Dictionary

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March 5, 2019

Definition 1 (Fibre product)**.** *Let X, Y, S are schemes and X, Y be schemes over S, i.e., there exists a morphism $s_1 : X \to S, s_2 : Y \to S$. We define* **fibre product** *of X and Y over S, denoted by $X \times_S Y$ as following;*

1. $X \times_S Y$ has two morphisms p_1 and p_2 in the below picture making the square in picture commutes.

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "s_1"'] & & Y \arrow[dl, "s_2"] \\ & S & \end{tikzcd}$$

2. For any Z with commutative diagram as below,

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[dr, "s_1"'] & & Y \arrow[dl, "s_2"] \\ & S & \end{tikzcd}$$

there exists a map $\theta : Z \to X \times_S Y$ such that triangles in the below diagram commutes.

$$\begin{tikzcd} & Z \arrow[d, dashed, "\theta"] \arrow[ddl, "g"'] \arrow[ddr, "f"] & \\ & X \times_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "s_1"'] & & Y \arrow[dl, "s_2"] \\ & S & \end{tikzcd}$$

Definition 2 (Closed immersion)**.** *A* **closed immersion** *is a scheme morphism $f : X \to Y$ such that*

1. *f induces $sp(X)$ be homeomorphic to a closed subset of $sp(Y)$*

2. *$f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$, a sheaf map induced by f is surjective.*

Definition 3 (Separated Morphism, p. 96)**.** *Let $f : X \to Y$ be a morphism of schemes. The* **diagonal morphism** *is the unique morphism $\Delta : X \to X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X \to X$ is the identity map of $X \to X$. We say that the morphism f is* **separated** *if the diagonal morphism Δ is a closed immersion. In that case we say X is* **separated over** *Y. A scheme X is separated if it is separated over Spec $\mathbb{Z}$.*

Diagonal morphism:

$$\begin{tikzcd}
& X \arrow[d, dashed, "\Delta"] \arrow[ddl, "1"'] \arrow[ddr, "1"] & \\
& X \times_Y X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "s_1"'] & & Y \arrow[dl, "s_2"] \\
& S &
\end{tikzcd}$$

Notes that all triangles and squares commutes.

$$f \text{ is separated over } Y \iff \Delta \text{ is closed immersion.} \implies X \text{ is separated over } Y.$$

$$X \text{ is separated} \iff X \text{ is separated over } \mathrm{Spec}\mathbb{Z}.$$

Remark 4 (Counterexample - Affine line with the point p doubled.)**.** *Let k be a field, let $X_1 = X_2 = \mathbb{A}^1_k = Speck[x]$, let P be a point in $\mathbb{A}^1_k$ corresponding to maximal ideal (x). Let $U_1 = U_2 = \mathbb{A}^1_k - \{(0)\}$, and $\psi : U_1 \to U_2$ is just identification map (which induces isomorphism of locally ringed space bewteen them). Then we can glue X_1 and X_2 via ψ, so we have X which is gluing of X_1 and X_2. In this case, Δ is not closed immersion; since $\Delta(sp(X_1)) = \Delta(sp(X_2))$ is a set containing one generic points. So its closure is whole space, containing other generic points, thus it is not closed in $sp(X)$.*

Proposition 5 (Proposition 4.1)**.** *If $f : X \to Y$ is any morphism of affine schemes, then f is separated.*

Corollary 6 (Corollary 4.2)**.** *$f : X \to Y$ is separated iff $\Delta(X)$ is a closed subset of $X \times_Y X$.*

Theorem 7 (Theorem 4.3 Valuative Criterion of Separatedness)**.** *$f : X \to Y$ is a morphism of schemes, X is noetherian. Then f is separated if and only if following holds. $\forall k$, field, and $\forall R$, valuative ring with quotient field k, $T := SpecR, U = Speck$, $i : U \to T$ induced by the inclusion $R \subseteq k$. For given maps making below commutative diagram,*

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\
T \arrow[r] & Y
\end{tikzcd}$$

there exists at most one morphism of $T \to X$ making the whole diagram commutative.

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\
T \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

Corollary 8.

1. *Open, closed immersion are separated.*
2. *composition of separated morphisms are separated.*
3. *product of separated morphisms are separated.*
4. *$g \circ f$ is separated morphism $\implies$ f is separated.*
5. *$f : X \to Y$ is separated $\iff$ Y covered by open subsets U_i such that $f^{-1}(U_i) \to V_i$ is separated for all i.*

Definition 9 (Proper morphism)**.** *$f : X \to Y$ a morphism of schemes is proper if*

1. *f is separated*

2. f is of finite type and

3. f is universally closed, i.e.

   (a) Closed means image of any closed subsets is closed

   (b) Universally closed means it is closed and for any $Y' \to Y$ a scheme morphism the corresponding morphism $f' : X' \to Y'$ obtained by base extension is also closed, where $X' = X \times_Y Y'$.

Remark 10 (Example of nonproper morphism)**.** *Let k be a field, X be an affine line over k, i.e., $X = \mathbb{A}^1_k$, then $f : X \to k$ is separated and of finite type but not proper. Since*

$$X \times_k X \to X$$

is not closed, when we sending $V(xy-1)$, it gives $X - \{0\}$.

Proposition 11 (Valuative Criterion of Properness, theorem 4.7)**.** *$f : X \to Y$ is a morphism of finite type, X is noetherian. Then f is proper if and only if following holds. $\forall k$, field, and $\forall R$, valuative ring with quotient field k, $T := SpecR, U = Speck$, $i : U \to T$ induced by the inclusion $R \subseteq k$. For given maps making below commutative diagram,*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] & Y \end{tikzcd}$$

there exists at most one morphism of $T \to X$ making the whole diagram commutative.

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Notes that we just change $f : X \to Y$ be any morphism of scheme to a morphism of finite type, and change separated to proper.

Corollary 12.

1. *Cosed immersion is proper*

2. *composition of proper morphisms is proper*

3. *product of proper morphisms is proper*

4. *$g \circ f$ is proper morphism $\implies$ f is proper.*

5. *Any projection morphism is proper.*

Definition 13 (Projective n-space)**.** *Let Y be a scheme.*

1. *The* **projective n-space over** *Y is denoted by $\mathbb{P}^n_Y$ and*

$$\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{Spec\mathbb{Z}} Y.$$

2. *$f : X \to Y$ is projective $\iff$ $f = i \circ \pi$ where $i : X \to \mathbb{P}^n_Y$ is closed immersion and $\pi : \mathbb{P}^n_Y \to Y$ is projection.*

3. *$f : X \to Y$ is quasi-projective $\iff$ $f = j \circ \pi$ where $j : X \to X'$ is open immersion and $\pi : \mathbb{P}^n_Y \to Y$ is projective morphism.*

Example 14. *Let A be a ring, S be a graded ring with $S_0 = A$, such that S is finitely generated as an A-algebra by S_1. Then $ProjS \to SpecA$ is projective morphism.*

Since S is quotient of $A[x_0, \cdots, x_n]$, thus $A[x_0, \cdots, x_n] \to S$ gives Proj $S \to$ Proj $A[x_0, \cdots, x_n] = \mathbb{P}^n_A$, and $A \to A[x_0, \cdots, x_n]$ gives Proj $A[x_0, \cdots, x_n] \to \mathrm{Spec}A$.

Theorem 15 (Theorem 4.9)**.** *Projective morphism of Noetherian Scheme is proper.*
Quasi-Projective morphism of Noetherian Scheme is of finite type and separated.

Proposition 16. *k is algebraically closed, and t is a functor from category of variety to category of scheme, which is fully faithful as we showed. Then, image of t is set of quasi-projective integral schemes over k. And image of set of projective variety is set of projective integral scheme, and image of any variety V is integral separted scheme of finite type over k.*

Definition 17. *Abstract variety is integral separated scheme of finite type over an algebraically closed field k. Complete abstract variety is an abstract variety which is proper over k.*

Modern counterexample which is an abstract variety but not mapped by t, i.e., nonquasiprojective variety is generated by toric variety. But I don't know, Laura said that it may be nonnormal toric variety.

References

[1] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.