Week in Review April 1st, 2015

- $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$
- $\log_a(x) + \log_a(y) = \log_a(xy)$

① Evaluate $\log_3(108) - \log_3(4)$ (*)

$\rightsquigarrow (*) = \log_3\left(\frac{108}{4}\right) = \log_3(27) = \underline{\underline{3}}$

② Evaluate: $\log_2(6) - \log_2(15) + \log_2(20) = \log_2\left(\frac{6 \cdot 20}{15}\right) = \log_2(2 \cdot 4)$

$= \log_2(8) = \underline{\underline{3}}$

- $\log_a(x^y) = y \cdot \log_a(x)$
- $\ln(e^x) = x$

③ Find the value of $\ln(\sqrt{e^3})$

$\rightsquigarrow \ln\sqrt{e^3} = \ln\left((e^3)^{1/2}\right) = \ln\left(e^{3/2}\right) = 3/2 \ln(e)$

$= 3/2 \cdot 1 = \underline{\underline{3/2}}$

- $(e^x)^y = e^{x \cdot y}$

④ Find the value of $e^{-2\ln(5)}$

$\rightsquigarrow e^{-2\ln(5)} = \left(e^{\ln(5)}\right)^{-2} = 5^{-2} = \underline{\underline{\frac{1}{25}}}$

⑤ Express $\log_8(x) - \log_8\sqrt{9x+2} + 5\log_8(x+1)$ as single logarithm

$\rightsquigarrow \log_8(x) - \log_8\sqrt{9x+2} + \log_8\left((x+1)^5\right) = \underline{\underline{\log_8\left(\frac{x \cdot (x+1)^5}{\sqrt{9x+2}}\right)}}$

⑥ Solve for x: $\log_a(x+3) + \log_a(x) = 1$

$\Leftrightarrow \log_a\left((x+3) \cdot x\right) = 1 \Leftrightarrow (x+3) \cdot x = a \Leftrightarrow x^2 + 3x - a = 0$

$\Rightarrow x_{1/2} = -\frac{3}{2} \pm \sqrt{9+4a}$ However, only the positive solution makes sense here, since the domain of $\log_a$ is $\mathbb{R}_{>0}$.

E.g.: $a = 10 \Rightarrow x_{1/2} = -\frac{3}{2} \pm \sqrt{9+40} = -\frac{3}{2} \pm 7 = -\frac{17}{2}$ or $\frac{11}{2}$

So, $\frac{11}{2}$ would be the correct solution here.

(7) Solve for x: $y = \ln(7x-9)$

$\rightsquigarrow y = \ln(7x-9) \Leftrightarrow e^{y} = 7x-9 \Leftrightarrow 7x = e^{y}+9 \Leftrightarrow x = \frac{e^{y}+9}{7}$

(8) Solve for x: $\ln(x) - \ln(x+1) = \ln(2) + \ln(3)$ (*)

$\rightsquigarrow$ (*) $\Leftrightarrow \ln(x) - \ln(2) = \ln(x+3) + \ln(3)$

$\Leftrightarrow \ln\left(\frac{x}{2}\right) = \ln\left((x+3)\cdot 3\right) \Leftrightarrow \frac{x}{2} = 3x+9 \Leftrightarrow \frac{5}{2}x = -9$

$\Leftrightarrow x = -\frac{18}{5}$

(9) Find the inverse of $f(x) = e^{6x-3}$

(i) Set $y = f(x)$
(ii) Solve for x
(iii) Interchange x and y

$\rightsquigarrow y = e^{6x-3} \Leftrightarrow \ln(y) = 6x-3$

$\Leftrightarrow x = \frac{\ln(y)+3}{6} \Rightarrow f^{-1}(x) = \frac{\ln(x)+3}{6}$

(10) Find $\lim_{x \to \infty} [\log(2x-1) - \log(3x+6)]$ (*)

$\rightsquigarrow$ (*) $= \lim_{x \to \infty} \log_a\left(\frac{2x-1}{3x+6}\right) = \lim_{x \to \infty} \log_a\left(\frac{2-\frac{1}{x}}{3+\frac{6}{x}}\right)$

$= \log_a\left(\frac{\lim_{x\to\infty} 2 - \lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty} 3 + \lim_{x\to\infty}\frac{6}{x}}\right) = \log_a\left(\frac{2}{3}\right) = \log_a(2) - \log_a(3)$

(11) Find $\lim_{x \to 3^+} \ln(x^2-9)$ (*)

$\cdot \lim_{x \to 0^+} \ln(x) = -\infty$

$\rightsquigarrow$ We have $\lim_{x \to 3^+} x^2 - 9 = 0$ and we will approach 0 from the right since $x \to 3^+$ and hence $x^2 > 9$ for every x in the sequence. Hence,

(*) $= \lim_{y \to 0^+} \ln(y) = -\infty$

(12) Find $\lim_{x \to 0} \ln(\cos(x))$ (*)

$\rightsquigarrow$ We have $\lim_{x \to 0} \cos(x) = 1$. Hence,

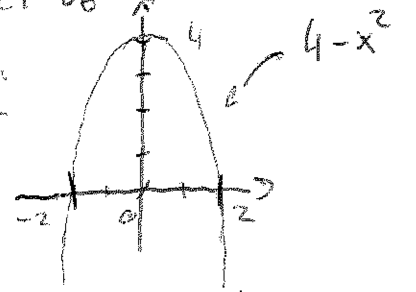
$$(*) = \lim_{y \to 1} \ln(y) = 0$$

- $\ln(1) = 0$

(13) What is the domain of $f(x) = \ln(4-x^2)$?

$\rightsquigarrow$ The domain of ln is $\mathbb{R}_{>0}$. Hence the domain of $f(x)$ will be the set of all x with $4-x^2>0$. This is the open intervall $(-2,2)$

Note:



(14) What is the domain of $f(x) = \ln(x) + \ln(3-x)$

$\rightsquigarrow$ We have $f(x) = \ln\big(x(3-x)\big)$. Thus, the domain of $f(x)$ contains all elements of $\mathbb{R}$ such that: (i) both x and $-x+3$ are greater 0.
(ii) both x and $-x+3$ are smaller than 0.

- For (i) we need $x>0$ and $x<3$, i.e., $x \in (0,3)$
- For (ii) we need $x<0$ and $x>3$, which is impossible.

Hence, the domain of $f(x)$ is $\underline{\underline{(0,3)}}$.

---

(15) Differentiate:

a) $f(t) = \cos^2(t \cdot \ln(t))$

b) $f(x) = \ln(\sin(2x))$

c) $f(x) = \log_5(e^{10x})$

d) $f(x) = 3^{\tan(7x)}$

- $\frac{d}{dx} \ln(x) = \frac{1}{x}$
- $\frac{d}{dx} a^x = a^x \cdot \ln(a)$
- $\frac{d}{dx} \log_a(x) = \frac{1}{x \cdot \ln(a)}$

⤳ a) By chain rule we get:

$$\frac{d}{dt}\cos^2(t\cdot\ln(t)) = 2\cdot\cos(t\cdot\ln(t))\cdot\frac{d}{dt}(\cos(t\cdot\ln(t))) =$$

$$2\cdot\cos(t\cdot\ln(t))\cdot(-\sin(t\cdot\ln(t)))\cdot\frac{d}{dt}(t\cdot\ln(t)) =$$ (Product rule)

$$2\cdot\cos(t\cdot\ln(t))\cdot(-\sin(t\cdot\ln(t)))\cdot\left(\frac{t}{t}+1\cdot\ln(t)\right)$$

b) $$\frac{d}{dx}\ln(\sin(2x)) = \frac{1}{\sin(2x)}\cdot\frac{d}{dx}\sin(2x) = \frac{1}{\sin(2x)}\cdot\cos(2x)\cdot\frac{d}{dx}2x$$

$$= \frac{\cos(2x)}{\sin(2x)}\cdot 2 = \underline{\underline{2\cdot\cot(2x)}}$$

c) $$\frac{d}{dx}\log_5(e^{10x}) = \frac{1}{e^{10x}\cdot\ln(5)}\cdot\frac{d}{dx}e^{10x}$$

$$= \frac{1}{e^{10x}\cdot\ln(5)}\cdot e^{10x}\cdot\frac{d}{dx}10x = \underline{\underline{\frac{10}{\ln(5)}}}$$

d) $$\frac{d}{dx}3^{\tan(7x)} = 3^{\tan(7x)}\cdot\ln(3)\cdot\frac{d}{dx}\tan(7x)$$

$$= 3^{\tan(7x)}\ln(3)\cdot\sec^2(7x)\cdot\frac{d}{dx}7x = \underline{\underline{3^{\tan(7x)}\cdot\ln(3)\cdot\sec^2(7x)\cdot 7}}$$

(16) Using logarithmic differentiation, find the derivative of

a) $y = x^{\sin(x)}$

b) $f(x) = \dfrac{e^{-x}\cdot\sin^2(x)}{x^2+x+1}$

> (i) Take ln on both sides
> (ii) Differentiate implicitly w.r.t. x ($\frac{d}{dx}$ both sides)
> (iii) Solve for $\frac{dy}{dx}$

⤳ a) $y = x^{\sin(x)}$ (*)

$\Leftrightarrow \ln(y) = \sin(x)\cdot\ln(x)$

$$\Leftrightarrow \frac{d}{dx}\ln(y) = \frac{d}{dx}\left(\sin(x)\cdot\ln(x)\right)$$

• Left: $\frac{d}{dx}\ln(y) = \frac{1}{y}\cdot\frac{d}{dx}y = \frac{y}{y}\cdot\underbrace{\frac{d}{dy}(y)}_{=1}\cdot\frac{dy}{dx}$

• Right: Apply product rule.

$$\Leftrightarrow \frac{1}{y}\cdot 1\cdot\frac{dy}{dx} = \left(\frac{d}{dx}\sin(x)\right)\cdot\ln(x) + \left(\frac{d}{dx}\ln(x)\right)\cdot\sin(x)$$

$$\Leftrightarrow \frac{dy}{dx} = y\cdot\left(\cos(x)\cdot\ln(x) + \frac{\sin(x)}{x}\right)$$

• Use (*) on page 4

$$\Leftrightarrow \frac{dy}{dx} = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$$

→ b) $y = \frac{e^{-x}\cdot\sin^2(x)}{x^2+x+1}$ (*)

$$\Leftrightarrow \ln(y) = \ln\left(\frac{e^{-x}\cdot\sin^2(x)}{x^2+x+1}\right)$$

$$\Leftrightarrow \ln(y) = -x + \ln(\sin^2(x)) - \ln(x^2+x+1)$$

$$\Leftrightarrow \frac{d}{dx}\ln(y) = \frac{d}{dx}(-x) + \frac{d}{dx}\ln(\sin^2(x)) - \frac{d}{dx}\ln(x^2+x+1)$$

$$\Leftrightarrow \frac{1}{y}\cdot\frac{dy}{dx} = -1 + \frac{1}{\sin^2(x)}\cdot 2\sin(x)\cdot\cos(x) - \frac{2x+1}{x^2+x+1}$$

$$\Leftrightarrow \frac{dy}{dx} = y\left(-1 + 2\frac{\cos(x)}{\sin(x)} - \frac{2x+1}{x^2+x+1}\right)$$

• Use (*) on page 5

$$\Leftrightarrow \frac{dy}{dx} = \frac{e^{-x}\cdot\sin^2(x)}{x^2+x+1}\cdot\left(-1 + 2\cot(x) - \frac{2x+1}{x^2+x+1}\right)$$

(17) Find the equation of the tangent line to the graph of $f(x) = x \cdot \ln(x)$ at $x = e^2$

⇝ We have:

(i) $f'(x) = \frac{d}{dx}(x \cdot \ln(x)) = x \cdot \frac{d}{dx}\ln(x) + \frac{d}{dx}(x) \cdot \ln(x) = \frac{x}{x} + 1 \cdot \ln(x) = 1 + \ln(x)$

Thus, in particular $f'(e^2) = 1 + \ln(e^2) = 1 + 2\ln(e) = 1 + 2 = \underline{3}$

(ii) $f(e^2) = e^2 \cdot \ln(e^2) = e^2 \cdot 2 \cdot 1 = 2e^2$

Hence, we get in total for the tangent line ℓ at $P(a, f(a))$ Set: $a = e^2$

$$\ell: \; y - f(a) = f'(a)(x - a)$$

$$\Rightarrow \ell: \; y - 2e^2 = 3 \cdot (x - e^2)$$

(18) What is the slope of the parametric curve $x = t\ln(t), y = 2^{3t}$ at the point $P(0, 8)$?

⇝ • $\frac{dx}{dt} = 1 + \ln(t)$ (see (17))

• $\frac{dy}{dt} = \frac{d}{dt} 2^{3t} = 2^{3t} \cdot \ln(2) \cdot \frac{d}{dt} 3t = 2^{3t} \cdot \ln(2) \cdot 3$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2^{3t} \cdot \ln(2) \cdot 3}{1 + \ln(t)}$$

We search the t^* which satisfies $x(t^*) = 0$ and $y(t^*) = 8$. This is $t^* = 1$ since $2^{3 \cdot 1} = 2^3 = 8$.

Thus we get for the tangent line ℓ at $P(0, 8)$

$$\ell: \; y - 8 = \frac{dy}{dx}(t^*) \cdot (x - 0) \Rightarrow y - 8 = \frac{2^{3 \cdot 1} \cdot \ln(2) \cdot 3}{1 + \underbrace{\ln(1)}_{=0}} \cdot x \Leftrightarrow \underline{y - 8 = 24 \cdot \ln(2) \cdot x}$$