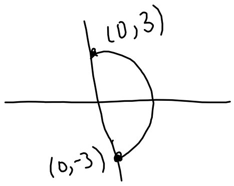
$$\int_c F\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt$$

$$\int_0^1 \langle t, t^8, -4t^2\rangle\cdot\langle 1, 2t, 4t^3\rangle\,dt$$

$$\int_0^1 (t + 2t^9 - 16t^5)\,dt = \dots$$

11. Find the work done by the force field $\mathbf{F}(x,y) = \langle x^2, xy\rangle$ in moving an object counterclockwise around the right half of the circle $x^2 + y^2 = 9$.

$$r(t) = \boxed{\langle 3\cos t, 3\sin t\rangle}$$



$$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

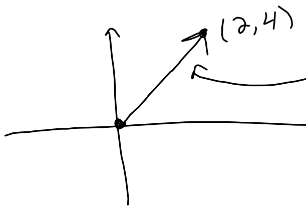
$$W = \int_c F\cdot dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overbrace{\boxed{\langle 9\cos^2 t, 9\cos t\sin t\rangle}}^{F(r(t))}\cdot\langle -3\sin t, 3\cos t\rangle\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cancel{-27\cos^2 t\sin t} + \cancel{27\cos^2 t\sin t})\,dt$$

$$= 0$$

12. Suppose we are moving a particle from the point $(0,0)$ to the point $(2,4)$ in a force field $\mathbf{F}(x,y) = \langle y^2, x\rangle$. Find $\displaystyle\int_c \mathbf{F}\cdot d\mathbf{r}$ where:

a.) The particle travels along the line segment from $(0,0)$ to $(2,4)$.



Path = line

$r(t) = \vec{r_0} + t\vec{v} = \langle 0,0\rangle + t\langle 2,4\rangle$

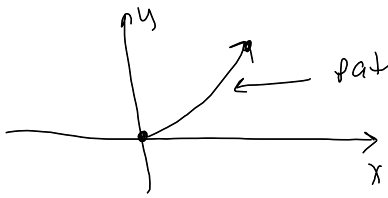
path → $r(t) = \langle 2t, 4t\rangle, \quad 0 \le t \le 1$

$\int_c F\cdot dr = \int_0^1 \langle 16t^2, 2t\rangle \cdot \langle 2,4\rangle = \int_0^1 32t^2 + 8t\,dt = \frac{32}{3} + 4$

$= \frac{32+12}{3}$

$= \frac{44}{3}$

b.) The particle travels along the curve $y = x^2$ from $(0,0)$ to $(2,4)$.



path = parabola

Path: $r(t) = \langle t, t^2\rangle$

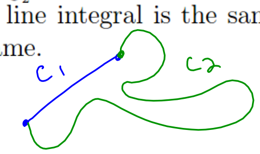
$0 \le t \le 2$

$\int F\cdot dr = \int_0^2 \langle t^4, t\rangle \cdot \langle 1, 2t\rangle\,dt$

$= \int_0^2 (t^4 + 2t^2)\,dt = \left.\frac{t^5}{5} + \frac{2t^3}{3}\right|_0^2$

$= \boxed{32/5 + 16/3}$

Definition: If $\mathbf{F}$ is a continuous vector field, we say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if and only if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 with the same starting and ending points. In other words, the line integral is the same **no matter what path** you travel on as long as the endpoints are the same.



Recall from chapter 14: The **gradient** of a function $f(x, y)$ is $\nabla f = \langle f_x(x, y), f_y(x, y)\rangle$. **Thus we can now think of the gradient as being a vector field.**

13. Find the gradient of $f(x, y) = \sqrt{x^2 + y^2}$.

$$\nabla f = \left\langle \underbrace{\frac{2x}{2\sqrt{x^2+y^2}}}_{f_x}, \underbrace{\frac{2y}{2\sqrt{x^2+y^2}}}_{f_y} \right\rangle$$

is a conservative vector field with potential function $f(x,y) = \sqrt{x^2+y^2}$

Definition: A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar funtion f, that is there exists a function f so that $\mathbf{F} = \nabla f$. We call f the **potential function**.

Recall the Fundamental Theorem of Calculus tells us that $\int_a^b f'(x)dx = f(b) - f(a)$.

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b$$
$$= f(b) - f(a)$$

Since $\nabla f = \langle f_x, f_y \rangle$, we can think of the potential function, f, as some sort of antiderivative of ∇f. Hence

$$\int \mathbf{F} \cdot d\mathbf{r} = \int \nabla f \cdot d\mathbf{r}$$

Fundamental Theorem for Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let $\mathbf{F}$ be a conservative vector field. Let f be a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C. Then

$$\int_c \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

endpoints of domain of r(t)

Note: **Line integrals of conservative vectors fields are independent of path** because in a **conservative** vector field, the line integral is computed by only using the **endpoints** (not the PATH). Therefore, if we are in a **conservative** vector field, the line integral along a curve C in that vector field will be the same **no matter what curve we travel across** that connects the endpoints together. **WHICH MEANS WE DON'T EVEN NEED TO PARAMETERIZE THE CURVE!**

14. Let $f(x, y) = 3x + x^2y - yx^2$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \nabla f$ and C is the curve given by $\boxed{\mathbf{r}(t) = \langle 2t, t^2 \rangle,\ 1 \le t \le 2.}$

$$\int_c F \cdot dr = \int_c \nabla f \cdot dr = f(r(b)) - f(r(a))$$

FTLI

$$= f(r(2)) - f(r(1))$$

$r(2) = \langle 4,4 \rangle$

$r(1) = \langle 2,1 \rangle$

$$= f(4,4) - f(2,1)$$

$f(x,y) = 3x + x^2y - yx^2$

- $f(4,4) = 12 + 64 - 64$

$f(4,4) = 12$

- $f(1,1) = 3 + 1 - 1$

$= 2$

$$= 12 - 2$$

$$= 10$$

Question: How do we determine if a vector field is conservative, and if so, how do we find the potential function? The 'test for conservative' we use depends on whether **F** is in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Theorem: $\boxed{\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle} = P\mathbf{i} + Q\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, if and only if $\boxed{\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}}$. Note: This above criteria to determine if a vector field is conservative works only for $\mathbb{R}^2$.

$F = \langle P, Q \rangle$

15. Is $\mathbf{F}(x,y) = \langle 3x^2 - 4y, 4y^2 - 2x\rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

$P = 3x^2 - 4y$, $\dfrac{\partial P}{\partial y} = -4$

$Q = 4y^2 - 2x$, $\dfrac{\partial Q}{\partial x} = -2$

no, not conservative!

16. Is $\mathbf{F}(x,y) = \langle \underset{P}{x+y}, \underset{Q}{x-2}\rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

$\dfrac{\partial P}{\partial y} = 1$, $\dfrac{\partial Q}{\partial x} = 1$

is conservative! How do you find f?

$\nabla f = \langle f_x, f_y \rangle = \langle x+y, x-2\rangle$

$\int (x+y)\,dx = \dfrac{x^2}{2} + xy + g(y)$

$\int (x-2)\,dy = xy - 2y + h(x)$

$f(x,y) = xy + \dfrac{x^2}{2} - 2y$

$\nabla f = \langle f_x, f_y \rangle = \langle y+x, x-2\rangle = F$!

17. Given $\mathbf{F}(x,y) = \langle \underset{P}{2xy^3}, \underset{Q}{3x^2y^2} \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by $\mathbf{r}(t) = \langle t^3 + 2t^2 - t, 3t^4 - t^2 \rangle$, $0 \le t \le 2$.

First check! Is F conservative?

$\frac{\partial P}{\partial y} = 6xy^2$, $\frac{\partial Q}{\partial x} = 6xy^2$ yes!

Find $f(x,y)$: $\int 2xy^3\,dx = x^2y^3 + g(y)$

$\int 3x^2y^2\,dy = x^2y^3 + h(x)$ $\quad 64 \; 3 \; 192$

$\boxed{f(x,y) = x^2y^3}$

$r(t) = \langle t^3 + 2t^2 - t, 3t^4 - t^2 \rangle$
$0 \le t \le 2$

so $\int_C F \cdot dr = \int_C \nabla f \cdot dr$

$= f(r(2)) - f(r(0))$

$= f(14, 188) - f(0,0)$

$= (14)^2 (188)^3$

$r(2) = \langle 8+8-2, 192-4 \rangle$
$= \langle 14, 188 \rangle$
$r(0) = \langle 0,0 \rangle$

18. Let $\mathbf{F}(x,y) = \langle 3 + \underset{P}{2xy^2}, \underset{Q}{2x^2y} \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the arc of the hyperbola $y = \frac{1}{x}$ from $(1,1)$ to $\left(4, \frac{1}{4}\right)$.

Path!

$\frac{\partial P}{\partial y} = 4xy, \quad \frac{\partial Q}{\partial x} = 4xy$:)

$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f\left(4, \frac{1}{4}\right) - f(1,1)$

find f: $\int (3 + 2xy^2)\, dx = 3x + x^2y^2 + g(y)$

$\int 2x^2y\, dy = x^2y^2 + h(x)$

$\boxed{f(x,y) = x^2y^2 + 3x}$

$= (16)\left(\frac{1}{16}\right) + 12 - (4)$

$= \boxed{9}$

19. Given $\mathbf{F}(x,y) = \langle \overset{P}{3x^2 - 4y}, \overset{Q}{4y^2 - 2x} \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by

path $\boxed{\mathbf{r}(t) = \langle t^2, t^2 + t - 2 \rangle,}\ 0 \le t \le 1$.

is it conservative? $\frac{\partial P}{\partial y} = -4 \quad \frac{\partial Q}{\partial x} = -2$

no! Path dependent

$$\int_C F \cdot dr = \int F(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^1 \langle 3(t^2)^2 - 4(t^2 + t - 2), 4(t^2 + t - 2)^2 - 2t^2 \rangle \cdot \langle 2t, 2t + 1 \rangle\, dt$$

$$= \ldots$$

20. Given that $\mathbf{F} = \langle \overset{f_x}{4xe^z}, \overset{f_y}{\cos(y)}, \overset{f_z}{2x^2e^z} \rangle$ is conservative and $\mathbf{r}(t) = \langle \overset{*}{\sin(t)}, t, \overset{*}{\cos(t)} \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $0 \le t \le \frac{\pi}{2}$. Note: We had to tell you $\mathbf{F}$ is conservative since we have not yet learned the testing criteria for conservativness in $\mathbb{R}^3$

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f\left(r\left(\tfrac{\pi}{2}\right)\right) - f(r(0))$$

$$= f\left(1, \tfrac{\pi}{2}, 0\right) - f(0, 0, 1)$$

$$\int 4xe^z\,dx = 2x^2e^z + \underline{g(y,z)}$$

$$\int \cos y\,dy = \underline{\sin y} + h(x,z)$$

$$\int 2x^2e^z\,dz = 2x^2e^z + \underline{k(x,y)}$$

$$f(x,y,z) = 2x^2e^z + \sin y$$

$$f\left(1, \tfrac{\pi}{2}, 0\right) = 2 + 1 = 3$$

$$f(0,0,1) = 0$$

$$= 3 - 0 = \boxed{3}$$