Section 5.5

1. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of its base is twice the width. Material for the base costs $10 per square meter. Material for the sides costs $6 per square meter. Find the cost of materials for the cheapest such container.

\[ V = 10 \text{ m}^3 \Rightarrow 2w^2h = 10 \]

(a) Picture

(b) What is the constraint?

\[ h = \frac{10}{2w^2} = \frac{5}{w^2} \]

(c) What do we want to do?

Minimize the cost

\[ C = C_{\text{base}} + C_{\text{sides}} \]

\[ C_{\text{base}} = 10(2w^2) = 20w^2 \]

\[ C_{\text{sides}} = 6(2wh + 2hw) = 12wh + 12hw \]

\[ C = 20w^2 + 12wh + 12hw \]

\[ C = 20w^2 + 12w(\frac{5}{w^2}) \]

\[ C = 20w^2 + \frac{180}{w} \]

(d) Find critical numbers

\[ C' = 40w - \frac{180}{w^2} \]

\[ C' = 40w^3 - 180 = 0 \]

\[ 40w^3 = 180 \]

\[ w^3 = \frac{180}{40} = \frac{9}{2} \]

\[ w = \sqrt[3]{\frac{9}{2}} \]

Proves minimum.

Total cost:

\[ C(\sqrt[3]{\frac{9}{2}}) = 20(\sqrt[3]{\frac{9}{2}})^2 + \frac{180}{\sqrt[3]{\frac{9}{2}}} \]

\[ \approx 163.54 \]
2. Find the point on the parabola \( x + y^2 = 0 \) that is closest to the point \((0, -3)\).

\( \text{picture} \quad x + y^2 = 0 \quad x = -y^2 \)

minimize \( d \).

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 0
\]

\[
d = \sqrt{x^2 + (y+3)^2}
\]

\[
d^2 = y^4 + (y+3)^2
\]

\[
ad \quad d' = 4y^3 + 2(y+3)
\]

\[
d' = \frac{4y^3 + 2(y+3)}{2y^4 + 2(y+3)^2}
\]

con:

\[
4y^3 + 2(y+3) = 0
\]

\[
4y^3 + 2y + 6 = 0
\]

\[
y = -1
\]

Point on parabola: \( y = -1 \)

\( x = -y^2 = -1 \)

Point = \((-1, -1)\)
3. A piece of wire 12 inches long is cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a circle. How should the wire be cut so that the total area enclosed is a maximum?

\[ \begin{align*}
A &= A_\triangle + A_\circ \\
A &= \frac{x^2}{12\sqrt{3}} + \frac{1}{4\pi} (12-x)^2 \\
A' &= \frac{2x}{12\sqrt{3}} + \frac{1}{2\pi} 2(12-x)(-1) \\
A' &= \frac{x}{6\sqrt{3}} - \frac{12-x}{2\pi} \\
\end{align*} \]

\[ x = 6\sqrt{3} \]

\[ A = \frac{x^2}{12\sqrt{3}} + \frac{1}{4\pi} (12-x)^2 \\
A(0) = \frac{144}{4\pi} \approx 11.45 \text{ largest} \\
A(12) = \frac{12}{\sqrt{3}} \approx 4.9 \\
A(\infty) \approx 4.32 \]

To make the enclosed area a minimum, cut 7.47 inches into a triangle.
To make the enclosed area a maximum, cut 12 inches into a circle.
4. What are the dimensions of the largest rectangle that can be inscribed in the area bounded by the curve $y = 12 - x^2$ and the $x$-axis?

![Diagram with equation and calculations]

Maximize the area of the rectangle

$$A = (2x)(12-x^2)$$

$$A = 24x - 2x^3$$

$$A' = 24 - 6x^2$$

$$A' = 0 \Rightarrow 24 - 6x^2 = 0$$

$$6x^2 = 24$$

$$x^2 = 4$$

$$x = \pm 2$$

Dimensions: $2x = 4$

$y = 12 - x^2 = 12 - 4 = 8$

Length $= 4$

Width $= 8$
Section 5.7

5. Given \( f''(x) = 2e^x - 4\sin(x) \), \( f(0) = 1 \), and \( f'(0) = 2 \), find \( f(x) \).

\[
\begin{align*}
\hat{f}'(x) &= \text{antiderivative of } f''(x) \\
\hat{f}'(x) &= 2e^x - 4(-\cos x) + C \\
\hat{f}'(0) &= 2 \Rightarrow 2e^0 + 4\cos(0) + C = 2 \\
6 + C &= 2 \Rightarrow C = -4
\end{align*}
\]

\[
\begin{align*}
\hat{f}'(x) &= 2e^x + 4\cos x - 4 \\
\hat{f}(x) &= \text{antiderivative of } \hat{f}'(x) \\
\hat{f}(x) &= 2e^x + 4\sin x - 4x + K \\
\hat{f}(0) &= 1 \Rightarrow 2e^0 + 4\sin(0) - 4(0) + K = 1 \\
2 + K &= 1 \Rightarrow K = -1
\end{align*}
\]

6. A particle accelerates according to the equation \( a(t) = .12t^2 + 4 \). If the initial velocity is 10 and the initial position is 0, find the position function \( s(t) \).

\[
\begin{align*}
a(t) &= .12t^2 + 4 \\
\hat{v}_0 &= 10 \\
s_0 &= 0
\end{align*}
\]

\[
\begin{align*}
v(t) &= .12\frac{t^3}{3} + 4t + v_0 \\
&\Rightarrow v(t) = .04t^3 + 4t + 10 \\
s(t) &= .04\frac{t^4}{4} + 4\frac{t^2}{2} + 10t + s_0 \\
&\Rightarrow s(t) = .01t^4 + 2t^2 + 10t
\end{align*}
\]
7. A stone is dropped from a 450 meter tall building.

a.) Derive a formula for the height of the stone at time \( t \). Note the acceleration due to gravity is \(-9.8\) meters per second squared.

b.) With what velocity does the stone hit the ground?

\[
\begin{align*}
    v(t) &= -9.8 - 9.8t \\
    s(t) &= -4.9t^2 + 450
\end{align*}
\]

\[
\begin{align*}
    v(t) &= -9.8t + v_0 \\
    s(t) &= -9.8t^2 + s_0
\end{align*}
\]

\[
\begin{align*}
    a(t) &= -9.8 \\
    s(t) &= 450
\end{align*}
\]

(b) \( \mathbb{Q} \) when does it hit the ground?

\[
\begin{align*}
    s(t) &= 0 \\
    -4.9t^2 + 450 &= 0 \\
    t^2 &= \frac{450}{4.9} \\
    t &= \sqrt{\frac{450}{4.9}} \text{ seconds} \\
    v\left(\sqrt{\frac{450}{4.9}}\right) &= -9.8 \left(\sqrt{\frac{450}{4.9}}\right) \text{ m/s}
\end{align*}
\]
8. A car is traveling at a speed of \( \frac{220}{3} \) feet per second when the brakes are fully applied thus producing a constant deceleration of \( 40 \) feet per second squared. How far does the car travel before coming to a stop?

\[
a(t) = -40, \quad v_0 = \frac{220}{3} \\
v(t) = -40t + \frac{220}{3} \\
s(t) = -40 \frac{t^2}{2} + \frac{220t}{3} + s_0 \\
s(t) = -20t^2 + \frac{220t}{3} \quad \text{feet} \\
s \left( \frac{11}{6} \right) = -20 \left( \frac{11}{6} \right)^2 + \frac{220}{3} \left( \frac{11}{6} \right) \quad \text{feet} \\
\approx 67 \text{ feet} \\
\]

Car stops when \( v(t) = 0 \)

\[
-40t + \frac{220}{3} = 0 \\
40t = \frac{220}{3} \\
t = \frac{\frac{220}{3}}{40} = \frac{11}{6} \text{ s}
\]
9. Find the vector functions that describe the velocity and position of a particle that has an acceleration of \( \mathbf{a}(t) = \langle 0, 2 \rangle \), initial velocity of \( \mathbf{v}(0) = \langle 1, -1 \rangle \) and an initial position of \( \mathbf{r}(0) = \langle 0, 0 \rangle \).

\[
\mathbf{a}(t) = \langle 0, 2 \rangle \\
\mathbf{v}(t) = \langle 1, 2t - 1 \rangle \\
\mathbf{v}(t) = \langle t + 2, t^2 - t + 2 \rangle \\
\mathbf{r}(t) = \langle t, t^2 - t \rangle
\]
Section 6.1

10. Compute \( \sum_{i=2}^{5} \frac{i}{i+1} \) 
\[ \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} + \frac{5}{5+1} \]
\[ i=2 \quad i=3 \quad i=4 \quad i=5 \]
\[ = \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} \]

11. Compute \( \sum_{i=1}^{500} (9) = \frac{9+9+9+\ldots+9}{500 \text{ times}} = 9(500) \)

12. Compute \( \sum_{i=3}^{300} (2) = \frac{2+2+2+\ldots+2}{298 \text{ times}} = 2(298) \)

13. Using the formula \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \), find \( \sum_{i=1}^{99} 4i \).
\[ = 4 \sum_{i=1}^{99} i = 4 \left( \frac{99(100)}{2} \right) \]

What if we wanted:
\[ \sum_{i=3}^{99} 4i = \sum_{i=1}^{99} 4i - \sum_{i=1}^{2} 4i \]
\[ = 4 \left( \frac{99(100)}{2} \right) - (4 + 8) \]
14. Write in sigma notation:

a.) \( \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7} = \sum_{i=3}^{7} \sqrt{i} \) or \( \sum_{i=1}^{5} \sqrt{i+2} \)

b.) \( 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \sum_{i=1}^{5} \frac{1}{i^2} \)

c.) \( 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 \) = \( \sum_{i=0}^{6} (-1)^i x^i \)