Exercise on the order of an entire function

An entire function has a Maclaurin series expansion $\sum_{n=0}^{\infty} c_n z^n$ that converges in the entire complex plane. Since the series coefficients $c_n$ uniquely determine the function, the order $\lambda$ should be computable directly from the $c_n$. This exercise provides a formula for computing the order of an entire function from the Maclaurin series coefficients.

Recall that when $f$ is an entire function, the quantity $M(r)$ denotes $\max\{|f(z)| : |z| = r\}$, and $\lambda$ equals $\lim\sup_{r \to \infty} \{\log \log M(r)\} / \{\log r\}$. In this exercise, you will prove that the order $\lambda$ also equals $\lim\sup_{n \to \infty} \frac{n \log n}{\log |c_n|}$ (interpret the fraction as 0 if $c_n = 0$).

For working purposes, denote the preceding quantity temporarily by $\beta$; you need to show that (a) $\beta \leq \lambda$ and (b) $\lambda \leq \beta$.

1. Fix a positive $\epsilon$. By the definition of order, $M(r) < e^{r^{\lambda+\epsilon}}$ for sufficiently large $r$. Bound $|c_n|$ for large $n$ by applying Cauchy’s estimate with $r = n^{1/(\lambda+\epsilon)}$, and deduce that $\beta \leq \lambda + \epsilon$. Let $\epsilon \downarrow 0$.

2. Fix a positive $\epsilon$. Then $|c_n| < n^{-n/(\beta+\epsilon)}$ for sufficiently large $n$, by the definition of $\beta$. Observe that $M(r) \leq \sum_{n=0}^{\infty} |c_n| r^n$. By splitting the sum where $n \approx (2r)^{\beta+\epsilon}$, show that $\lambda < \beta + 2\epsilon$. Let $\epsilon \downarrow 0$.

The proof of Hadamard’s factorization theorem depends on the theory of infinite products. Your work above shows that one can also use infinite series to construct entire functions with prescribed growth.

3. Let $s$ be an arbitrary positive real number, and suppose that

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1/s}}.$$ 

Show that $f$ is an entire function of order $s$. 

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Theory of Functions of a Complex Variable II Dr. Boas