Integrate $|xy(x+y)|_p$ over $\mathbb{Z}_p^2$ with Haar measure.

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- This is one of two exercises in my life related to zeta functions.
- These gives me both pain and joy simultaneously, since this gave my first time painful working...
- Thus I just want to share those two things, emphasizing on the 'joy' part.

Want to understand

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p} |xy(x+y)|_p d\mu =? \tag{1}$$

I saw it from Undergraduate Analysis 3 Final exam, and later, I knew it is related to Sautoy's Zeta function paper [1].
Then, we should deal with the meaning of

$$(1)\ \mathbb{Z}_p,\ (2)\ |\cdot|_p,\ (3)\ \mu \text{ and } (4) \int_{\mathbb{Z}^p\times\mathbb{Z}^p}.$$

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Definition ($\mathbb{Q}_p$)

For any $x \in \mathbb{Q}$, $x = p^m \frac{a}{b}$ for some unique $m \in \mathbb{Z}$ with $(a, b) = 1$. Define

$$ord_p(x) := m, |x|_p := \frac{1}{p^m}.$$

$|\cdot|_p$ is a non-archimedean norm, i.e.,

1. $|x|_p \geq 0, \forall x, |x| = 0 \iff x = 0.$
2. $|xy| = |x||y|.$
3. $|x + y| \leq \max\{|x|, |y|\}$ (not usual Triangle inequality.)

And $d(x, y) = |x - y|_p$ induces metric on $\mathbb{Q}$.

Definition

$\mathbb{Q}_p$ is a completion of $\mathbb{Q}$ with $|\cdot|_p$, i.e., set of all equivalent classes of Cauchy sequence.

Thus any $x \in \mathbb{Q}_p$, x has the unique Laurent series expansion, i.e.,

$$x = \sum_{i=m} a_i p^i, a_m \neq 0,$$

where $m = ord_p(x), a_i \in \{0, 1, \cdots, p-1\}, \forall i \geq m$.
Exercise: What is Laurent expansion of -1?
Exercise: It is totally disconnected but locally compact.

Definition (p-adic integer)

$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x_p| \leq 1\}$.

Why? Any expansion of p-adic integer starts with p^m with $m \geq 0$. So it has no Laurent part.

Proposition

$\mathbb{Z}_p$ is open and closed, compact, and decomposed with disjoint union of $(p\mathbb{Z}_p + 0), \cdots, (p\mathbb{Z}_p + p - 1)$.

Proof.

$\mathbb{Z}_p$ is actually closed ball. So it suffices to show that boundary is open. Take $\epsilon < 1$. Then, $\forall x \in S_{0,1}$, $B_{x,\epsilon} \subseteq S$, since

$$\epsilon = |x - y| < |x - 0| = 1 \implies |x| = |y|.$$

for any $y \in B_{x,\epsilon}$. (Compare coefficients.) □

Proof.

To see compact, let (x_n) be sequence in $\mathbb{Z}_p$. By pigeonhall principle, $\exists b_0 \in \{0, 1, \cdots, p-1\}$ infinitely many elements of (x_n) has zeroth digit (coefficient of 1) is b_0. Take x_{n_0} be the smallest index elements from those infinite elements. Then use induction to get subsequence converging to $x = \sum_{i=0}^{\infty} b_i$. $\square$

Decomposition part is clear from the above proof.
If you love algebra more, then you can construct $\mathbb{Q}_p$ using DVR, $\mathbb{Z}_p$ m-adic completion. Maybe next Gigem I will deal with such algebra things..

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Definition

Topological group G is a group with topology such that multiplication and inverse map is continuous.

Then, it is known that if G is abelian and locally compact, then $\exists$ nonzero translation invariant measure μ (which is called "Haar measrue") which is unique up to scalar. Also it has good properties that

1. $f : G \to \mathbb{C}$ is continuous $\implies$ f is μ-integrable.
2. $\int_G f(x)d\mu = \int_G f(gx)d\mu$, for all $g \in G$.
3. Every borel subsets of G is μ-measurable.
4. $\mu(A)$ is finite $\iff$ A is compact borel open nonempty subset.

From this, let μ be Haar measure for $\mathbb{Q}_p$such that

$$\mu(\mathbb{Z}_p) = 1.$$

Exercise. What is $\mu(p\mathbb{Z}_p)$? $1/p$, since

$$\mathbb{Z}_p = (p\mathbb{Z}_p + 0) \cup \cdots \cup (p\mathbb{Z}_p + p - 1)$$

and $\mu(p\mathbb{Z}_p + i) = \mu(p\mathbb{Z}_p)$ from translation invariance.
By similar way, $\mu(p^m\mathbb{Z}_p) = \frac{1}{p^m}$.

How can we integrate on $\mathbb{Z}_p$? Notes that image of $f : \mathbb{Z}_p \to \mathbb{C}$ is countable, thus we can take a level set, then preimage of level set is closed, thus borel measurable set. For example, if A is a measurable set and $c \in im(f)$, then

$$A_f(c) := \{x \in A : f(x) = c\}$$

thus

$$\int_A f(x) d\mu = \sum_{c \in im(f)} \int_{A_f(c)} f(x) d\mu = \sum_{c \in C} c\mu(A_f(c)).$$

Example: $\int_{\mathbb{Z}_p} |x^d|^s d\mu = \frac{p-1}{p-p^{-ds}}$, for $d \in \mathbb{N} \cup \{0\}, s \in \mathbb{R}_{\geq 0}$ To see this, notes that

- $|x^d|^s = 1$ for $x \in \mathbb{Z}_p - p\mathbb{Z}_p$,
- $|x^d|^s = \frac{1}{p^{ds}}$ for $x \in p\mathbb{Z}_p - p^2\mathbb{Z}_p$,

and so on, thus

$$\begin{aligned}
\int_{\mathbb{Z}_p} |x^d|^s d\mu &= 1 \cdot \mu(\mathbb{Z}_p - p\mathbb{Z}_p) + \frac{1}{p^{ds}} \mu(p\mathbb{Z}_p - p^2\mathbb{Z}_p) + \cdots \\
&= 1 \cdot (1-p) + \frac{1}{p^{ds}} \left(\frac{1}{p} - \frac{1}{p^2} \right) + \frac{1}{p^{2ds}} \left(\frac{1}{p^2} - \frac{1}{p^3} \right) + \cdots \\
&= \left(1 + \frac{1}{p^{ds+1}} + \frac{1}{p^{2ds+2}} + \cdots \right) \left(1 - \frac{1}{p} \right) \\
&= \left(1 - \frac{1}{p} \right) \frac{1}{1 - p^{-ds-1}} = \frac{p-1}{p-p^{-ds}}.
\end{aligned}$$

And, to deal with integration over $\mathbb{Z}_p^2$, we need the Fubini-Tonelli theorem. (Notes that $(\mathbb{Q}_p, \mu)$ is σ-finite.)

Theorem (Fubini's theorem for $p-$adic version)

Suppose $f(x,y) : \mathbb{Q}_p^{n+m} \to \mathbb{R}$ is integrable, such that $\int_{\mathbb{Q}_p^{n+m}} f(x,y)d\mu_{x,y} < \infty$. Then, for almost every $y \in \mathbb{Q}_p^m$,

1. *The slice f^y is integrable on $\mathbb{Q}_p^n$.*
2. *The function defined by $\int_{\mathbb{Q}_p^n} f^y(x)d\mu_x$ is integrable on $\mathbb{Q}_p^m$*
3. $\int_{\mathbb{Q}_p^n}\left(\int_{\mathbb{Q}_p^m} f(x,y)d\mu_y\right)d\mu_x = \int_{\mathbb{Q}_p^{n+m}} f(x,y)d\mu_{x,y} = \int_{\mathbb{Q}_p^m}\left(\int_{\mathbb{Q}_p^n} f(x,y)d\mu_x\right)d\mu_y$

Proof.

Quite standard procedure on $(\mathbb{Q}_p, \mu)$ and its product measure. □

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Now we are ready for dealing with

$$\int_{\mathbb{Z}_p^2} |xy(x+y)|_p d\mu.$$

From Fubini-Tonelli,

$$\int_{\mathbb{Z}_p^2} |xy(x+y)|_p d\mu = \int_{\mathbb{Z}_p} |x|^s \int_{\mathbb{Z}_p} |y|^s |x+y|^s d\mu_y d\mu_x.$$

First of all, fix $x = \sum_{i=l}^{\infty} a_i p^i$ with $a_l \neq 0$. Then, $|x| = p^{-l}$. Now we can decompose y-part of $\mathbb{Z}_p$ as follow; for any $y \in \mathbb{Z}_p$ with $y = \sum_{i=k}^{\infty} b_i p^i$ with $b_k \neq 0$,

1. $|x| > |y| \implies |x+y| = |x|$
2. $|x| < |y| \implies |x+y| = |y|$
3. $|x| = |y|$ and $a_l + b_l = p$ implies $|x+y| < p^{-l}$, otherwise $|x+y| = |y| = |x|$.

To see this, compare Laurent expansions.

From above, we can decompose $\mathbb{Z}_p$ of y as below;

1. $\{y \in \mathbb{Z}_p : |x| > |y|\} = \{y = \sum_{i=k}^{\infty} b_i p^i, b_k \neq 0, k > l\} = p^{l+1}\mathbb{Z}_p$.
2. $\{y \in \mathbb{Z}_p : |x| < |y|\} = \{y = \sum_{i=k}^{\infty} b_i p^i, b_k \neq 0, k < l\} = \mathbb{Z}_p - p^l\mathbb{Z}_p$.
3. $\{y \in \mathbb{Z}_p : |x| = |y|\} = p^l\mathbb{Z}_p - p^{l+1}\mathbb{Z}_p$.

And third one can be decomposed with

- $(p^{l+1}\mathbb{Z}_p + (p - a_l))$, giving $|x + y| < p^{-l}$
- $p - 2$ components of translation of $p^{l+1}\mathbb{Z}_p$ giving $|x + y| = p^l$.

Also, $(p^{l+1}\mathbb{Z}_p + (p - a_l))$ is decomposed with $(p^{l+2}\mathbb{Z}_p + (p - a_{l+1}))$ and other $p - 2$ translation of $(p^{l+2}\mathbb{Z}_p)$ part, so on.

Thus finally, we have

1. $\{y \in \mathbb{Z}_p : |x| > |y|\} = p^{l+1}\mathbb{Z}_p$.
2. $\{y \in \mathbb{Z}_p : |x| < |y|\} = \mathbb{Z}_p - p^{l}\mathbb{Z}_p$.
3. $\{y \in \mathbb{Z}_p : |x| = |y|, |x+y| = p^{-r}\} = p-2$ part of $p^{r+1}\mathbb{Z}_p$, for every $r \in \{l, l+1, \cdots\} \subset \mathbb{Z}$.

So do the first part of integration, case $|x| > |y|$. Then $|x+y| = |x|$, thus

$$\int_{p^{l+1}\mathbb{Z}_p} |y|^s |x+y|^s d\mu = |x|^s \int_{p^{l+1}\mathbb{Z}_p} |y|^s d\mu$$

Then use partition $p^{l+1}\mathbb{Z}_p - p^{l+2}\mathbb{Z}_p$, $p^{l+2}\mathbb{Z}_p - p^{l+3}\mathbb{Z}_p$, $\cdots$, level set of $|y|^s = \frac{1}{p^{(l+1)s}}, \frac{1}{p^{(l+2)s}}, \cdots$, thus

$$\begin{aligned}
&=|x|^s \sum_{j=1}^{\infty} \frac{1}{p^{(l+j)s}} \mu \left(p^{l+j}\mathbb{Z}_p - p^{l+j+1}\mathbb{Z}_p \right) \\
&=|x|^s \sum_{j=1}^{\infty} \frac{1}{p^{(l+j)s}} \left(1 - \frac{1}{p} \right) \mu \left(p^{l+j+1}\mathbb{Z}_p \right) \\
&=|x|^s \sum_{j=1}^{\infty} \frac{1}{p^{(l+j)s}} \left(1 - \frac{1}{p} \right) \frac{1}{p^{l+j+1}}
\end{aligned}$$

$$
\begin{aligned}
=&|x|^s \sum_{j=1}^{\infty} \frac{1}{p^{(l+j)s}} \left(1 - \frac{1}{p}\right) \frac{1}{p^{l+j+1}} \\
=&|x|^s \frac{1}{p^{ls}} (1 - p^{-1}) p^{-l-1} \sum_{j=1}^{\infty} p^{-j(s+1)} \\
=&|x|^{2s+1} \left(\frac{1}{p} - \frac{1}{p^2}\right) \sum_{j=1}^{\infty} p^{-j(s+1)} \\
=&|x|^{2s+1} \left(\frac{1}{p} - \frac{1}{p^2}\right) \left(\frac{p^{-s-1}}{1 - p^{-s-1}}\right) \\
=&|x|^{2s+1} \left(\frac{1}{p} - \frac{1}{p^2}\right) \left(\frac{1}{p^{s+1} - 1}\right).
\end{aligned}
$$

For the second part, $|x| < |y|$, we should integrate over $\mathbb{Z}_p - p^l\mathbb{Z}_p$, which has decomposition $\mathbb{Z}_p - p\mathbb{Z}_p$, $p\mathbb{Z}_p - p^2\mathbb{Z}_p$, $\cdots, p^{l-1}\mathbb{Z}_p - p^l\mathbb{Z}_p$, thus

$$\begin{aligned}\int_{\mathbb{Z}_p - p^l\mathbb{Z}^p} |y|^s|x+y|^s d\mu &= \int_{\mathbb{Z}_p - p^l\mathbb{Z}^p} |y|^{2s} d\mu \\ &= \sum_{j=0}^{l-1} \int_{p^j\mathbb{Z}_p - p^{j+1}\mathbb{Z}_p} |y|^{2s} d\mu \\ &= \sum_{j=0}^{l-1} \frac{1}{p^{2js}} \mu\left(p^j\mathbb{Z}_p - p^{j+1}\mathbb{Z}_p\right)\end{aligned}$$

$$= \sum_{j=0}^{l-1} \frac{1}{p^{2js}} \mu\left(p^j\mathbb{Z}_p - p^{j+1}\mathbb{Z}_p\right)$$

$$= \sum_{j=0}^{l-1} \frac{1}{p^{2js}} \left(1 - \frac{1}{p}\right) \mu\left(p^{j+1}\mathbb{Z}_p\right)$$

$$= \sum_{j=0}^{l-1} \frac{1}{p^{2js}} \left(1 - \frac{1}{p}\right) \frac{1}{p^{j+1}}$$

$$= \left(\frac{1}{p} - \frac{1}{p^2}\right) \sum_{j=0}^{l-1} \frac{1}{p^{j(2s+1)}}$$

$$= \left(\frac{1}{p} - \frac{1}{p^2}\right) \frac{1 - p^{-(l-1)(2s+1)}}{1 - p^{-2s-1}} = \left(\frac{1}{p} - \frac{1}{p^2}\right) \frac{1 - |x|^{2s+1} p^{(2s+1)}}{1 - p^{-2s-1}}$$

For the rest part, for any $r \in \{l, l+1, \cdots, \} \subset \mathbb{Z}$, $|x+y| = p^{-r}$ for $(p-2)$ parts of $p^{r+1}\mathbb{Z}_p$, hence if we denote A for such parts for fixed r, then from assumption $|y| = p^{-l} = |x|$,

$$\int_A |y|^s|x+y|^s = |x|^s(p-2)\int_{p^{r+1}\mathbb{Z}_p} p^{-r}d\mu = |x|^s(p-2)\frac{1}{p^{2r+1}}$$

Thus,

$$\int_{\substack{p^l\mathbb{Z}_p - p^{l+1}\mathbb{Z}_p\\ |x|=|y|}} |y|^s|x+y|^s$$

$$\begin{aligned}
=&|x|^s(p-2)\frac{1}{p}\sum_{r=l}^{\infty}\frac{1}{p^{2r}} = |x|^s(p-2)\frac{1}{p}\frac{p^{-2l}}{1-p^{-2}}\\
=&|x|^{s+2}(p-2)\frac{1}{p}\cdot\frac{1}{1-p^{-2}}
\end{aligned}$$

Thus, for fixed $|x|^s$, the inside integral value is

$$\int_{\mathbb{Z}_p} |y|^s|x+y|^s d\mu_y = |x|^{2s+1}\left(\frac{1}{p}-\frac{1}{p^2}\right)\left(\frac{1}{p^{s+1}-1}\right) + \left(\frac{1}{p}-\frac{1}{p^2}\right)\frac{1-|x|^{2s+1}p^{(2s+1)}}{1-p^{-2s-1}} + |x|^{s+2}(p-2)\frac{1}{p}\cdot\frac{1}{1-p^{-2}}$$

i.e.,

$$= C_1(p)|x|^{2s+1} + C_2(p)|x|^{s+2} + C_3(p).$$

where $C_1(p) = \left(\frac{1}{p}-\frac{1}{p^2}\right)\left(\frac{1}{p^{s+1}-1}-\frac{p^{2s+1}}{1-p^{-2s-1}}\right)$, $C_2(p) = \frac{p-2}{p(1-p^{-2})}$,
and $C_3(p) = \left(\frac{1}{p}-\frac{1}{p^2}\right)\frac{1}{1-p^{-2s-1}}$.

Thus,

$$\int_{\mathbb{Z}_p} |x|^s \int_{\mathbb{Z}_p} |y|^s |x+y|^s d\mu_y d\mu_x$$

$$= C_1(p) \int_{\mathbb{Z}_p} |x|^{3s+1} d\mu + C_2(p) \int_{\mathbb{Z}_p} |x|^{2s+2} d\mu + C_3(p) \int_{\mathbb{Z}_p} |x|^s d\mu$$

$$= \left(1 - \frac{1}{p}\right) \left(C_1(p) \frac{1}{1-p^{-3s-2}} + C_2(p) \frac{1}{1-p^{-2s-3}} + C_3(p) \frac{1}{1-p^{-s-1}} \right)$$

Using simplifyFraction in Matlab you can get...

The numerator part is

$$\begin{aligned} -p^{2s}(2p^s - 2p^{s+1} + p^{3s+1} - 2p^{s+2} + 2p^{s+3} - 2p^{s+5} + p^{s+6} + 2p^4 \\ -p^5 - p^{3s+2} - 3p^{3s+3} - p^{4s+2} + 3p^{3s+4} + p^{4s+3} + 2p^{3s+5} + p^{4s+4} \\ -4p^{3s+6} - p^{4s+5} - p^{5s+4} + p^{3s+7} + p^{5s+5} + 2p^{4s+7} + p^{5s+6} + p^{6s+5} \\ -p^{4s+8} - p^{5s+7} - p^{6s+6} - p^{6s+7} + p^{6s+8}) \end{aligned}$$

and the denominator part is

$$p(p^{2s+3} - 1)(p^{3s+2} - 1)(p^{s+1} - 1)(p + 1).$$

Thank you for listening!

References

- 1) Mihnea Popa, CHAPTER 3. p-ADIC INTEGRATION, `https://sites.math.northwestern.edu/~mpopa/571/chapter3.pdf` Actually, his lecture notes series on Algebraic Geometry is interesting!
- 2) M. P. F. du Sautoy, Zeta functions of groups: the quest for order versus the flight from ennui, Groups St. Andrews 2001 in Oxford. Vol. I, London Math. Soc. Lecture Note Ser. 304, Cambridge Univ. Press, 2003, 150189.