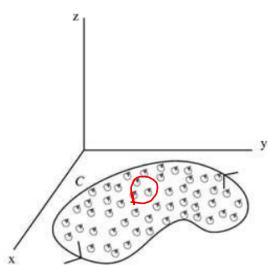


Spring 2020 Math 251

Week in Review 7
courtesy: Amy Austin
 (covering sections 16.4-16.9)

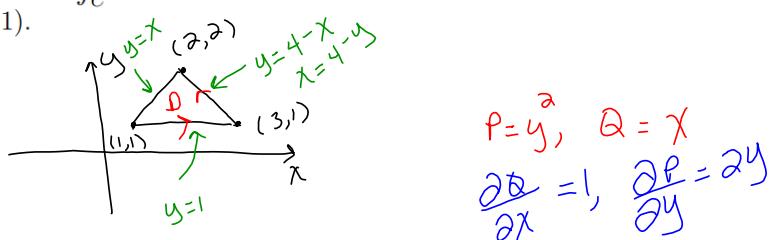
Green's Theorem: Let C be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then

$$\oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



C is the boundary
 curve of D

1. Evaluate $\oint_C y^2 dx + x dy$ where C is the triangular path from $(1, 1)$ to $(3, 1)$ to $(2, 2)$ then back to $(1, 1)$.



since C is closed,

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

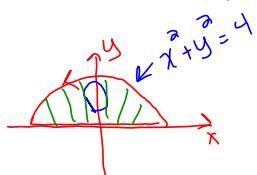
$$\begin{aligned} \text{D: } & 1 \leq y \leq 2 \\ & y \leq x \leq 4-y \end{aligned}$$

$$\begin{aligned} & = \iint_D (1-2y) dA \quad dA = dx dy \\ & = \int_1^2 \int_y^{4-y} (1-2y) dx dy \\ & = \boxed{-\frac{5}{3}} \end{aligned}$$

2. Evaluate $\oint_C y^2 dx + x^2 dy$ where C is the boundary of the region bounded by the semicircle $y = \sqrt{4 - x^2}$ and the x axis. Assume positive orientation. ccw

Green's Theorem: $\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

C = boundary curve of D



$$D: 0 \leq r \leq 2$$

$$0 \leq \theta \leq \pi$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$dA = r dr d\theta$$

$$\iint_D (2x - 2y) dA$$

$$\iint_D (2r \cos \theta - 2r \sin \theta) r dr d\theta$$

$$\int_0^\pi \int_0^2 (2r^2 \cos^2 \theta - 2r^2 \sin^2 \theta) dr d\theta$$

$$\boxed{-\frac{32}{3}}$$

3. Suppose a particle travels one revolution ^{*}clockwise around the unit circle under the force field $\mathbf{F}(x, y) = \langle e^x - y^3, \cos(y) + x^3 \rangle$. Find the work done.

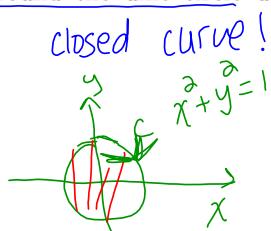
P Q

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$- \iint_D (3x^2 + 3y^2) dA$$

$$- \iint_0^{\pi} (3r^2) r dr d\theta$$

$$\boxed{-\frac{9\pi}{4}}$$



clockwise
means
negate
your answer!

D in polar is

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$x^2 + y^2 = r^2$$

$$dA = r dr d\theta$$

4. Find the divergence and curl of $\mathbf{F} = \langle xy, xz, xyz^2 \rangle$.

Del operator is $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

① Divergence of vector field is

$$\text{Div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \langle xy, xz, xyz^2 \rangle$$

$$\boxed{\text{Div}(\mathbf{F}) = y + 0 + 2xyz}$$

$$\text{② } \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & xyz^2 \end{vmatrix}$$

$$\boxed{\text{curl}(\mathbf{F}) = \langle xz^2 - x, -(yz^2 - 0), z - x \rangle}$$

5. If $\mathbf{F} = \langle x, e^y \sin z, e^y \cos z \rangle$, Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle t^4, t, 2t^2 \rangle$, for $1 \leq t \leq 2$.

If $\text{curl}(\mathbf{F}) = \vec{0}$ \mathbf{F} is conservative

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & e^y \sin z & e^y \cos z \end{vmatrix} = \langle e^y \cos z - e^y \cos z, 0, 0 \rangle = \langle 0, 0, 0 \rangle$$

\mathbf{F} is conservative

\mathbf{F} conservative means $\mathbf{F} = \nabla f$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} \quad f = \text{potential function}$$

$$\mathbf{F} = \langle x, e^y \sin z, e^y \cos z \rangle = \langle f_x, f_y, f_z \rangle$$

$$\text{To find } f, \int x \, dx = \frac{x^2}{2} + g(y, z)$$

$$\int e^y \sin z \, dy = e^y \sin z + h(x, z)$$

$$\int e^y \cos z \, dz = e^y \cos z + k(x, y)$$

$$\boxed{f(x, y, z) = \frac{x^2}{2} + e^y \sin z}$$

5. If $\mathbf{F} = \langle x, e^y \sin z, e^y \cos z \rangle$, Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle t^4, t, 2t^2 \rangle$, for $1 \leq t \leq 2$.

end points: $\mathbf{r}(2) + \mathbf{r}(1)$

$$\boxed{\begin{aligned} \mathbf{r}(2) &= (16, 2, 8) \\ \mathbf{r}(1) &= (1, 1, 2) \end{aligned}}$$

FTLI: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$

$$(f(x, y, z) = \frac{x^2}{2} + e^y \sin z) = f(\mathbf{r}(2)) - f(\mathbf{r}(1)) \quad f(\mathbf{r}(2)) = f(16, 2, 8)$$

$$= \frac{16^2}{2} + e^2 \sin(8) - \left(\frac{1}{2} + e \sin(2) \right) \quad f(\mathbf{r}(1)) = f(1, 1, 2)$$

Definition: If a smooth parametric surface S is given by $\mathbf{r}(u, v)$, and S is covered just once as (u, v) ranges throughout the parametric domain D , then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

D = domain of parameterization
of the surface

6. Find the surface area of the part of the surface $z = 2x^2 + y + 3$ that lies above the triangle with vertices $(0, 0)$, $(2, 0)$ and $(2, 4)$

let $\begin{cases} x = x \\ y = y \end{cases}$ $\mathbf{r}(x, y) = \langle x, y, 2x^2 + y + 3 \rangle$ where

$z = 2x^2 + y + 3$

$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 4x \\ 0 & 1 & 1 \end{vmatrix}$

$D: 0 \leq x \leq 2$
 $0 \leq y \leq 2x$

$A(S) = \iint_D |\mathbf{r}_x \times \mathbf{r}_y| dA = \langle -4x, -1, 1 \rangle$

$= \iint_D \sqrt{16x^2 + 2} dA$ need $dA = dy dx$ $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{16x^2 + 2}$

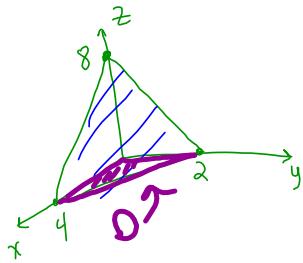
$= \int_0^2 \int_0^{2x} \sqrt{16x^2 + 2} dy dx$

$= \int_0^2 \sqrt{16x^2 + 2} y \Big|_{y=0}^{y=2x} dx$

$= \int_0^2 2x \sqrt{16x^2 + 2} dx$ u-sub!

$= \frac{1}{24} \left(66 - 2 \right)$

7. Find the surface area of the part of the plane $2x + 4y + z = 8$ that lies in the first octant.



Step 1 Parameterize the plane

$$z = 8 - 4y - 2x$$

$$\begin{aligned} x &= x \\ y &= y \\ z &= 8 - 4y - 2x \end{aligned}$$

$$r(x, y) = \langle x, y, 8 - 4y - 2x \rangle$$

where D:

$$dy/dx: \begin{aligned} 0 &\leq x \leq 4 \\ 0 &\leq y \leq 2 - \frac{1}{2}x \end{aligned}$$

$$r_x = \langle 1, 0, -2 \rangle$$

$$r_y = \langle 0, 1, -4 \rangle$$

$$dx/dy: \begin{aligned} 0 &\leq y \leq 2 \\ 0 &\leq x \leq 4 - 2y \end{aligned}$$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -4 \end{vmatrix}$$

$$r_x \times r_y = \langle 2, 4, 1 \rangle$$

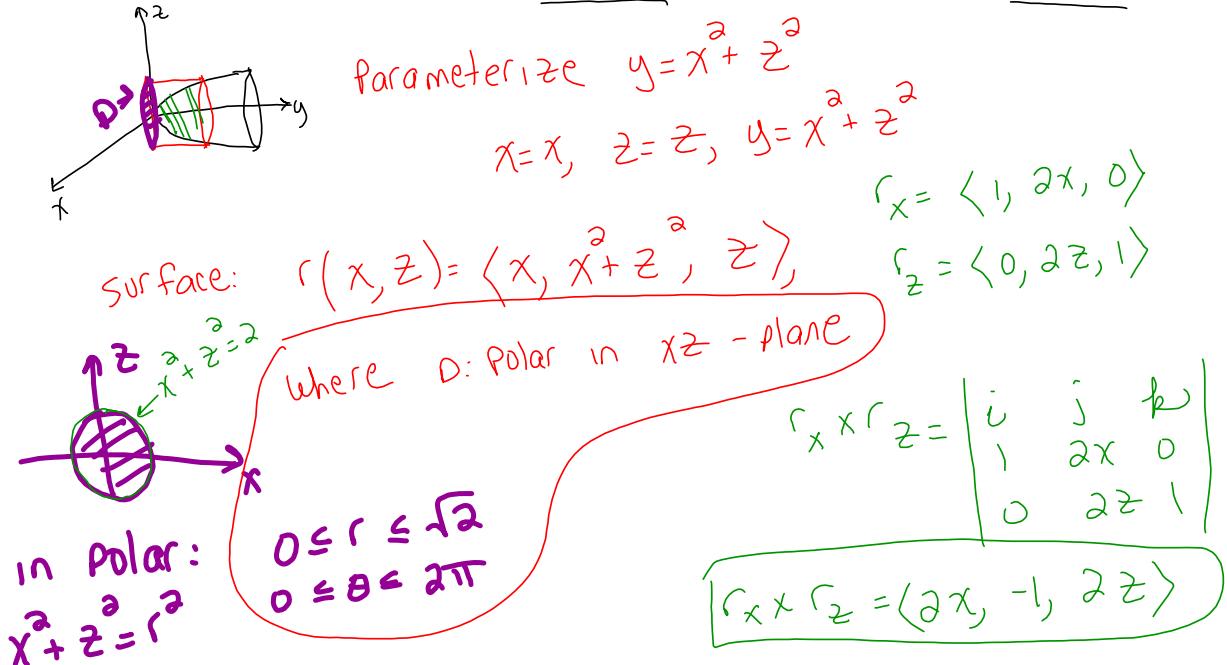
$$A(S) = \iint_D |r_x \times r_y| dA$$

$$= \iint_D \sqrt{4 + 16 + 1} dA$$

$$= \iint_D \sqrt{21} dA$$

$$= \iint_D 4 - 2y \sqrt{21} dA = 4\sqrt{21}$$

8. Find the area of the part of the surface $y = x^2 + z^2$ that lies inside the cylinder $x^2 + z^2 = 2$.



$$\begin{aligned} A(S) &= \iint_D |r_x \times r_z| dA \\ &= \iint_D \sqrt{4x^2 + 1 + 4z^2} dA \\ &= \iint_D \sqrt{4x^2 + 4z^2 + 1} dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1} r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r \sqrt{4r^2 + 1} dr \end{aligned}$$

Answer $\frac{13\pi}{3}$

Recall from spherical coordinates, we can parameterize a sphere as $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$ and

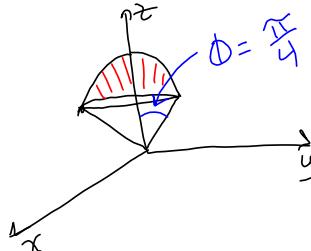
$z = \rho \cos \phi$. Thus $\mathbf{r}(\theta, \phi) = \langle \rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi \rangle$, and

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle \rho^2 \sin^2 \phi \cos \theta, \rho^2 \sin^2 \phi \sin \theta, \rho^2 \sin \phi \cos \phi \rangle$$

and

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \rho^2 \sin(\phi)$$

9. Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies above the cone $z = \sqrt{x^2 + y^2}$



$$A(S) = \iint |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \frac{\pi}{4}$$

$$A(S) = \iint_0^{\frac{\pi}{4}} 16 \sin \phi d\phi d\theta$$

$$16 \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \cos \phi d\phi$$

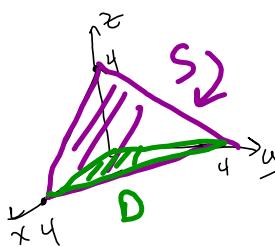
$$\boxed{32\pi \left(1 - \frac{\sqrt{2}}{2}\right)}$$

Definition: Suppose we want to integrate a function $f(x, y, z)$ over a surface S defined by the equation $\mathbf{r}(u, v)$ and S is covered just once as (u, v) ranges throughout the parametric domain D , then the **surface integral** of f over S is

$$\iint_S f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$$

10. Evaluate $\iint_S (y + z) dS$ where S is the part of the plane $x + y + z = 4$ that lies in the first octant.

Step 1 Parameterize S $\mathbf{r}(x, y) = \langle x, y, 4-x-y \rangle$



where $D: y = 4 - x$

$$\begin{aligned} & \iint_S f(x, y, z) dS \\ &= \iint_D f(\mathbf{r}(x, y)) \left| \mathbf{r}_x \times \mathbf{r}_y \right| dA \quad \mathbf{r}_x \times \mathbf{r}_y = \vec{n} \\ &= \iint_D (y + 4 - x - y) \left| \langle 1, 1, 1 \rangle \right| dA \\ &= \iint_D (4 - x) \sqrt{3} dA \quad 0 \leq x \leq 4 \\ &= \sqrt{3} \int_0^4 \int_0^{4-x} (4 - x) dy dx \quad 0 \leq y \leq 4 - x \\ &= \boxed{\frac{64}{\sqrt{3}}} \end{aligned}$$

11. Set up but do not evaluate $\iint_S (y^2 + z^2) dS$ where S is part of the paraboloid $x = 4 - y^2 - z^2$ that lies in front of the plane $x = 0$

Parameterize Surface:

$$\mathbf{r}(y, z) = \langle 4 - y^2 - z^2, y, z \rangle$$

D in yz -plane

$\mathbf{r}_y \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2y & 1 & 0 \\ -2z & 0 & 1 \end{vmatrix}$

D in polar

$0 \leq r \leq 2$

$0 \leq \theta \leq 2\pi$

$y^2 + z^2 = r^2$

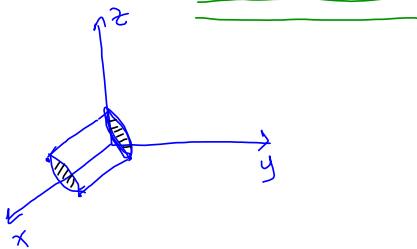
$dA = r dr d\theta$

$(\mathbf{r}_y \times \mathbf{r}_z) = \langle 1, 2y, 2z \rangle$, $|(\mathbf{r}_y \times \mathbf{r}_z)| = \sqrt{1 + 4y^2 + 4z^2}$

$\iint_S (y^2 + z^2) dS = \iint_D (y^2 + z^2) \sqrt{1 + 4y^2 + 4z^2} dA$

$= \int_0^{2\pi} \int_0^2 r^2 \sqrt{1 + 4r^2} r dr d\theta$

12. Evaluate $\iint_S (z + x^2y) dS$ where S is the part of the cylinder in the first octant that lies between the planes $x = 0$ and $x = 4$.



on a cylinder of radius 3

$$\begin{aligned} y &= 3\cos\theta \\ z &= 3\sin\theta \end{aligned}$$

$$y^2 + z^2 = 9$$

$$\begin{aligned} x &= x \\ y &= 3\cos\theta \\ z &= 3\sin\theta \end{aligned}$$

$$\begin{aligned} 0 &\leq \theta \leq \frac{\pi}{2} \\ 0 &\leq x \leq 4 \end{aligned}$$

$$r(x, \theta) = (x, 3\cos\theta, 3\sin\theta)$$

$$r_x \times r_\theta = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & -3\sin\theta & 3\cos\theta \end{vmatrix} = \langle 0, -3\cos\theta, -3\sin\theta \rangle$$

$$\begin{aligned} |r_x \times r_\theta| &= \sqrt{0 + 9\cos^2\theta + 9\sin^2\theta} \\ &= \sqrt{9} = 3 \end{aligned}$$

$$\iiint_S (z + x^2y) dS = \iint_D (3\sin\theta + x^2 \cdot 3\cos\theta) 3 dA$$

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \int_0^4 (3\sin\theta + x^2 \cdot 3\cos\theta) 3 dx d\theta \\ &= 3 \int_0^{\frac{\pi}{2}} \left[(3\sin\theta)x + x^3 \cos\theta \right] \Big|_{x=0}^{x=4} d\theta \\ &= 3 \int_0^{\frac{\pi}{2}} (12\sin\theta + 64\cos\theta) d\theta \end{aligned}$$

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13. Evaluate $\iint_S z \, dS$, where S is the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$.

SKIP!

Green's theorem: $\oint_C P \, dx + Q \, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$F = \langle P, Q \rangle$ $\oint_C F \cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

C = closed curve in xy plane

Stokes' Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive (counterclockwise) orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S .

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

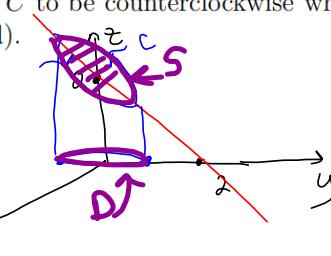
14. Use Stokes' Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle z^2, 2x, y^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. Orient C to be counterclockwise when looking from above (which ensures the normal vector points upward).

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot dS$$

Parameterize S : S is the part of the plane that is inside $x^2 + y^2 \leq 1$

$$z = 2 - y \quad \text{let } x = x, y = y, z = 2 - y$$

$$S: r(x, y) = \langle x, y, 2 - y \rangle, \quad x^2 + y^2 \leq 1$$



$C = \text{boundary of } S$

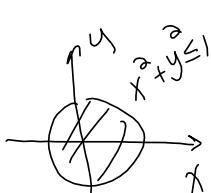
$$\text{Plane! } \vec{n} = \langle 0, 1, 1 \rangle$$

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 2x & y^2 \end{vmatrix} \end{aligned}$$

$$\iint_S \operatorname{curl} \mathbf{F} \cdot dS = \iint_D \operatorname{curl} \mathbf{F}(r(x, y)) \cdot (r_x \times r_y) dA$$

$$= \iint_{x^2 + y^2 \leq 1} \langle 2y, 2(z-y), 2 \rangle \cdot \langle 0, 1, 1 \rangle dA$$

$$= \iint_{x^2 + y^2 \leq 1} (0 + 4 - 2y + 2) dA$$



$$= \iint_{x^2 + y^2 \leq 1} (6 - 2y) dA$$

Polar in xy plane
 $y = r \sin \theta$
 $dA = r dr d\theta$

$$= \iint_0^{2\pi} \int_0^1 (6 - 2r \sin \theta) r dr d\theta$$

- your #14 15. Use Stokes' Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle z^2, y^2, xy \rangle$ where C is the boundary of the plane $2x + y + 2z = 2$ in the first octant. Orient C to be counterclockwise when looking from above.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot dS, \quad C = \text{boundary curve of } S.$$

Step 1: Parameterize S

$$= \iint_D \text{curl } \mathbf{F}(r(x, z)) \cdot (r_x \times r_z) dA$$

$$\mathbf{F} = \langle z^2, y^2, xy \rangle$$

$$\text{curl } \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y & xy \end{vmatrix}$$

$$= \langle x, -(y - 2z), 0 \rangle = \langle x, 2z - y, 0 \rangle$$

$$\iint_S \text{curl } \mathbf{F} \cdot dS = \iint_D \text{curl } \mathbf{F}(r(x, z)) \cdot (r_x \times r_z) dA$$

$$= \iint_D \langle x, 2z - (2 - 2x - 2z), 0 \rangle \cdot \langle 2, 1, 2 \rangle dA$$

$$\begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq z \leq 1-x \end{array}$$

$$= \iint_D \langle x, 4z + 2x - 2, 0 \rangle \cdot \langle 2, 1, 2 \rangle dA$$

$$= \iint_D (2x + 4z + 2x - 2) dA$$

$$= \iiint_D (4x + 4z - 2) dz dx$$

$S = \text{surface with boundary curve } C$

Parameterization of S is
let $x = x, z = z, y = 2 - 2x - 2z$

$$r(x, z) = \langle x, 2 - 2x - 2z, z \rangle$$

D: xz -plane

$$\begin{cases} 0 \leq x \leq 1 \\ 0 \leq z \leq 1-x \end{cases}$$

$$\begin{cases} 0 \leq x \leq 1 \\ 0 \leq z \leq 1-x \end{cases}$$

16. Use Stokes' Theorem to find $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle x^2 \sin z, y^2, xy \rangle$ and S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy plane, oriented upward.

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \boxed{\int_C \mathbf{F} \cdot d\mathbf{r}}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(r(t)) \cdot r'(t) dt$$

C was parameterized by $r(t)$, $a \leq t \leq b$

$C: r(\theta) = \langle \cos \theta, \sin \theta, 0 \rangle$

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(r(\theta)) \cdot r'(\theta) d\theta \\ &= \int_0^{2\pi} \langle 0, \sin \theta, \cos \theta \sin \theta \rangle \cdot \langle -\sin \theta, \cos \theta, 0 \rangle d\theta \\ &= \int_0^{2\pi} (0 + \sin \theta \cos \theta + 0) d\theta \\ &= \int u^2 du = \frac{\sin^3 \theta}{3} \Big|_0^{2\pi} = 0 \end{aligned}$$

Divergence Theorem Let E be a simple solid region whose boundary surface has positive (outward) orientation. Let \mathbf{F} be a vector field whose component functions have continuous partial derivatives on an open region that contains E . Then

$S = \text{boundary of } E$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} dV$$

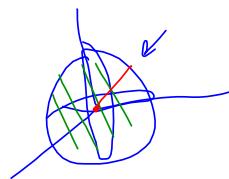
17. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle x + \sin z, 2y + \cos x, 3z + \tan y \rangle$ over the sphere $x^2 + y^2 + z^2 = 4$.

$$\operatorname{div} \mathbf{F} = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot \mathbf{F} = 1+2+3$$

divergence theorem: $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \cdot dV$ $\operatorname{div} \mathbf{F} = 6$

What is E ?

$E = \text{interior of sphere}$ using: spherical coordinates!



$$E: \begin{aligned} 0 &\leq r \leq 2 \\ 0 &\leq \theta \leq 2\pi \\ 0 &\leq \phi \leq \pi \end{aligned}$$

$$dV = r^2 \sin\theta dr d\theta d\phi$$

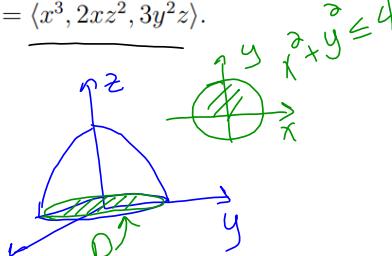
$$\int_0^{2\pi} \int_0^\pi \int_0^2 6r^2 \sin\theta dr d\theta d\phi$$

$$\mathbf{F} = \langle$$

$$\int_0^{2\pi} d\theta \int_0^\pi \sin\theta d\phi \int_0^2 6r^2 dr$$

18. Let S be the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy -plane. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle x^3, 2xz^2, 3y^2z \rangle$.

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F} \, dV \\
 &= \iiint_E (3x^2 + 3y^2) \, dV \\
 &= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 3r^2 \, dz \, r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^2 3r^3 (4-r^2) \, dr \, d\theta
 \end{aligned}$$

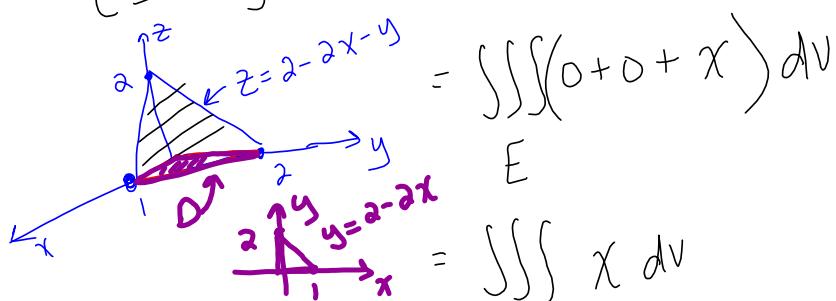


$$\begin{aligned}
 0 \leq z &\leq 4-x^2-y^2 = 4-r^2 \\
 0 \leq r &\leq 2 \\
 0 \leq \theta &\leq 2\pi \\
 dV &= dz \, r \, dr \, d\theta
 \end{aligned}$$

19. Using the Divergence Theorem, find the flux of the vector field $\mathbf{F} = \langle z \cos y, x \sin z, xz \rangle$ where S is the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$, and $2x + y + z = 2$.

$$\text{Flux } F = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

E = region S encloses



$$E : \begin{aligned} 0 &\leq z \leq 2 - 2x - y \\ 0 &\leq y \leq 2 - 2x \\ 0 &\leq x \leq 1 \end{aligned} \quad E = \iint_D \left[\int_0^{2-2x-y} x \, dz \right] \, dA$$

$$= \int_0^1 \int_0^{2-2x} \int_0^{2-2x-y} x \, dz \, dy \, dx$$

