A PROOF OF THE INF-SUP CONDITION FOR THE STOKES EQUATIONS ON LIPSCHITZ DOMAINS

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Abstract. The purpose of this paper is to present a rather simple proof of an inequality of Nečas [9] which is equivalent to the inf-sup condition. This inequality is fundamental in the study of the Stokes equations. The boundary of the domain is only assumed to be Lipschitz.

1. Introduction

One of the most important inequalities in the theory of incompressible fluids is the so-called inf-sup condition, cf. [2] [3] [5] [10] . Let Ω be a bounded domain in R^N with a Lipschitz boundary. Let $L^2(\Omega)$ and $H_0^1(\Omega)$ be the standard Hilbert spaces of distributions on Ω, with norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H_0^1(\Omega)}$. The L^2-inner product, as well as the pairing between $H_0^1(\Omega)$ and its dual, $H^{-1}(\Omega)$ is denoted by $(\cdot,\cdot)$. Note that the space $\mathcal{D}(\Omega)$, of infinitely differentiable functions with compact support in Ω, is dense in both $L^2(\Omega)$ and $H_0^1(\Omega)$.

Let $L_0^2(\Omega) = \{u \in L^2(\Omega) | \int_\Omega u \,\mathrm{d}x = 0\}$. Denote by $\mathbf{v}$ vector functions with components $v_i, i = 1,\ldots,N$. In the following, C denotes a constant which depends only on Ω unless otherwise stated. The important inf-sup condition is

$$\inf_{u\in L_0^2(\Omega)} \sup_{\mathbf{v}\in[H_0^1(\Omega)]^N} \frac{(u,\nabla\cdot\mathbf{v})}{\|u\|_{L^2(\Omega)}\|\mathbf{v}\|_{[H^1(\Omega)]^N}} \geq C > 0, \tag{1.1}$$

where $\nabla\cdot\mathbf{v} = \sum_{i=1}^N \frac{\partial v_i}{\partial x_i}$ is the standard divergence operator and $\|\mathbf{v}\|^2_{[H_0^1(\Omega)]^N} = \sum_{i=1}^N \|v_i\|^2_{H^1(\Omega)}$. This is the same as

$$C\|u\|_{L^2(\Omega)} \leq \sup_{\mathbf{v}\in[H_0^1(\Omega)]^N} \frac{(u,\nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_{[H^1(\Omega)]^N}} \quad \text{for all} \quad u\in L_0^2(\Omega). \tag{1.2}$$

It is easy to see that both (1.1) and (1.2) are equivalent to

$$C\|u\|_{L^2(\Omega)} \leq \sum_{i=1}^N \sup_{v\in[H_0^1(\Omega)]} \frac{(u,\frac{\partial v}{\partial x_i})}{\|v\|_{H^1(\Omega)}} \quad \text{for all} \quad u\in L_0^2(\Omega). \tag{1.3}$$

The norm on $H^{-1}(\Omega)$ is given by

$$\|u\|_{H^{-1}(\Omega)} = \sup_{v\in[H_0^1(\Omega)]} \frac{(u,v)}{\|v\|_{H^1(\Omega)}}. \tag{1.4}$$

Date: October 5, 2007.

This work was supported in part under the National Science Foundation Grant No. DMS-9973328 .

Noting that, for $u \in L^2(\Omega)$, the distributional derivative, $\frac{\partial u}{\partial x_i} \in H^{-1}(\Omega)$, (1.3) is the same as

$$C\|u\|_{L^2(\Omega)} \leq \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{H^{-1}(\Omega)}, \quad \text{for all} \quad u \in L_0^2(\Omega). \tag{1.5}$$

The following inequality
(1.6)

$$C\|u\|_{L^2(\Omega)} \leq \left(\|u\|^2_{H^{-1}(\Omega)} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|^2_{H^{-1}(\Omega)} \right)^{1/2} \equiv \|u\|_{X(\Omega)}, \text{ for all } u \in L^2(\Omega),$$

was proved by Nečas [9].

It is not difficult to show that (1.6) is equivalent to (1.3) and therefore equivalent to (1.2). We will show in Appendix I that existence and uniqueness in the Stokes problem follows easily from (1.2). Thus (1.6) is of fundamental importance. To emphasize this point even more, it is easy to see that Korn's second inequality follows from (1.6), (cf. Duvault-Lions [3]) which is fundamental in the theory of elasticity. Direct proofs of Korn's second inequality, in the case of nonsmooth domains are given in, e.g., [4], [6] and [8]. The proofs are not elementary. As is done in [3], we will show in Appendix II that Korn's second inequality is a corollary of (1.6).

The purpose of this note is to provide a proof of (1.6). More precisely we shall give a proof of the following.

Theorem 1.1. *Let $\Omega \in R^N$ be a connected domain with a Lipschitz boundary. Then there exists a constant C such that, for all $u \in L^2(\Omega)$,*

$$C\|u\|_{L^2(\Omega)} \leq \|u\|_{X(\Omega)}. \tag{1.7}$$

Clearly if $\Omega = \cup_{j=1}^{M} \Omega_j$ and if (1.7) holds for each $\Omega_j, j = 1, \ldots, M < \infty$, then it holds for Ω.

We will use the fact that any Lipschitz domain can be written as the union of strongly star shaped Lipschitz domains. Hence we may assume, without loss, that Ω is strongly star shaped. A weaker version of this statement is proved by Bernardi [1]. The stronger version follows by a slight modification of her proof. A strongly star shaped Lipschitz domain is a domain which can be represented as follows.

$$\Omega = \{x \in R^N | r < g(x/r)\} \quad \text{and} \quad \partial\Omega = \{x \in R^N | r = g(x/r)\}$$

where $r = |x|$ and $g \in W^1_\infty$ with $\|g\|_{W^1_\infty} \leq \Lambda$. Evidently we have assumed that Ω is star shaped with respect to the origin. By x/r we mean $(x_1/r, \ldots, x_N/r)$ and note that x_i/r is independent of r away from the origin.

2. A transformation of B to Ω

The plan is to map the unit ball B to Ω with ∂B mapped to the boundary $\partial\Omega$ and then to show that Theorem 1.1 holds if it holds for $\Omega = B$. We will use the variables y in B and set $|y| = \rho$. The boundary $\partial B = \{y \in R^N | \rho = 1\}$.

The simplest transformation would be defined by $x_i = y_i g(y/\rho)$. This transformation is not smooth enough to do the job unless $\partial\Omega$ is sufficiently smooth, so we follow the idea of Nečas and smooth g. We do this as follows.

Let $\Phi \in \mathcal{D}(R^N)$ with $\Phi(x) = 0$ if $|x| \geq 1$, $\Phi(x) \geq 0$ and $\int \Phi(x)\, dx = 1$. For $0 < h < 1/4$ and $|x| > 1/2$, define

$$g(h, x) = \frac{1}{h^N} \int_{R^N} g(y/\rho)\Phi((x - y)/h)\, dy.$$

The following are some important properties of $g(h, x)$.

(1) $\lim_{h\to 0} g(h, y/\rho) = g(y/\rho)$
(2) $|\frac{\partial}{\partial h} g(h, x)| \leq C$
(3) $|\frac{\partial}{\partial x_i} g(h, x)| \leq C$
(4) $|D^\alpha g(h, x)| \leq C/h, |\alpha| = 2$
(5) For $|x| = 1, |g(h, x) - g(x)| \leq \bar{C}h$, where $\bar{C}$ depends only on Λ, the Lipschitz bound, and bounds for $g(x/r)$.

We next set $h = \epsilon(1 - \rho)$. From the last inequality, 5, we obtain easily that, for ϵ small enough

$$g(\epsilon(1 - \rho), y/\rho) \geq g(y/\rho) - \bar{C}\epsilon \geq C > 0.$$

We are now in a position to define the transformation $\mathcal{T}$ from the unit ball to Ω as follows. For $y \in B$ define $\mathcal{T}$ by

$$x_i = y_i g(\epsilon(1 - \rho), y/\rho), \tag{2.1}$$

for $i = 1, \ldots, N$ and ϵ small enough.

We next check that $\mathcal{T}$ maps B to Ω. To this end notice that $r = \rho g(\epsilon(1 - \rho), y/\rho)$ and set $f(\rho) = \rho g(\epsilon(1 - \rho), y/|y|)$ for $y \in B$ fixed. Now $f(0) = 0$ and $f(1) = g(y/|y|)$. Now

$$f'(\rho) = g(\epsilon(1 - \rho), y/|y|) - \epsilon\rho \frac{\partial}{\partial h} g(\epsilon(1 - \rho), y/|y|) > 0$$

if ϵ is small enough. Thus $\mathcal{T}$ maps B to Ω.

We next compute the Jacobian of $\mathcal{T}$. For ease of notation, set $G = g(\epsilon(1 - \rho), y/\rho)$. From (2.1) the Jacobian matrix a_{ij} is

$$a_{ij} = \frac{\partial x_i}{\partial y_j} = \delta_{ij} G + y_i \frac{\partial G}{\partial y_j}.$$

Let $\mathbf{v}^1$ be the vector in R^N whose components are $v_j^1 = y_j/\rho$. Then

$$\sum_{j=1}^N a_{ij} v_j^1 = G v_i^1 + \rho v_i^1 \frac{\partial G}{\partial \rho} = (G - \epsilon\rho \frac{\partial G}{\partial h}) v_i^1,$$

so that $\mathbf{v}^1$ is an eigenvector of a_{ij} with eigenvalue $(G - \epsilon\rho\frac{\partial G}{\partial h})$. To find the rest of the eigenvalues, let $\{\mathbf{v}^k\}$ be an orthonormal set of vectors. Then $A_{kl} = \sum_{i,j} a_{ij} v_i^k v_j^l$ has the same eigenvalues as a_{ij}. Now

$$A_{kl} = \delta_{kl} G + \sum_i v_i^k v_i^1 \rho \sum_j v_j^l \frac{\partial G}{\partial y_j}.$$

Hence A_{kl} is an upper diagonal matrix with $A_{kk} = G$ if $k > 1$. Thus we have found that the Jacobian determinant $J = (G - \epsilon\rho\frac{\partial G}{\partial h})G^{N-1}$.

Now from Property 4, $|D^\alpha g(1-\rho, y/\rho)| \leq C/(1-\rho)$, for any $|\alpha| = 2$. We have for two constants c_0 and c_1 that $c_0 \leq g(y/\rho) \leq c_1$. Hence

$$c_1(1-\rho) \geq (1-\rho)g(y/\rho) = g(x/r) - r + \rho(g(\epsilon(1-\rho), y/\rho) - g(y/\rho)),$$

where we used the fact that $y/\rho = x/r$. Thus, using Property 5,

$$c_1(1-\rho) \geq (1-\rho)g(y/\rho) \geq g(x/r) - r - \rho\epsilon\bar{C}(1-\rho).$$

Similarly

$$c_0(1-\rho) \leq (1-\rho)g(y/\rho) \leq g(x/r) - r + \rho\epsilon\bar{C}(1-\rho).$$

Hence, for ϵ small enough we have

$$c_0(1-\rho) \leq (g(x/r) - r) \leq c_1(1-\rho).$$

It follows then that for $|\alpha| = 2$

$$|D^\alpha g(1-\rho, y/\rho)| \leq C/(g(x/r) - r)$$

and therefore for $|\alpha| = 1$

$$|D^\alpha J| \leq C/(g(x/r) - r),$$

where D^α is any partial derivative of order one with respect to x_i or y_i.

3. Some lemmas

We will need a few lemmas.

Lemma 3.1. *Let $u \in L^2(R^N)$. Then*

$$\|u\|_{L^2(R^N)} = \|u\|_{X(R^N)}.$$

Proof. Using the Fourier transform, $\hat{u}$

$$\|u\|^2_{L^2(R^N)} = \|\hat{u}\|^2_{L^2(R^N)} = \|(1+|\xi|^2)^{-1/2}\hat{u}\|^2_{L^2(R^N)} + \sum_{i=1}^N \|(1+|\xi|^2)^{-1/2}\xi_i\hat{u}\|^2_{L^2(R^N)} = \|u\|^2_{X(R^N)}.$$

□

Lemma 3.2. *Let $\Omega \in R^N$ be a bounded domain with. Let $\gamma \in \mathcal{D}(\Omega)$ be fixed. Then there exists a constant C_γ depending only on γ such that, for all $u \in L^2(\Omega)$,*

$$\|\gamma u\|_{L^2(\Omega)} \leq C_\gamma \|u\|_{X(\Omega)}.$$

Proof. Using Lemma 3.1 $\|\gamma u\|_{L^2(\Omega)} = \|\gamma u\|_{L^2(R^N)} = \|\gamma u\|_{X(R^N)}$. The lemma follow from the fact that multiplication by γ is a bounded operator from $X(\Omega)$ to $X(R^N)$. □

We next give an elementary proof of a Hardy-type inequality.

Lemma 3.3. *Let $\phi \in \mathcal{D}(\Omega)$. Then*

$$\int_\Omega \frac{\phi^2}{(g(x/r)-r)^2}\,\mathrm{d}x \le 4\|\phi\|^2_{H^1(\Omega)}. \tag{3.1}$$

Proof. Let $\Omega_\delta = \{x \in \Omega \,|\, |x| > \delta > 0\}$. Then

$$\int_{\Omega_\delta} \frac{\phi^2}{(g(x/r)-r)^2}\,\mathrm{d}x = \int_{\Omega_\delta} \frac{\partial}{\partial r}\left(\frac{\phi^2}{g(x/r)-r}\right)\mathrm{d}x - \int_{\Omega_\delta} \frac{1}{g(x/r)-r}\frac{\partial(\phi^2)}{\partial r}\,\mathrm{d}x.$$

Now

$$\begin{aligned}\int_{\Omega_\delta} \frac{\partial}{\partial r}\left(\frac{\phi^2}{g(x/r)-r}\right)\mathrm{d}x &= \sum_{i=1}^N \int_{\Omega_\delta} \frac{x_i}{r}\frac{\partial}{\partial x_i}\left[\frac{\phi^2}{g(x/r)-r}\right]\mathrm{d}x \\ = -(N-1)\int_{\Omega_\delta} \frac{\phi^2}{r(g(x/r)-r)}\,\mathrm{d}x &- \int_{|x|=\delta} \frac{\phi^2}{g(x/r)-\delta}\,\mathrm{d}s,\end{aligned}$$

where the last integral is taken over the surface of the N-sphere of radius δ. Hence, for $\delta < \min g(x/r)$,

$$\int_{\Omega_\delta} \frac{\partial}{\partial r}\left(\frac{\phi^2}{g(x/r)-r}\right)\mathrm{d}x \le 0.$$

Thus

$$\int_{\Omega_\delta} \frac{\phi^2}{(g(x/r)-r)^2}\,\mathrm{d}x \le -2\int_{\Omega_\delta} \frac{1}{g(x/r)-r}\phi\frac{\partial\phi}{\partial r}\,\mathrm{d}x.$$

Using Schwarz's inequality and letting δ go to zero we obtain Lemma 3.3. □

We now use this to prove the following.

Lemma 3.4. *Let $f \in C^1(\Omega) \cap C^0(\bar\Omega)$ and satisfy $|D^\alpha f| \le \tilde C/(g(x/r)-r)^{|\alpha|}$ for $x \in \Omega$ and $|\alpha| \le 1$. Then there is constant C depending only on $\tilde C$ such that*

$$\|f\phi\|_{H^1(\Omega)} \le C\|\phi\|_{H^1(\Omega)}, \quad \textit{for all} \quad \phi \in \mathcal{D}(\Omega). \tag{3.2}$$

Proof. This follows immediately from the Lemma 3.3. □

4. Reduction to the unit ball

Let $B_\delta = \{y \in B| \quad |y| > \delta\}$. In view of Lemma 3.2 it suffices to prove (1.6) for $\tilde v \in \mathcal{D}(\mathcal{T}(B_\delta))$. Thus we want to prove the following proposition.

Proposition 4.1. *Let $v \in \mathcal{D}(B_\delta)$ and $\tilde v(x) := v(\mathcal{T}^{-1}(x))$. Then*

$$\|v\|_{X(B)} \le C\|\tilde v\|_{X(\Omega)}$$

and

$$\|\tilde v\|_{L^2(\Omega)} \le C\|v\|_{L^2(B)}.$$

Proof. Let $\eta \in \mathcal{D}(B \setminus B_\delta)$ be such that $\eta(y) = 1$ for $|y| \le \delta/2$. Since $v \in \mathcal{D}(B_\delta)$ we note that $\eta\frac{\partial v}{\partial y_i} = 0$. Hence

$$(\frac{\partial v}{\partial y_i}, \phi) = ((1-\eta)\frac{\partial v}{\partial y_i}, \phi) = (\frac{\partial v}{\partial y_i}, (1-\eta)\phi).$$

Thus

$$\sup_{\phi\in\mathcal{D}(B)} \frac{(\frac{\partial v}{\partial y_i}, \phi)}{\|\phi\|_{H^1(B)}} = \sup_{\phi\in\mathcal{D}(B)} \frac{(\frac{\partial v}{\partial y_i}, (1-\eta)\phi)}{\|(1-\eta)\phi\|_{H^1(B)}} \frac{\|(1-\eta)\phi\|_{H^1(B)}}{\|\phi\|_{H^1(B)}}$$

$$\le C \sup_{\psi\in H_0^1(B_{\delta/2})} \frac{(\frac{\partial v}{\partial y_i}, \psi)}{\|\psi\|_{H^1(B)}} = C \sup_{\psi\in\mathcal{D}(B_{\delta/2})} \frac{(\frac{\partial v}{\partial y_i}, \psi)}{\|\psi\|_{H^1(B)}}.$$

Next we make a change of variables.

$$(\frac{\partial v}{\partial y_i}, \psi) = \int_B \frac{\partial v}{\partial y_i}\psi\, dy = \sum_{j=1}^N \int_\Omega \frac{\partial \tilde{v}}{\partial x_j}\frac{\partial x_j}{\partial y_i}\tilde{\psi} J^{-1}\, dx,$$

where $J = [g(\epsilon(1-\rho), y/\rho) - \epsilon\rho\frac{\partial g}{\partial h}(\epsilon(1-\rho), y/\rho)][g(\epsilon(1-\rho), y/\rho)]^{N-1}$. Set $f_{ij} = \frac{\partial x_j}{\partial y_i}J^{-1}$. Then

$$(\frac{\partial v}{\partial y_i}, \psi) \le \sum_{j=1}^N \|\frac{\partial \tilde{v}}{\partial x_j}\|_{H^{-1}(\Omega)}\|f_{ij}\tilde{\psi}\|_{H^1(\Omega)}.$$

Since

$$|D^\alpha f_{ij}| \le C/(g(x/r) - r)$$

for $|\alpha| = 1$, it follows from Lemma 3.4 that

$$\|f_{ij}\tilde{\psi}\|_{H^1(\Omega)} \le C\|\tilde{\psi}\|_{H^1(\Omega)} \le C\|\psi\|_{H^1(B)}.$$

Hence

$$\|\frac{\partial v}{\partial y_i}\|_{H^{-1}(B)} \le C\sum_{j=1}^N \|\frac{\partial \tilde{v}}{\partial x_j}\|_{H^{-1}(\Omega)}.$$

Clearly it follows in the same way that

$$\|v\|_{H^{-1}(B)} \le C\|\tilde{v}\|_{H^{-1}(\Omega)}.$$

This proves the first inequality of the proposition. Since J is bounded, the second inequality of the proposition follows.□

5. The Lemma for the unit ball

We want to prove that for the unit ball B in R^N

$$\|u\|_{L^2(B)} \le C(\|u\|_{H^{-1}(B)} + \sum_{j=1}^N \|\frac{\partial u}{\partial x_j}\|_{H^{-1}(B)}).$$

Let $\eta \in \mathcal{D}(B)$ be such that $\eta(x) = 1$ for $|x| \le 1/3$ and $\eta(x) = 0$ for $|x| \ge 1/2$. Then

$$\|\eta u\|_{L^2(B)} = \|\eta u\|_{L^2(R^N)} \le \|\eta u\|_{X(R^N)}.$$

Now

$$\|\eta u\|_{X(R^N)} \le \sup_{\phi\in\mathcal{D}(R^N)} \frac{(\eta u,\phi)}{\|\phi\|_1} + \sum_{j=1}^N \sup_{\phi\in\mathcal{D}(R^N)} \frac{(\frac{\partial}{\partial x_j}(\eta u),\phi)}{\|\phi\|_1} \le C\|u\|_{X(B)}.$$

Hence we need to bound $v = (1-\eta)u \in \mathcal{D}(B_{1/4})$ where $B_{1/4} = \{x|1/4 < |x| < 1\}$. In order to prove that

$$\|v\|_{L^2(B)} \le C\|u\|_{X(B)},$$

we define the following three operators, each of which maps $\mathcal{D}(R^N)$ to $H^1_D(B_{1/4})$, where $H^1_D(B_{1/4}) = \{v \in H^1(B_{1/4})|v(x) = 0, |x| = 1\}$. For $\phi \in \mathcal{D}(R^N)$, define

$$P\phi = \phi(x) + 3\phi\left(\frac{(2-r)}{r}x\right) - 4\phi\left(\frac{(3-2r)}{r}x\right),$$

$$\tilde{P}\phi = \phi(x) - 3\phi\left(\frac{(2-r)}{r}x\right) + 2\phi\left(\frac{(3-2r)}{r}x\right)$$

and

$$\hat{P}\phi = \phi(x) + 3[r/(2-r)]\phi\left(\frac{(2-r)}{r}x\right) - 4[r/(3-2r)]\phi\left(\frac{(3-2r)}{r}x\right).$$

The following relations hold:

$$\frac{\partial}{\partial r}(x_i/r) = 0, \left(x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}\right)r = 0.$$

Also

$$\frac{\partial}{\partial x_i} = \frac{x_i}{r}\frac{\partial}{\partial r} + \sum_{j=1}^N \frac{x_j}{r^2}\left(x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}\right).$$

By construction we have

$$P\frac{\partial\phi}{\partial r} = \frac{\partial}{\partial r}(\tilde{P}\phi)$$

and

$$P\left[\left(x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}\right)\phi\right] = \left(x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}\right)(\hat{P}\phi).$$

Define P^t by $(P^t v,\phi) := (v, P\phi)$ and note that for $v \in \mathcal{D}(B_{1/4})$ we have that $P^t v \in \mathcal{D}(R^N)$ and $P^t v = v$ in B. In fact the support of $P^t v$ is contained in the annulus $1/4 < r < 5/2$. Now we have

$$(\frac{\partial}{\partial x_i}(P^t v),\phi) = -(v, P(\frac{\partial\phi}{\partial x_i})) = -\left(v, P\left[\frac{\partial}{\partial r}\left(\frac{x_i}{r}\phi\right)\right]\right) - \sum_{j=1}^N \left(v, P\left[\frac{x_j}{r^2}\left(x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}\right)\right]\right).$$

Notice that if f is a function such that $f(x) = h(x/r)$ then for any $\alpha > 0, f(\alpha x) = f(x)$, i.e. $f(x)$ is homogeneous of degree 0. Hence for such a function $P[f\phi] = fP\phi$. Taking $f(x) = \frac{x_i}{r}$ we see that $P\left[\frac{x_i}{r}\phi\right] = \frac{x_i}{r}P\phi$. Thus

$$\left(v, P\left[\left(\frac{x_i}{r}\right)\frac{\partial\phi}{\partial r}\right]\right) = \left(v, \frac{x_i}{r}\left(\frac{\partial}{\partial r}\tilde{P}\phi\right)\right)$$

and

$$\left(v, P\left[\frac{x_j}{r}\left(\frac{x_j}{r}\frac{\partial}{\partial x_i} - \frac{x_i}{r}\frac{\partial}{\partial x_j}\right)\phi\right]\right) = \left(v, \frac{x_j}{r}\left(\frac{x_j}{r}\frac{\partial}{\partial x_i} - \frac{x_i}{r}\frac{\partial}{\partial x_j}\right)(\hat{P}\phi)\right).$$

We now see that

$$(\frac{\partial}{\partial x_i}(P^t v), \phi) = \left(\frac{x_i}{r}\frac{\partial v}{\partial r}, (\tilde{P}\phi - \hat{P}\phi)\right) + (\frac{\partial v}{\partial x_i}, \hat{P}\phi).$$

The mapping defined by $y_i = \frac{(2-r)}{r}x_i, \quad r = |x|$, for $x \in B_{1/4}$ (and similarly the mapping $y_i = \frac{(3-2r)}{r}x_i, \quad r = |x|$, for $x \in B_{1/4}$) is smooth. We verify this at the end of this section by computing its Jacobian. Therefore

$$\|P\phi\|_{H^1(B_{1/4})} + \|\tilde{P}\phi\|_{H^1(B_{1/4})} + \|\hat{P}\phi\|_{H^1(B_{1/4})} \le C\|\phi\|_{H^1(R^N)}.$$

Hence

$$\begin{aligned}\|v\|_{L^2(B)} = \|P^t v\|_{L^2(B)} \le \|P^t v\|_{L^2(R^N)} &\le \|P^t v\|_{H^{-1}(R^N)} + \sum_{i=1}^N \|\frac{\partial}{\partial x_i}(P^t v)\|_{H^{-1}(R^N)} \\ \le \quad & \|v\|_{H^{-1}(B_{1/4})} + \sum_{i=1}^N \|\frac{\partial v}{\partial x_i}\|_{H^{-1}(B_{1/4})}.\end{aligned}$$

Thus we finally conclude that

$$\|v\|_{L^2(B)} \le C\|v\|_{X(B)} \le C\|u\|_{X(B)},$$

which completes the proof provided we check the smoothness of the the above-mentioned mappings.

To this end let

$$B_0 = \{x \in R^N | 0 < |x| < 1\}$$

and

$$B_1 = \{x \in R^N | 1 < |x| < 2\}.$$

Define

$$\mathcal{M} : B_0 \mapsto B_1$$

by

$$y_i = \frac{(2-r)}{r}x_i,$$

for $x \in B_0$ and $y \in B_1$. Now

$$b_{ij} := \frac{\partial y_i}{\partial x_j} = \frac{(2-r)}{r}\delta_{ij} - 2\frac{x_i x_j}{r^3}$$

and the Jacobian of $\mathcal{M}$ is

$$J = det(\frac{\partial y_i}{\partial x_j}) = -((2-r)/r)^{N-1}.$$

To see this, note that the vector $\mathbf{v}^1$, with components $\{v_i^1 = x_i/r, i = 1, \ldots, N\}$, is an eigenvector with eigenvalue $\lambda_1 = -1$ of the matrix with entries b_{ij}; i.e.

$$\sum_{j=1}^{N} b_{ij} v_j^1 = \frac{(2-r)}{r} v_i^1 - 2/r v_i^1 \sum_{j=1}^{N} v_j^1 v_j^1 = -v_i^1.$$

Now let $\{\mathbf{v}^k, k = 1, \ldots, N\}$ be N orthonormal vectors, with entries $\{v_i^k, i = 1, \ldots, N\}$ for each k; i.e.

$$\sum_{i=1}^{N} v_i^k v_i^l = \delta_{kl}.$$

Then for $k > 1$

$$\sum_{j=1}^{N} b_{ij} v_j^k = \frac{(2-r)}{r} v_i^k - 2/r v_i^1 \sum_{j=1}^{N} v_j^1 v_j^k = \frac{(2-r)}{r} v_i^k.$$

Hence $\lambda_k = \frac{(2-r)}{r}$, for $k > 1$ and therefore

$$J = det(\frac{\partial y_i}{\partial x_j}) = \lambda_1 \cdots \lambda_N = -((2-r)/r)^{N-1}.$$

6. Appendix I: Existence and uniqueness in the Stokes problem

We will show how (1.2) may be used to easily solve the following Stokes problem: For $\mathbf{f} \in [H^{-1}(\Omega)]^N$ and $g \in L_0^2(\Omega)$ find $\mathbf{u} \in [H_0^1(\Omega)]^N$ such that

$$D(\mathbf{u}, \mathbf{v}) + (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in [H_0^1(\Omega)]^N \tag{6.1}$$

and

$$(\nabla \cdot \mathbf{u}, q) = (g, q) \quad \text{for all} \quad q \in L_0^2(\Omega). \tag{6.2}$$

Here $D(\cdot, \cdot)$ is the Dirichlet integral. Let T be the solution operator for the Dirichlet problem: $T : H^{-1}(\Omega) \to H_0^1(\Omega)$ defined by

$$D(Tf, \phi) = (f, \phi) \quad \text{for all} \quad \phi \in H_0^1(\Omega).$$

The operator T is extended to vectors component-wise. Then it is easy to check that, taking $\mathbf{v} = T\nabla q$ in the first equation, a necessary condition is

$$(-\nabla \cdot T\nabla p, q) = (g - \nabla \cdot Tf, q) \quad \text{for all} \quad q \in L_0^2(\Omega). \tag{6.3}$$

Noting that $D^{1/2}(\mathbf{v}, \mathbf{v})$ is equivalent to the norm on $[H_0^1(\Omega)]^N$ it follows easily that

$$\begin{aligned}(-\nabla \cdot T\nabla q, q)^{1/2} &= (T\nabla q, \nabla q)^{1/2} = D^{1/2}(T\nabla q, T\nabla q) \\ &= \sup_{\mathbf{v} \in [H_0^1(\Omega)]^N} \frac{(q, \nabla \cdot \mathbf{v})}{D^{1/2}(\mathbf{v}, \mathbf{v})} \geq C\|q\|_{L^2(\Omega)} \quad \text{for all} \quad q \in L_0^2(\Omega).\end{aligned}$$

The bilinear form on the left hand side of (6.3) is continuous on $L_0^2(\Omega) \times L_0^2(\Omega)$ and by the last inequality it is coercive. Since $g - \nabla \cdot Tf \in L_0^2(\Omega)$ we may apply the Lax-Milgram Theorem and obtain the existence and uniqueness of $p \in L_0^2(\Omega)$

satisfying (6.3). With this p we can now solve (6.1) with $\mathbf{u} = Tf + T\nabla p$ and it follows that (6.2) is also satisfied. This solves the Stokes problem (6.1), (6.2).

7. Appendix II: Korn's second inequality

Let $\mathbf{u} \in [H^1(\Omega)]^N$ with components u_i and define the "strain tensor" $\epsilon_{ij} = 1/2(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$.

Theorem 7.1. *Let $\Omega \in R^N$ be a connected domain with a Lipschitz boundary. Then there exists a constant C such that,*

$$C(\sum_{i=1}^{N} \|u_i\|^2_{H^1(\Omega)})^{1/2} \leq (\sum_{i=1}^{N} \|u_i\|^2_{L^2(\Omega)})^{1/2} + (\sum_{i,j=1}^{N} \|\epsilon_{ij}\|^2_{L^2(\Omega)})^{1/2},$$

for all $\mathbf{u} \in [H^1(\Omega)]^N$.

Proof. Since smooth functions are dense in $L^2(\Omega)$ and $H^1(\Omega)$ it suffices to consider only smooth vectors $\mathbf{u}$. The theorem follows by noting that, for i, j fixed and any k,

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial \epsilon_{ik}}{\partial x_j} + \frac{\partial \epsilon_{ij}}{\partial x_k} - \frac{\partial \epsilon_{jk}}{\partial x_i}$$

and applying the Theorem 1.1 to $u = \frac{\partial u_i}{\partial x_j}$. $\square$

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