# Spring 2020 Math 251

**Week in Review 7**
*courtesy: Amy Austin*
(covering sections 16.4-16.9)

**Green's Theorem:** Let $C$ be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then

$$\oint_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

1. Evaluate $\oint_C y^2\, dx + x\, dy$ where $C$ is the triangular path from $(1,1)$ to $(3,1)$ to $(2,2)$ then back to $(1,1)$.

2. Evaluate $\oint_C y^2\, dx + x^2\, dy$ where $C$ is the boundary of the region bounded by the semicircle $y = \sqrt{4 - x^2}$ and the $x$ axis. Assume positive orientation.

3. Suppose a particle travels one revolution clockwise around the unit circle under the force field $\mathbf{F}(x, y) = \langle e^x - y^3, \cos(y) + x^3 \rangle$. Find the work done.

4. Find the divergence and curl of $\mathbf{F} = \langle xy, xz, xyz^2 \rangle$.

5. If $\mathbf{F} = \langle x,\ e^y \sin z,\ e^y \cos z \rangle$, Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle t^4, t, 2t^2 \rangle$, for $1 \leq t \leq 2$.

   **Definition:** If a smooth parametric surface $S$ is given by $\mathbf{r}(u, v)$, and $S$ is covered just once as $(u, v)$ ranges throughout the parametric domain $D$, then the **surface area** of $S$ is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

6. Find the surface area of the part of the surface $z = 2x^2 + y + 3$ that lies above the triangle with vertices $(0,0)$, $(2,0)$ and $(2,4)$

7. Find the surface area of the part of the plane $2x + 4y + z = 8$ that lies in the first octant.

8. Find the area of the part of the surface $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 2$.

Recall from spherical coordinates, we can parameterize a sphere as $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ and

$z = \rho \cos\phi$. Thus $r(\theta, \phi) = \langle \rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi \rangle$, and

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \left\langle \rho^2 \sin^2\phi \cos\theta, \rho^2 \sin^2\phi \sin\theta, \rho^2 \sin\phi \cos\phi \right\rangle$$

and

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \rho^2 \sin(\phi)$$

9. Find the surface area of the part of the sphere $x^2+y^2+z^2 = 16$ that lies above the cone $z = \sqrt{x^2+y^2}$

**Definition:** Suppose we want to integrate a function $f(x, y, z)$ over a surface $S$ defined by the equation $\mathbf{r}(u, v)$ and $S$ is covered just once as $(u, v)$ ranges throughout the parametric domain $D$, then the **surface integral of $f$ over $S$ is**

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

10. Evaluate $\iint_S (y + z)\, dS$ where $S$ is the part of the plane $x + y + z = 4$ that lies in the first octant.

11. Set up but do not evaluate $\iint_S \left(y^2 + z^2\right) dS$ where $S$ is part of the paraboloid $x = 4 - y^2 - z^2$ that lies in front of the plane $x = 0$

12. Evaluate $\iint_S \left(z + x^2 y\right) dS$ where $S$ is the part of the cylinder $y^2 + x^2 = 9$ in the first octant that lies between the planes $x = 0$ and $x = 4$.

13. Evaluate $\iint_S z\, dS$, where $S$ is the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$.

**Stokes' Theorem:** Let $S$ be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve $C$ with positive (counterclockwise) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\Re^3$ that contains $S$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

14. Use Stokes' Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle z^2, y^2, xy \rangle$ where $C$ is the boundary of the plane $2x + y + 2z = 2$ in the firsrt octant. Orient $C$ to be counterclockwise when looking from above.

15. Use Stokes' Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle z^2, 2x, y^2 \rangle$ and $C$ is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. Orient $C$ to be counterclockwise when looking from above (which ensures the normal vector points upward).

16. Use Stokes' Theorem to find $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle x^2 \sin z, y^2, xy \rangle$ and $S$ is the part of the paraboliod $z = 1 - x^2 - y^2$ that lies above the $xy$ plane, oriented upward.

A surface integral over a **closed surface** can be evaluated as a triple integral over the volume enclosed by the surface.

**Divergence Theorem** Let $E$ be a simple solid region whose boundary surface has positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains $E$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV$$

17. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle x + \sin z, 2y + \cos x, 3z + \tan y \rangle$ over the sphere $x^2 + y^2 + z^2 = 4$.

18. Let $S$ be the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the $xy$-plane. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle x^3, 2xz^2, 3y^2 z \rangle$.

19. Using the Divergence Theorem, find the flux of the vector field $\mathbf{F} = \langle z \cos y, x \sin z, xz \rangle$ where $S$ is the tetrahendron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 2$.