Examination 1

1. Suppose $u(x, y)$ is a twice continuously differentiable real-valued function on an open subset of the plane. Show that u is harmonic if and only if $\partial u/\partial z$ is holomorphic.

   Hint: Recall that a harmonic function is one for which $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, and the operator $\dfrac{\partial}{\partial z}$ is an abbreviation for $\dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$.

2. Suppose $f(z)$ is a holomorphic function whose real part is $u(x, y)$ and whose imaginary part is $v(x, y)$. Show that the gradient vector of the function u and the gradient vector of the function v are orthogonal to each other.
   (This problem says—in the language of real calculus—that the level curves of u and the level curves of v are families of orthogonal trajectories.)

3. Let (p_n) denote the sequence of prime numbers: namely, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, and so forth. For which values of the complex number z does the infinite series $\displaystyle\sum_{n=1}^{\infty} z^{p_n}$ converge? Explain how you know.

4. Determine all values of the complex number z for which the infinite series
   $$\sum_{n=1}^{\infty} \frac{z^n}{1 + z^{2n}}$$
   converges. (This series is *not* a power series, so the convergence region need not be a disk.) Justify your answer.

5. State some version of each of the following theorems (with all hypotheses and conclusions correct).
   (a) Cauchy's theorem (about integrals being equal to zero)
   (b) Cauchy's integral formula
   (c) Morera's theorem

6. Explain why $\displaystyle\oint_C \left(z + \frac{1}{z}\right)^{102} dz$ equals 0 whenever C is a circle for which the integral makes sense (that is, the origin does not lie on the integration path).