Section 4.8: L’Hospital’s Rule

**Indeterminate form:** If \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \) or \( \frac{\infty}{\infty} \), then we say the limit is in indeterminate form.

**L’Hospital’s Rule:** If \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \) or \( \frac{\infty}{\infty} \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \).

Some common misconceptions: If \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \) or \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{0} \), the limit is NOT indeterminate! For example,

(i) \( \lim_{x \to 0^+} \frac{\ln x}{\sqrt{x}} \)  

(ii) \( \lim_{x \to 0^+} \frac{x}{\ln x} \)

**Example 1:** Find the following limits, if they exist. If the limit does not exist, explain why.

(i) \( \lim_{x \to 1} \frac{\ln x}{x - 1} \)

(ii) \( \lim_{x \to 0} \frac{\sin x - x}{x^3} \)

(iii) \( \lim_{x \to 0} \frac{\sin mx}{\sin nx} \)
(iv) \[ \lim_{x \to \infty} \frac{(\ln x)^2}{x} \]

**Indeterminate Products:** If \( \lim_{x \to a} f(x)g(x) = 0 \cdot \infty \), this limit is an indeterminate product. Why do we call the product indeterminate?

\[
\begin{align*}
\lim_{x \to \infty} \frac{1}{x^2} \cdot x &= \lim_{x \to \infty} \frac{1}{x} \cdot x^2 &= \lim_{x \to \infty} \frac{1}{x^2} \cdot 6x^2
\end{align*}
\]

All three of these limits are of the form \( 0 \cdot \infty \), yet they all have different limits. The goal is to try to manipulate the product to get the limit in the form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \), then use L'Hospital’s rule.

**Example 2:** Find the following limits, if they exist. If the limit does not exist, explain why.

(i) \( \lim_{x \to 0^+} x^3 \ln x \)

(ii) \( \lim_{x \to 1^+} (x - 1) \tan(\pi x/2) \)
**Indeterminate Powers**: If $\lim_{x \to a} f(x)^{g(x)}$ is of the form $0^0$, $\infty^0$ or $1^\infty$, then the limit is an indeterminate power. To solve such a limit, take the natural logarithm, which converts the indeterminate power into an indeterminate product.

**Example 3**: Find the following limits, if they exist. If the limit does not exist, explain why.

(i) $\lim_{x \to \infty} x^{\frac{3}{x}}$

(ii) $\lim_{x \to \infty} \left( \frac{2x - 3}{2x + 5} \right)^{2x+1}$

**Indeterminate difference**: If $\lim_{x \to a} (f(x) - g(x)) = \infty - \infty$, this limit is an indeterminate difference.

**Example 4**: Find $\lim_{x \to 1} \left( \frac{1}{\ln x} - \frac{1}{x - 1} \right)$