On pie and noodles

Gregory Berkolaiko

Department of Mathematics

Aggieland Saturday, 18 Feb 2012
Buffon’s Needle

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- Take a sheet of ruled paper (spacing 2)
- Take a needle of length 2
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- Drop the needle at random
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- What’s the probability the needle intersects a line?
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- Even more famous as a biologist, defined species, discussed evolution.
- Was made Comte de Buffon in 1773.
- He was not “Buffon” when he invented “Buffon’s needle”.
A calculus solution

\[
\text{Probability} = \frac{\text{Successful outcomes}}{\text{Total outcomes}}
\]

\(x\) is the distance from needle's center to the nearest line:
\(x \in [0, 1]\).

\(\phi\) is the angle with vertical:
\(\phi \in [0, \pi/2]\).

Given \(x\), success if \(\phi \leq \arccos x\).

Given \(x\), the probability is \(\frac{\arccos x}{\pi/2}\).
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  \[ \frac{\arccos x}{\pi/2}. \]
Add for all possible values of $x$:

$$\int_0^1 \frac{\arccos x}{\pi/2} \, dx = \frac{2}{\pi} \int_0^1 \arccos x \, dx.$$
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Use substitution $x = \cos \alpha$, then integrate by parts:

\[ \int_0^1 \arccos x \, dx = 1. \]
A calculus solution (cont)

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Probability is $\frac{\pi}{2}$.
Some remarks

“The solution [...] was obtained by using integral calculus, for the first time in the history of the development of probability” — A.M. Mathai
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The constant $\pi$ is related to things round. But everything in this problem is straight! What is going on?
A bet with dice: On 1,2,3,4 you get $2. On 5,6 you lose $6. Should you take it?

$$\text{Probability to win is } \frac{2}{3}.$$

$$\text{Expected payoff is } 2 \times \frac{2}{3} + (-6) \times \frac{1}{3} = -\frac{2}{3}.$$
Probabilities and averages

- A bet with dice: On 1,2,3,4 you get $2. On 5,6 you lose $6. Should you take it?
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Another bet: On even (2,4,6) get $2. On odd (1,3,5) lose $1.

- Expected payoff is \( \frac{1}{2} \).

Now you can play both games on the same roll!

- Probability to win, well, complicated (see below).
- Expected payoff is \(-\frac{1}{6}\).

Which is just \(-\frac{2}{3} + \frac{1}{2}\)!
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Roll

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Mathematical notation is \( \mathbb{E}x \), for **expectation** of random variable \( x \).

In our example,

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\mathbb{E}(g_1 + g_2) = \mathbb{E}g_1 + \mathbb{E}g_2,
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where \( g_1 \) and \( g_2 \) are our (random) winnings in game 1 and 2.
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$$\mathbb{E}(g_1 + g_2) = \mathbb{E}g_1 + \mathbb{E}g_2,$$

where $g_1$ and $g_2$ are our (random) winnings in game 1 and 2.

This is a general mathematical law.
Let $n$ be the random variable that counts the number of intersections of the needle with the lines.
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The needle can have 0 or 1 intersection, so

\[ n = \begin{cases} 
1 & \text{with probability } p, \\
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\end{cases} \]

This probability $p$ is the answer to the Buffon’s problem!
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Instead of finding probability we can look for the expectation.
Put a mark on the needle that separates the needle into two parts, part 1 and part 2.
Bending the needle

- Put a mark on the needle that separates the needle into two parts, part 1 and part 2.
- Let $n_1$ be the number of intersections that part 1 has with the lines; $n_2$ be the number of intersections that part 2 has.
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- Put a mark on the needle that separates the needle into two parts, part 1 and part 2.
- Let $n_1$ be the number of intersections that part 1 has with the lines; $n_2$ be the number of intersections that part 2 has.
- Obviously, $n = n_1 + n_2$. Therefore $\mathbb{E}n = \mathbb{E}n_1 + \mathbb{E}n_2$. 
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- Obviously, $n = n_1 + n_2$. Therefore $\mathbb{E}n = \mathbb{E}n_1 + \mathbb{E}n_2$.
- Now bend the needle at the mark. Let $\hat{n}$ be the number of intersections of the bent needle.
Put a mark on the needle that separates the needle into two parts, part 1 and part 2.

Let \( n_1 \) be the number of intersections that part 1 has with the lines; \( n_2 \) be the number of intersections that part 2 has.

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Now **bend the needle** at the mark. Let \( \hat{n} \) be the number of intersections of the bent needle.

The probability distribution of \( \hat{n} \) is different from that of \( n \): for example the bent needle can have 2 intersections now.
Put a mark on the needle that separates the needle into two parts, part 1 and part 2.

Let $n_1$ be the number of intersections that part 1 has with the lines; $n_2$ be the number of intersections that part 2 has.

Obviously, $n = n_1 + n_2$. Therefore $E[n] = E[n_1] + E[n_2]$.

Now bend the needle at the mark. Let $\hat{n}$ be the number of intersections of the bent needle.

The probability distribution of $\hat{n}$ is different from that of $n$: for example the bent needle can have 2 intersections now.

But part 1 is not bent, so $E[n_1]$ is unchanged; same for $E[n_2]$. 
\[ \hat{n} \text{ is different from } n, \text{ but } \mathbb{E}n_1 \text{ and } \mathbb{E}n_2 \text{ are unchanged.} \]
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Therefore

\[ \mathbb{E} \hat{n} = \mathbb{E} n_1 + \mathbb{E} n_2 = \mathbb{E} n = p. \]
\(\hat{n}\) is different from \(n\), but \(E_n^1\) and \(E_n^2\) are unchanged.

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The answer to Buffon’s problem is still equal to the expected number of intersections, even for the bent needle!
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 Sanity test: fold the needle in half. It is now twice shorter, so the probability to have any intersection should be \( p/2 \). But each intersection is double, so \( \mathbb{E}\hat{n} = 2 \times p/2 = p \). Good!
Symmetry makes the world go round

- By the same reasoning, we can bend the needle again
Symmetry makes the world go round

By the same reasoning, we can bend the needle again and again and again! We can bend it into any shape! Of course the length must stay the same, 2. What shape should we bend it into? The more symmetry the better!

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- Triangle? Square? Hexagon? Circle!
We bent the needle into a circle.
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The circumference is \( L = 2\pi R = 2 \), so radius is \( R = \frac{1}{\pi} \).
We bent the needle into a circle.

The circumference is $L = 2\pi R = 2$, so radius is $R = 1/\pi$.

Let $x$ be the distance from the center of the circle to the nearest line: $x \in [0, 1]$. 

Calculate the answer

$$E_n = 2 \times \frac{1}{\pi} + 0 \times (1 - \frac{1}{\pi}) = \frac{2}{\pi}.$$
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We have an intersection when $x \leq R$. In fact, we have two!
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Calculate $E_n = 2 \times \frac{1}{\pi} + 0 \times (1 - \frac{1}{\pi}) = \frac{2}{\pi}$. 

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Calculate $\mathbb{E} n = 2 \times \frac{1}{\pi} + 0 \times (1 - \frac{1}{\pi}) = \frac{2}{\pi}$.

Easy as 1, 2, $\pi$!
Advanced mathematics simplifies things.
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And where is pie? Where is noodles?
Conclusions

- Advanced mathematics simplifies things.
- And where is pie? Where is noodles?
- Well, pie is $\pi$. 
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And where is pie? Where is noodles?

Well, pie is $\pi$.

And a needle that bends this way and that is a noodle!
Advanced mathematics simplifies things. And where is pie? Where is noodles? Well, pie is $\pi$. And a needle that bends this way and that is a noodle!

So the title was:

**On $\pi$ and Buffon’s noodle.**