

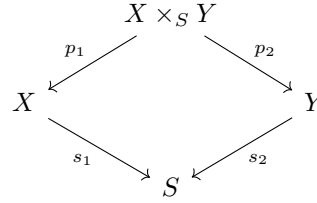
Dictionary

Byeongsu Yu

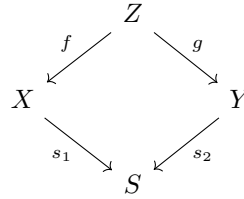
March 5, 2019

Definition 1 (Fibre product). *Let X, Y, S be schemes and X, Y be schemes over S , i.e., there exists a morphism $s_1 : X \rightarrow S, s_2 : Y \rightarrow S$. We define **fibre product** of X and Y over S , denoted by $X \times_S Y$ as following;*

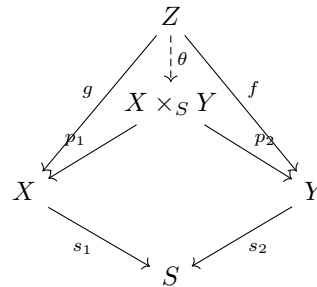
1. $X \times_S Y$ has two morphisms p_1 and p_2 in the below picture making the square in picture commutes.



2. For any Z with commutative diagram as below,



there exists a map $\theta : Z \rightarrow X \times_S Y$ such that triangles in the below diagram commutes.

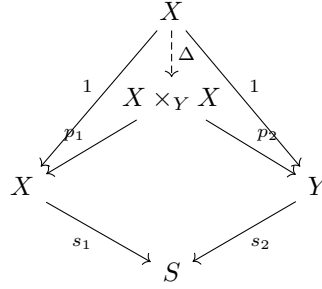


Definition 2 (Closed immersion). *A **closed immersion** is a scheme morphism $f : X \rightarrow Y$ such that*

1. f induces $\text{sp}(X)$ be homeomorphic to a closed subset of $\text{sp}(Y)$
2. $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$, a sheaf map induced by f is surjective.

Definition 3 (Separated Morphism, p. 96). *Let $f : X \rightarrow Y$ be a morphism of schemes. The **diagonal morphism** is the unique morphism $\Delta : X \rightarrow X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X \rightarrow X$ is the identity map of $X \rightarrow X$. We say that the morphism f is **separated** if the diagonal morphism Δ is a closed immersion. In that case we say X is **separated over Y** . A scheme X is separated if it is separated over $\text{Spec } \mathbb{Z}$.*

Diagonal morphism:



Notes that all triangles and squares commutes.

$$f \text{ is separated over } Y \iff \Delta \text{ is closed immersion.} \implies X \text{ is separated over } Y.$$

$$X \text{ is separated} \iff X \text{ is separated over } \text{Spec} \mathbb{Z}.$$

Remark 4 (Counterexample - Affine line with the point p doubled.). *Let k be a field, let $X_1 = X_2 = \mathbb{A}_k^1 = \text{Spec} k[x]$, let P be a point in \mathbb{A}_k^1 corresponding to maximal ideal (x) . Let $U_1 = U_2 = \mathbb{A}_k^1 - \{(0)\}$, and $\psi : U_1 \rightarrow U_2$ is just identification map (which induces isomorphism of locally ringed space between them). Then we can glue X_1 and X_2 via ψ , so we have X which is gluing of X_1 and X_2 . In this case, Δ is not closed immersion; since $\Delta(\text{sp}(X_1)) = \Delta(\text{sp}(X_2))$ is a set containing one generic points. So its closure is whole space, containing other generic points, thus it is not closed in $\text{sp}(X)$.*

Proposition 5 (Proposition 4.1). *If $f : X \rightarrow Y$ is any morphism of affine schemes, then f is separated.*

Corollary 6 (Corollary 4.2). *$f : X \rightarrow Y$ is separated iff $\Delta(X)$ is a closed subset of $X \times_Y X$.*

Theorem 7 (Theorem 4.3 Valutive Criterion of Separatedness). *$f : X \rightarrow Y$ is a morphism of schemes, X is noetherian. Then f is separated if and only if following holds. $\forall k$, field, and $\forall R$, valutive ring with quotient field k , $T := \text{Spec} R, U = \text{Spec} k, i : U \rightarrow T$ induced by the inclusion $R \subseteq k$. For given maps making below commutative diagram,*

$$\begin{array}{ccc}
 U & \longrightarrow & X \\
 i \downarrow & & \downarrow f \\
 T & \longrightarrow & Y
 \end{array}$$

there exists at most one morphism of $T \rightarrow X$ making the whole diagram commutative.

$$\begin{array}{ccc}
 U & \longrightarrow & X \\
 i \downarrow & \nearrow & \downarrow f \\
 T & \longrightarrow & Y
 \end{array}$$

Corollary 8.

1. Open, closed immersion are separated.
2. composition of separated morphisms are separated.
3. product of separated morphisms are separated.
4. $g \circ f$ is separated morphism $\implies f$ is separated.
5. $f : X \rightarrow Y$ is separated $\iff Y$ covered by open subsets U_i such that $f^{-1}(U_i) \rightarrow V_i$ is separated for all i .

Definition 9 (Proper morphism). *$f : X \rightarrow Y$ a morphism of schemes is proper if*

1. f is separated

2. f is of finite type and

3. f is universally closed, i.e.

(a) Closed means image of any closed subsets is closed

(b) Universally closed means it is closed and for any $Y' \rightarrow Y$ a scheme morphism the corresponding morphism $f' : X' \rightarrow Y'$ obtained by base extension is also closed, where $X' = X \times_Y Y'$.

Remark 10 (Example of nonproper morphism). Let k be a field, X be an affine line over k , i.e., $X = \mathbb{A}_k^1$, then $f : X \rightarrow k$ is separated and of finite type but not proper. Since

$$X \times_k X \rightarrow X$$

is not closed, when we sending $V(xy - 1)$, it gives $X - \{0\}$.

Proposition 11 (Valuative Criterion of Properness, theorem 4.7). $f : X \rightarrow Y$ is a morphism of finite type, X is noetherian. Then f is proper if and only if following holds. $\forall k$, field, and $\forall R$, valutive ring with quotient field k , $T := \text{Spec} R$, $U = \text{Spec} k$, $i : U \rightarrow T$ induced by the inclusion $R \subseteq k$. For given maps making below commutative diagram,

$$\begin{array}{ccc} U & \longrightarrow & X \\ i \downarrow & & \downarrow f \\ T & \longrightarrow & Y \end{array}$$

there exists at most one morphism of $T \rightarrow X$ making the whole diagram commutative.

$$\begin{array}{ccc} U & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow f \\ T & \longrightarrow & Y \end{array}$$

Notes that we just change $f : X \rightarrow Y$ be any morphism of scheme to a morphism of finite type, and change separated to proper.

Corollary 12.

1. Closed immersion is proper
2. composition of proper morphisms is proper
3. product of proper morphisms is proper
4. $g \circ f$ is proper morphism $\implies f$ is proper.
5. Any projection morphism is proper.

Definition 13 (Projective n -space). Let Y be a scheme.

1. The **projective n -space over Y** is denoted by \mathbb{P}_Y^n and

$$\mathbb{P}_Y^n = \mathbb{P}_{\mathbb{Z}}^n \times_{\text{Spec} \mathbb{Z}} Y.$$

2. $f : X \rightarrow Y$ is projective $\iff f = i \circ \pi$ where $i : X \rightarrow \mathbb{P}_Y^n$ is closed immersion and $\pi : \mathbb{P}_Y^n \rightarrow Y$ is projection.
3. $f : X \rightarrow Y$ is quasi-projective $\iff f = j \circ \pi$ where $j : X \rightarrow X'$ is open immersion and $\pi : \mathbb{P}_Y^n \rightarrow Y$ is projective morphism.

Example 14. Let A be a ring, S be a graded ring with $S_0 = A$, such that S is finitely generated as an A -algebra by S_1 . Then $\text{Proj} S \rightarrow \text{Spec} A$ is projective morphism.

Since S is quotient of $A[x_0, \dots, x_n]$, thus $A[x_0, \dots, x_n] \rightarrow S$ gives $\text{Proj } S \rightarrow \text{Proj } A[x_0, \dots, x_n] = \mathbb{P}_A^n$, and $A \rightarrow A[x_0, \dots, x_n]$ gives $\text{Proj } A[x_0, \dots, x_n] \rightarrow \text{Spec } A$.

Theorem 15 (Theorem 4.9). *Projective morphism of Noetherian Scheme is proper. Quasi-Projective morphism of Noetherian Scheme is of finite type and separated.*

Proposition 16. *k is algebraically closed, and t is a functor from category of variety to category of scheme, which is fully faithful as we showed. Then, image of t is set of quasi-projective integral schemes over k . And image of set of projective variety is set of projective integral scheme, and image of any variety V is integral separated scheme of finite type over k .*

Definition 17. *Abstract variety is integral separated scheme of finite type over an algebraically closed field k . Complete abstract variety is an abstract variety which is proper over k .*

Modern counterexample which is an abstract variety but not mapped by t , i.e., nonquasiprojective variety is generated by toric variety. But I don't know, Laura said that it may be nonnormal toric variety.

References

- [1] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.