# Effective Resistance Preserving Isomorphism

GSO Presentation, Oct. 26, 2017

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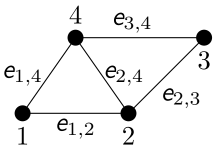
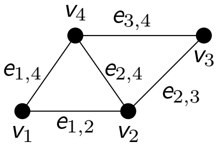



Figure: An example of a graph

### Definition (Graph)

*Let $G = (V, E)$ be a finite (undirected)* ***graph*** *where $V = \{1, 2, \cdots, n\}$ be a finite set of vertices and $E = \{e_{i,j} : i \neq j, 1 \leq i, j \leq n\}/\sim$ be a set of edges connecting $v_i$ with $v_j$, satisfying the equivalence relation $e_{i,j} \sim e_{j,i}$. An weight $w_{i,j}$ is a number in some field (ex. $\mathbb{R}$) assigned for each edge $e_{i,j}$. An* ***unweighted*** *graph is when $w_{i,j} = 1$ for any $i, j \in V$.*

For example, the below (unweighted) graph $G$ has




$$D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

**Definition (Degree, Adjacency, and Laplacian)**

*Suppose* $V = [n]$*. Let an* $n \times n$ *matrix* $D$ *be a* ***Degree matrix*** *of* $G$*, where* $D_{i,i} = \deg v_i$ *for each* $i \in [n]$*.* $\deg v_i$ *means the number of edges connected at* $v_i$*. Let an* $n \times n$ *matrix* $A$ *be* ***Adjacency matrix*** *of* $G$*, where* $A_{i,j} = \begin{cases} w_{i,j} & \textit{if } e_{i,j} \in E \\ 0 & \textit{if } e_{i,j} \notin E \end{cases}$*. Then, we can define the* ***Laplacian*** *of* $G$ *as*

$$L := D - A$$

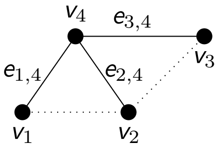



Figure: An example of spanning tree

## Definition (Spanning tree and tree number)

*Let $T$ be a **spanning tree** of given (connected) graph $G$ if $T$ contains every vertices of $G$. The number of spanning trees of $G$, denoted by $\kappa(G)$, is called a **tree number** or **complexity** of $G$ meaning that the number of spanning trees in a unweighted finite graph $G$. For the weighted graph $G$, let $S_G$ be the set of spanning trees of the underlying unweighted graph of $G$. Then, for each $T \in S_G$, $\chi_T := \prod_{e_{ij} \in E(T)} W_{ij}$. Then,*

$$\kappa(G) := \sum_{T \in S_G} \chi_T.$$

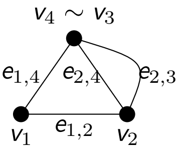



Figure: An example of contraction with respect to $v_3$ and $v_4$

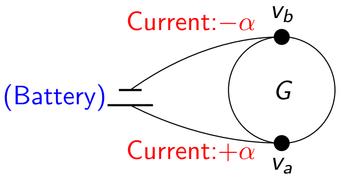
The intuition of contraction is identifying two vertices and deleting an edge between the vertices.

**Definition (Contraction of a graph)**

*The **contraction of** $G$ with respect to $v_i$ and $v_j$, $G_{v_i * v_j}$, is a graph obtained by shrinking the edge $e_{ij}$ to a point, i.e., identifying the vertex $v_i$ with $v_j$ and remove $e_{ij}$. For the unweighted graph, $G_{v_i * v_j}$ is a graph $(V_{new}, E_{new}, W_{new})$ such that*

$$V_{new} = (V \setminus \{v_i, v_j\}) \cup \{v_{n+1}\}, E_{new} = (E \setminus \{e_{i,j}\}) / \sim$$

*where $e_{k,j} \sim e_{k,n+1}, e_{i,k} \sim e_{k,n+1}$ for any $1 \leq k \leq n$.*




### Definition (Effective Resistance, definition in Physics)

*Suppose $G$ is a circuit. Then, an* ***Effective Resistance*** *of two vertices $a$ and $b$ is the total resistance of a circuit when we gives a (positive) current $\alpha > 0$ on a vertex $i$ and the negative current $-\alpha$ on a vertex $j$. Then, from the Ohm's Law, i.e., $V = IR$, where $V$ is electrical potential, $I$ is current, and $R$ is a resistance, the effective resistance as $R_{a,b}$ is*

$$R_{a,b} = \frac{V_a - V_b}{\alpha}.$$

**Theorem (Kirchoff's Law)**

*Suppose $G([n], E)$ is a circuit, $a, b \in V = [n]$. Let $r_{a,b}$ be a **resistance** of an edge $e_{a,b}$. Then, $c_{a,b} = r_{a,b}^{-1}$ is called **conductance** of an edge $e_{a,b}$. Let $V_a$ be an **electrical potential** at the node a, and $I_a$ be the **net current flowing into the circuit** at the node a with constraint $\sum_{i=1}^{n} I_i = 0$. Then,*

$$\sum_{j=1}^{n} c_{i,j}(V_i - V_j) = I_i \text{ for each } i \in V.$$

Note that

$$\sum_{j=1}^{n} c_{i,j}(V_i - V_j) = \sum_{j\neq i}^{n} c_{i,j}V_i - \sum_{j\neq i}^{n} c_{i,j}V_j.$$

Thus, we can reformulate this sum using the matrix notion.

$$\begin{pmatrix} -\sum_{j\neq 1}^{n} c_{1,j} & c_{1,2} & c_{1,3} & \cdots & c_{1,n} \\ c_{2,1} & -\sum_{j\neq 2}^{n} c_{2,j} & c_{2,3} & \cdots & c_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n,1} & c_{n,2} & c_{n,3} & \cdots & -\sum_{j\neq n}^{n} c_{n,n} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \cdots \\ V_n \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ \cdots \\ I_n \end{pmatrix}$$

If we assign $c_{i,j}$ as an weight for $e_{i,j}$, the above big matrix is $L = L(G)$, a Laplacian of given circuit $G$. Thus, we can conclude that

$$L \cdot V = I.$$

However, it is well-known that $L$ is always singular, i.e., not invertible. Thus, there may be infinitely many solutions or no solutions. Some mathematicians concern a solution which related to a graph invariant, which is a tree number.

**Theorem (Temperley, H. N. V. ( 1964)**

*Let $J = (t_{i,j})$ be a $n \times n$ matrix where $t_{i,j} = 1$ for all $i, j \in [n]$. Let $G = ([n], E)$ be a connected finite graph, $L = L(G)$. Then,*

$$\det(L + J) = n^2 \kappa(G)$$

## Theorem (Temperley, H. N. V. ( 1964)

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$$\det(L + J) = n^2 \kappa(G)$$

## Idea of proof.

From the multi linear expansion of determinant, which is the antisymmetric multi linear function, we can get

$$\mu \cdot \delta = \det(L + J),$$

where $\mu$ is a cofactor of $L$ and $\delta$ is sum of all entries in $J$, thus $\delta = n^2$. Now the Kirchoff's matrix tree theorem implies that $\mu = \kappa(G)$, the tree number of $G$. See W. Kook (2010) for more detail. □

To calculate an effective resistance between a vertex $a$ and $b$, we need to solve the above linear system when $I = (I_i)$ as below;

$$I_i = \begin{cases} \alpha & \text{if } i = a \\ -\alpha & \text{if } i = b \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $I \perp 1$, where $1$ is a $n \times 1$ vector consisting of all $1$'s. Then

**Lemma**

*If $I \perp 1$, then $X = (L + J)^{-1} I$ is a solution of $LV = I$.*

## Lemma

*If $I \perp 1$, then $X = (L+J)^{-1}I$ is a solution of $LV = I$.*

## Proof.

It suffices to show that $JX = 0$. Note that

$$(L+J)X = I \implies J(L+J)X = JI = 0,$$

since $I \perp 1$. And, $JL = 0$ by definition of Laplacian matrix, $D - A$. And since $J^2 = nJ$, $JX = 0$. Thus,

$$I = (L+J)(X) = LX + JX = LX.$$

□

Thus, by taking a solution $(L+J)^{-1}I$ for $LV = I$, we can redefine the effective resistance.

**Definition (A new definition of Effective Resistance)**

*Let $(L+J)^{-1} = (g_{i,j})_{i,j\in[n]}$. An effective resistance between two vertices $a, b$ is defined as*

$$R_{a,b} := \frac{V_a - V_b}{\alpha} = g_{aa} + g_{bb} - 2g_{ab}$$

**Proof.**

By direct calculation using $LV = I$,

$$V_a = (g_{aa} - g_{ab})\alpha, V_b = (g_{ba} - g_{bb})\alpha.$$

Plug it in the above definition. □

In fact, W. Kook (2010) shows that the effective resistance can be fully determined by the tree number of graph and its contraction.

**Theorem (W. Kook (2010))**

$$R_{a,b} = \frac{\kappa(G_{a*b})}{\kappa(G)}.$$

**Idea of proof.**

For any $n \times n$ matrix $M$,

$$\det M_{i*j} = \mu_{ii} + \mu_{jj} - \mu_{ij} - \mu_{ji},$$

where $\mu_{ij}$ is the $(i,j)$-cofactor of $M$, i.e., $\mu_{ij} = (-1)^{i+j} \det M_{ij}$, where $M_{ij}$ is obtained by removing $i$-th row and $j$-th column of $M$, using a little bit tricky calculation. $\square$

## Idea of proof.

Let $\mu_{ij}$ be the $(i,j)$-cofactor of $L+J$. Then, $g_{ij} = \frac{\mu_{ij}}{\det(L+J)}$ by the relationship between adjugate matrix and inverse matrix. Thus,

$$
\begin{aligned}
g_{ii} + g_{jj} - g_{ij} - g_{ji} &= \frac{\mu_{ii} + \mu_{jj} - \mu_{ij} - \mu_{ji}}{\det(L+J)} = \frac{\det(M_{i*j})}{\det(L+J)} \\
&= \frac{\det((L+J)_{i*j})}{n^2\kappa(G)} \text{ from Temperley's theorem} \\
&= \frac{\det(L_{i*j} + J_{i*j})}{n^2\kappa(G)} \\
&= \frac{\kappa(G_{i*j})}{\kappa(G)}
\end{aligned}
$$

(Actually, to show last equality we should checks some details.) □

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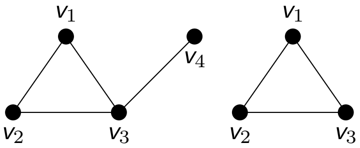
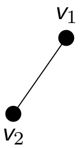


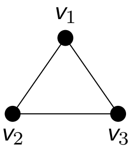

(a) a graph G (b) a graph G'

For example, if we have a graph $G$ and $G'$ as an above figure, we can think $G'$ as a subgraph of $G$. Thus, its effective resistances are

$$Eff(G) = \begin{pmatrix} 0 & \frac{2}{3} & \frac{2}{3} & \frac{5}{3} \\ \frac{2}{3} & 0 & \frac{2}{3} & \frac{5}{3} \\ \frac{2}{3} & \frac{2}{3} & 0 & 1 \\ \frac{5}{3} & \frac{5}{3} & 1 & 0 \end{pmatrix}, \text{ and } Eff(G'|_G) = \begin{pmatrix} 0 & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & 0 \end{pmatrix} = Eff(G'),$$




(a) a graph H




(b) a graph G'

However, this equality cannot be hold in general; for example, let $H$ be a subgraph of $G'$. Then,

$$Eff(H) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq Eff(H|_{G'}) = \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & 0 \end{pmatrix}.$$

**Definition (Subgraph isomorphism)**

*Let $G = (V, E)$ and $G' = (V', E')$ be graphs. Then, $G$ and $G'$ has an* ***isomorphism*** *$f$ if there exists a bijective map $f : V \to V'$ such that*

$$\{e_{f(i),f(j)} : i, j \in V \ s.t. \ e_{i,j} \in E\} = E'.$$

*For any two graph $G$ and $H$, we say $G$ and $H$ have the* ***subgraph isomorphism*** *if there exists a subgraph $G'$ of $G$ such that $G'$ is isomorphic to $H$.*

Actually, a subgraph isomorphism problem, devising an algorithm of judging whether given two graphs have subgraph isomorphism or not, is one of well-known problem.

**Definition (Effective resistance matrix)**

*Let* ***Effective resistance matrix*** *of a graph* $G = ([n], E)$ *be*

$$Eff(G) = (a_{ij}), \;\; where \; a_{ij} = R_{i,j}(G),$$

*where* $i, j \in [n]$. *Let* $G'([k], E')$ *be subgraph of* $G$, *with assumption that* $k \leq n$ *and* $E' \subset E$. *Then, the* ***effective resistance matrix*** *of* $G'$ *embedded on* $G$ *denoted as* $Eff(G'|_G)$, *which is a* $k \times k$ *submatrix of* $Eff(G)$ *including first* $k$ *columns and first* $k$ *rows.*

This is just collection of effective resistances of a graph, using a matrix form.

## Definition (Effective resistance preserving isomorphism)

*Let $G$ and $H$ be graphs. Suppose that there exists a subgraph $G'$ of $G$ such that $G'$ is isomorphic to $H$ as a graph. Then, one can say the graph isomorphism is an effective resistance preserving isomorphism if*

$$Eff(G'|_G) = Eff(H).$$

## Main Question.

Is the effective resistance (matrix) invariant under which subgraph isomorphisms? □

## (Partial) Answer.

Every subgraph isomorphisms between two trees, say $T_1$ and $T_2$, also preserves the effective resistances, i.e.,

$$Eff(T_1|_{T_2}) = Eff(T_2).$$

Also, a subgraph isomorphisms induced from topological operation, called ***attach as a wing***, is weakly preserves the effective resistances, i.e., if $G$ is a graph attaching $H$ to $K$ as a wing on two vertices of $H$, say $a, b$, then

$$Eff(K|_G) = Eff(H \cup e_{a,b})_{a,b} \cdot Eff(K)$$

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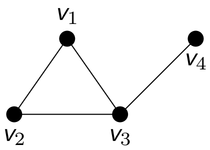
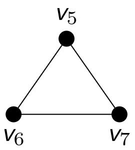
## Definition (Attaching two graphs on a vertex)

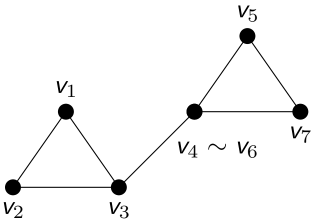
*For any two finite connected graph $G(V_G, E_G), H(V_H, E_H)$, denote $G \cup_{v \sim w} H$ be an **attaching** $G$ **and** $H$ **on a vertex.** This is a union of two graphs with identifying $v \in V_G$ with $w \in V_H$. Formally, $G \cup_{v \sim w} H$ is a graph $(V, E)$ such that $V = V_G \cup V_H / v \sim w$ and $E = E_G \cup E_H / (e_{av} \sim e_{aw}, e_{wb} \sim e_{vb})$ for any $a \in V_G, b \in V_H$.*




(a) a graph G




(b) a graph H




(c) a graph $G \cup_{v_4 \sim v_6} H$

## Lemma (Tree number and cutting the graph)

*For any two finite connected graphs $G, H$,*

$$\kappa(G) \cdot \kappa(H) = \kappa(G \cup_{v \sim w} H),$$

*for any vertices $v$ of $G$ and $w$ of $H$.*

## Idea of proof.

Main idea is that cutting a tree for identified points gives two subtree of given two parts of subgraphs. Also, attaching two trees from each part of two subgraphs are tree in a whole graphs. You can show bijection using

$$(\cdot)_{v \sim w} : S_G \times S_H \to S_{G \cup_{v \sim w} H} \text{ and } cut_v(\cdot) : S_{G \cup_{v \sim w} H} \to S_G \times S_H,$$

where $S_G$ is a set of all spanning trees of $G$. □

## Theorem (Main result 1)

*Let $G$ and $H$ be trees. If $G$ and $H$ have a subgraph isomorphism, then it is an effective resistance preserving isomorphism.*

## Proof.

Without loss of generality, let $G = ([n], E_G)$ and $H = ([k], E_H)$, by renumbering vertices if necessary, thus $G = H \cup_{k \sim k} H'$. It suffices to show that

$$Eff(G)_{i,j} = Eff(H)_{i,j} \text{ for all } 1 \leq i, j \leq k.$$

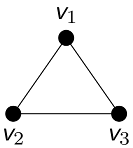
From applying lemma to the tree, we have $\kappa(H_{i*j})\kappa(H') = \kappa(G_{i*j})$ and $\kappa(H)\kappa(H') = \kappa(G)$. Thus,

$$Eff(G)_{i,j} = \frac{\kappa(G_{v_i * v_j})}{\kappa(G)} = \frac{\kappa(H_{v_i * v_j})\kappa(H')}{\kappa(H)\kappa(H')} = \frac{\kappa(H_{v_i * v_j})}{\kappa(H)} = Eff(H)_{i,j}$$

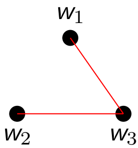
**Definition (Weak version of Effective resistance preserving isomorphism)**

*Let $G$ and $H$ be graphs. Suppose that there exists a subgraph $G'$ of $G$ such that $G'$ is isomorphic to $H$ as a graph. We can say that the subgraph isomorphism is an* ***effective resistance matrix preserving isomorphism up to scalar*** *if there exists $r \in \mathbb{R}$ such that*

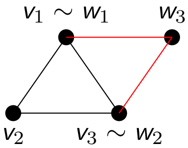
$$Eff(G'|_G) = rEff(H).$$



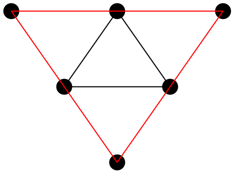

(a) a graph G




(b) a graph H




(c) a graph $G \bigcup_{\substack{v_1 \sim w_1 \\ v_3 \sim w_2}} H$



(d) a graph $G \curlywedge_{w_1, w_2} H$

## Definition (Attaching a graph on two vertices)

*Suppose there exist two finite connected graphs* $G(V_G, E_G), H(V_H, E_H)$, *with* $v_i, v_j \in V_G$, $w_k, w_q \in V_H$ *for some* $1 \leq i, j \leq ||V_G|, 1 \leq k, q \leq |V_H|$. *Then, denote* $G \bigcup_{\substack{v_i \sim w_k \\ v_j \sim w_q}} H$ *be an attaching* $G$ *and* $H$ *on two vertices. This is a union of two graphs with identifying* $v_i \in V_G$ *with* $w_k \in V_H$ *and* $v_j \in V_G$ *with* $w_q \in V_H$ *respectively. Formally,* $G \bigcup_{\substack{v_i \sim w_k \\ v_j \sim w_q}} H$ *is a graph* $(V, E)$ *such that* $V = V_G \cup V_E / \{\substack{v_i \sim w_k \\ v_j \sim w_q}\}$ *and* $E = E_G \cup E_H / \{\substack{e_{av_i} \sim e_{w_k b} \\ e_{av_j} \sim e_{w_q b}}\}$ *for any* $a \in V_G, b \in V_E$, *such that* $e_{av_i}, e_{av_j} \in E_G$ *and* $e_{w_k b}, e_{w_q b} \in E_H$.

Definition is little bit messy, but the main idea is just identifying two pairs of vertices from $G$ and $H$.

## Definition (Attaching a graph as a wing)

*Let $G$ and $H$ be two finite connected graphs, with $w_k, w_q \in V_H$ for some $1 \leq k, q \leq |V_H|$. Then, define $G \curlywedge_{w_k,w_q} H$ be an attaching $H$ to $G$ as a wing by attaching copies of $H$, say $H_i$, $1 \leq i \leq |E_G|$, for every edge in $G$. If $E_G$ has an order such that $E_G = \{e_i\}_{i=1}^{|E_G|}$, with $e_i = (v_j, v_k)$ for some $j < k$,*

$$G \curlywedge_{w_k,w_q} H := \left( \left( \left( G \bigcup_{e_1 \sim (w_k,w_q)} H_1 \right) \bigcup_{e_2 \sim (w_k,w_q)} H_2 \right) \cdots \bigcup_{e_{|E_G|} \sim (w_k,w_q)} H_{|E_G|} \right)$$

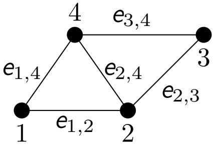
Definition is little bit messy, but the main idea is just attaching $H$ to $G$ with respect to each edges.

### Theorem (Main result 2)

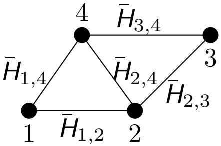
*Let $G, H$ be two finite unweighted connected graphs with $w_a, w_b \in V(H)$ for some $1 \leq a, b \leq |V(H)|$, and $|V(G)| \geq 2$. Then,*

$$Eff(G|_{G \curlywedge_{a,b} H}) = Eff(H \cup w_a w_b)_{a,b} \cdot Eff(G).$$

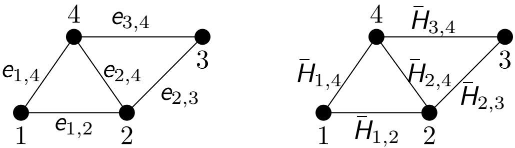
For example, we can think an pseudo-graph, where $\bar{H}_{i,j}$ is assigned to each edge of a graph $G$, as below. Note that $\bar{H}_{i,j}$ is a graph isomorphic to $H \cup e_{w_a,w_b}$.




(a) A graph $G$




(b) A (pseudo)graph $G \curlywedge_{a,b} H$




(a) A graph $G$ (b) A graph $G \curlywedge_{a,b} H$

An algorithm for calculate the effective resistance is to

1) Fix a spanning tree of $G$ (or $G_{a*b}$), say $T$.
2) For each edge $e_{i,j}$ in $T$, count the number of spanning trees in $\bar{H}_{i,j} = H \cup \{e_{i,j}\}$.
3) For each edge $e_{i,j}$ not in $T$, count the number of possible forests in $\bar{H}_{i,j} = H \cup \{e_{i,j}\}$.
4) Sum up all of possible number of choosing trees.
5) Do from 1) to 4) to get $\kappa(G \curlywedge_{a,b} H)$ and $\kappa((G \curlywedge_{a,b} H)_{a*b})$ respectively.

## Idea of proof.

For 1)-step, the number of spanning trees in $G$ is

$$\kappa(G).$$

For 2)-step, for each edge in a spanning tree of $G$, the spanning tree of $\bar{H}_{i,j}$ is

$$\kappa(H \cup w_a w_b).$$

For 3)-step, for $e_{i,j} \notin T$, we should count a forest, consists of two trees, and each tree contains exactly one vertex $i$ or $j$, respectively. Actually, this forest corresponds to a spanning tree containing $e_{i,j}$. Thus, by deletion-contraction principle, the number of such forests is

$$\kappa((H \cup w_a w_b)_{a*b}) = \kappa(H_{a*b}).$$

$\square$

**Idea of proof.**

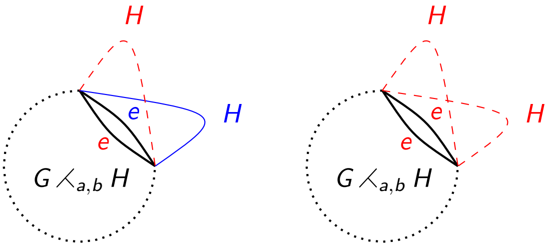
For 4)-step, to sum up, a spanning tree is consists of $|V(G)| - 1$ edges by definition. Also, there are $|E(G)| - |V(G)| + 1$ edges which is not contained in a spanning tree of $G$. Thus,

$$\kappa(G \curlywedge_{a,b} H) = \kappa(G)\kappa(H \cup ab)^{|V(G)|-1}\kappa(H_{a*b})^{|E_G|-|V_G|+1}.$$

□

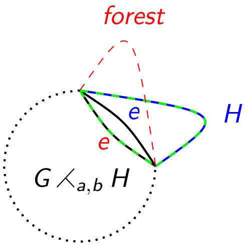
To apply this procedure to $(G \curlywedge_{a,b} H)_{a*b}$, we should deal with the counting problem when a weight of edges are greater than 1.

Consider the case when a multi-edge $e$, i.e., $w(e) = 2$ in $(G \curlywedge a, bH)_{a*b}$ is contained in a spanning tree $T$ of $G_{a*b}$. Then, only one $H \cup e$ should have a spanning tree and the other $H$ should have a forests consisting of two trees, thus the forest should not contain $e$.

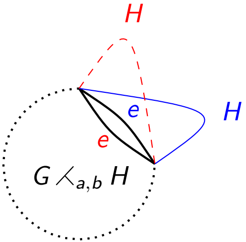



(a) When $e \in T$ (b) When $e \notin T$

The problematic case occurs when we pick a spanning tree $T$ of $H$ containing $e$. and replace $e$ by $e$, and pick a forest from $H$. However, note that the forest $+$ $e$ is a spanning tree in $H$, and the remaining of $T$ is a forests in $H$. Thus, this case is symmetry from choosing color, thus it can be counted by counting $T$ twice.



Also, when we pick a spanning tree of $H$ not containing $e$ with a forests from $H$, it seems should be counted once. However, by symmetry from choosing color, we should count this case twice, for the case when exchanging color.



Since every spanning tree does contain $e$ or does not contain $e$, each case of choosing a spanning tree of $H$ and a forest of $H$ should be counted twice. Since $\kappa(G_{a*b})$, as a complexity of $G_{a*b}$ as a weighted graph, count this case as twice weights, this is well counted in $\kappa(G_{a*b})$.
Thus, From step 1) we can choose $\kappa(G_{a*b})$ spanning trees, from step 2) we can choose

$$\kappa(H \cup ab)^{|V(G)|-2}$$

spanning trees of each $H$ attached to spanning tree of $G_{a*b}$, and from step 3) we can choose

$$\kappa(H \cup ab)_{a*b}^{|E(G)|-|V(G)|+2} = \kappa(H_{a*b})^{|E(G)|-|V(G)|+2}$$

spanning forests of edges not in a spanning tree.

Thus,

$$\kappa(G_{v*w} \curlywedge_{a,b} H) = \kappa(G_{v*w})\kappa(H \cup ab)^{|V(G)|-2}\kappa(H_{a*b})^{|E(G)|-|V(G)|+2}.$$

(It is omitted to differentiating cases whether $e_{v,w} \in E(G)$ or not.
In any case, the above equation holds.)
Now, we can derive the main result.

$$Eff(G \curlywedge_{w_a,w_b} H)_{i,j}$$

$$\begin{aligned}
&= \frac{\kappa\left(\left(G \curlywedge_{w_a,w_b} H\right)_{v_i * v_j}\right)}{\kappa\left(G \curlywedge_{w_a,w_b} H\right)} \text{ from definition} \\
&= \frac{\kappa\left(G_{v_i * v_j}\right)\kappa\left(H \cup w_a w_b\right)^{|V(G)|-2}\kappa\left(H_{w_a * w_b}\right)^{|E_G|-|V_G|+2}}{\kappa(G)\kappa\left(H \cup w_a w_b\right)^{|V(G)|-1}\kappa\left(H_{w_a * w_b}\right)^{|E_G|-|V_G|+1}} \\
&\quad \text{from counting results} \\
&= \frac{\kappa\left(H_{w_a * w_b}\right)}{\kappa\left(H \cup w_a w_b\right)} \cdot \frac{\kappa(G_{v_i * v_j})}{\kappa(G)} \\
&= \frac{\kappa\left(\left(H \cup w_a w_b\right)_{w_a * w_b}\right)}{\kappa\left(H \cup w_a w_b\right)} \cdot \frac{\kappa\left(G_{v_i * v_j}\right)}{\kappa\left(G\right)} \\
&\quad \text{from } \left(H \cup w_a w_b\right)_{w_a * w_b} \cong H_{w_a * w_b}, \\
&= Eff(H \cup w_a w_b)_{a,b} \cdot Eff(G)_{i,j}
\end{aligned}$$

## Corollary

*Let $G$ be a finite unweighted connected graph, and $H$ be an unweighted path graph with $k$ vertices, $w_1, \cdots, w_k$, for any $k \geq 3$.*

$$Eff(G|_{G \curlywedge_{w_1, w_k} H}) = \frac{k-1}{k} Eff(G).$$

## Proof.

Note that $H \cup w_1 w_k$ is a cycle with $k$ vertices. Also, $(H \cup w_1 w_k)_{w_1 * w_k}$ is a cycle with $k-1$ vertices. Hence,

$$Eff(H \cup w_1 w_k)_{1,k} = \frac{\kappa((H \cup w_1 w_k)_{w_1 * w_k}}{\kappa(H \cup w_1 w_k)} = \frac{k-1}{k}.$$

Thus, using the above theorem,

$$Eff(G|_{G \curlywedge_{w_1, w_k} H}) = Eff(H \cup w_1 w_k)_{1,k} \cdot Eff(G) = \frac{k-1}{k} Eff(G).$$

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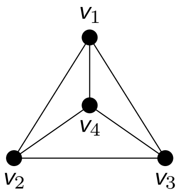
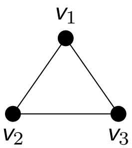
**Proposition**

*There is a effective resistance preserving isomorphism up to scalar which is not induced from an attaching a graph as a wing.*



(a) a graph G



(b) a graph $H$

**Proof.**

Let $G$ be a 1-faces of a 3-simplex and $H$ be a 2-simplex as shown figure. It seems clear that for any connected graph $H'$, $G$ cannot be an attaching $H'$ to $H$ as a wing. However, every subgraph isomorphism preserves effective resistance matrix up to scalar $\frac{3}{4}$, since its effective resistance matrices are

$$Eff(G) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, Eff(H) = \begin{pmatrix} 0 & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & 0 \end{pmatrix}$$

□

## Remark (Implication)

*1) Tree number and its variant, effective resistance are closely related to the topological operation on a graph, a ways of gluing graphs.*

*2) Thus effective resistance preserving isomorphisms may give a topological information of graphs.*

## Remark (Future works)

*1) To figure out the relationship clearly, we need to classify every effective resistance preserving isomorphisms.*

*2) Also, other geometry or topological concepts can be discretized (ex. OKnills work) thus can be applied to the future work.*

Thank you for listening!

References

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