Graph Theory I - Lecture Note

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Abstract

This note is based on the course, Graph Thoery I given by professor Catherine Yan on Spring 2017 at Texas A&M University. Much part of this note was TEX-ed after class. Every blemish on this note is Byeongsu's own.

Contents

1 Terminology

Definition 1.1 (Graph)**.** *A* **graph** $G = (V, E)$*, a pair of two set (or multiset) such that* V *is called* **a set (or mulitset) of vertices** *and* $E \subseteq \left(\binom{V}{2}\right)$ *is called* **a set of edges**.

Notation 1.2 (Chooseing from set with whether allowing repeat or not)**.** $\binom{V}{2}$ *is* a set of two elements subset of V, without allowing repeat. *However,* $\left(\binom{V}{2}\right)$ *is* a set of two elements subset of V, allowing repeat. *Thus, if* $x \in V$, $(x, x) \notin \binom{V}{2}$ *but* $(x, x) \in \left(\binom{V}{2}\right)$.

Definition 1.3 (Digraph and Arc)**.** *A* **digraph** $G = (V, D)$*, a pair of two set (or multiset) such that* V *is called a set (or multiset) of vertices and* D *is a collection of pairs of* V*. In the digraph, we call elements of* D *as* **arc**. *So,* $(x, y) \in D$ *means* $x \to y$*, which is directed, in contrast to the fact that* $\{x, y\} \in E$ *means undirected edge.*

There are several ways of representing a graph.

- $G = (V, E)$
- picture
- matrix

  Definition 1.4 (Order and Size)**.** *Let* $G = (V, E)$ *be a graph and* $|V| = n$*,* $|E| = m$*. Then, the* **order** *of* G *is* n *and the* **size** *of* G *is* m*.*

  Definition 1.5 (Adjacency matrix and Incident Matrix)**.** *Suppose* $|V| = n$*,* $|E| = m$*, with* $V = \{x_i\}_{i=1}^n$ *and* $E = \{e_i\}_{i=1}^n$*, indexed sets. Then, the* **Adjacency matrix** *of* G *is*

$$A_G(i, j) = \begin{cases} 1 \text{ if } \{x_i, x_j\} \in E \\ 0 \text{ otherwise.} \end{cases}$$

  Also, **Incidence matrix** *of* G *is*

$$I_G = \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & \cdots & e_m \end{array} \\ \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \end{array}$$

  such that $I_G(i, j) = \begin{cases} 1 \text{ if } v_i \in e_j \in E \\ 0 \text{ otherwise.} \end{cases}$

  Note that if G contains a loop, then one should specify how to express loop in the incidence matrix, whether gives 2 or 1 for the loop and its vertex.

Example 1.6 (Example of an adjacency matrix and an incident matrix)**.** *For the figure 1, its adjacency matrix and incident matrix are*

$$A_G = \begin{array}{c} \\ x \\ y \\ z \\ w \\ u \end{array} \begin{array}{c} \begin{array}{ccccc} x & y & z & w & u \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}, I_G = \begin{array}{c} \\ x \\ y \\ z \\ w \\ u \end{array} \begin{array}{c} \begin{array}{cccccc} e_{xy}^1 & e_{xy}^2 & e_{xy}^3 & e_{xw} & e_{zu} & e_{uu} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

if we specify that just give 1 *for* (u, e_{uu}).

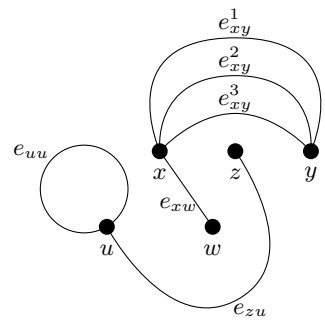


Figure 1: Example graph $G = (V, E)$, with $V = \{x, y, z, w, u\}$ and $E = \{e^1_{xy}, e^2_{xy}, e^3_{xy}, e_{xw}, e_{zu}, e_{uu}\}$.

Definition 1.7 (Complete graph and Cycle)**.** *Define $K_{|V|} = (V, \binom{V}{2})$ be a* **complete graph***. Also, let $C_n = (V, E)$, where $V = \{v_i\}_{i=1}^n$ and $E = \{\{v_i, v_{i+1}\}\}_{i=1}^{n-1} \cup \{\{v_n, v_1\}\}$ be a* **cycle of length** *n.*

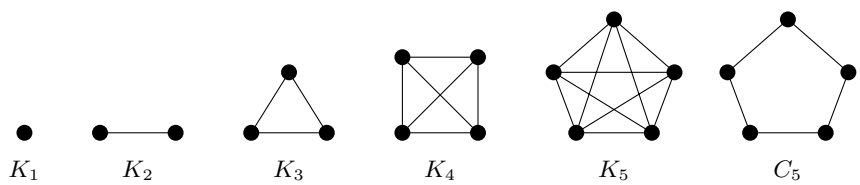


Figure 2: Example of complete graphs and cycle of length 5.

Definition 1.8 (Simple graph)**.** *G is called* **simple** *if it has no loop or multiple edges.*

Notation 1.9 (Abuse of notation)**.** *We use $v \in G$ for abbreviation of $v \in V \in G = (V, E)$.*

Definition 1.10 (Degree of a vertex, neighbor of vertex)**.** *Let $G = (V, E)$ be a graph. For any $x \in V$, the* **degree** *of x, $\deg(x)$, is defined as the number of edges that incident with x.* **Neighbor** *of $V' \subseteq V$ is $N_G(V') \subseteq V$ such that $\forall y \in N_G(V'), \exists x \in V'$ such that $\{x, y\} \in E$, namely, the set of all vertices which are incidents with any vertex in V'.*

Remark 1.11. *Note that in case of G has a loop, then we will count the loop as 2 for degree of a vertex.*

Definition 1.12 (Regular graph)**.** *A graph $G = (V, E)$ is* **regular** *if $\forall x \in V$, $\deg(x) = d$ for some fixed integer d.*

Thus empty graph is regular since it has no edge, therefore, $deg(v) = 0$.

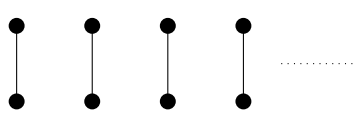
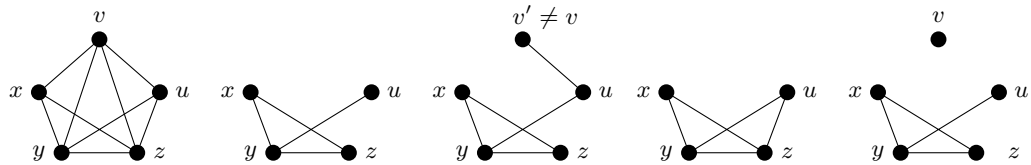


Figure 3: Example of 1-regular graphs; 1- regular graph with vertex set V is called called ***mathcing*** on V

Example 1.13 (Example of regular graph)**.** *1-Regular graph is just union of line segments, as shown in figure 3. 2-regular graph is union of cycles. It is difficult to classify n-regular graph with $n \geq 3$.*

Definition 1.14 (Subgraph, spanning subgraph, and induced subgraph)**.** *Let $G = (V, E)$ be a graph. Then $G' = (V', E')$ is called* **subgraph** *if G' is a graph itself and $V' \subseteq V$ and $E' \subseteq E$. The subgraph G' is called* **spanning subgraph** *if $V = V'$. Also, it is called* **induced** *subgraph if $E' = E \cap \binom{V'}{2}$.*



(a) Graph G (b) a subgraph not induced not spanning (c) not a subgraph (d) a subgraph induced not spanning (e) a subgraph not induced spanning

Figure 4: Example of subgraph, induced subgraph, and spanning subgraph.

Definition 1.15 (Complement of a graph)**.** *Let $G = (V, E)$, a graph. Then define $\bar{G} = (V, \binom{V}{2} - E)$ be a* **complement** *of G.*

Definition 1.16 (Clique of a graph)**.** *Let $G = (V, E)$, a graph. Define* **clique** *as a complete subgraph of G.*

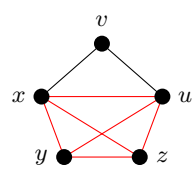


Figure 5: In this graph, the subgraph with red edges is a clique, since it is K_4.

Definition 1.17 (Independent Set)**.** *In $V \in G = (V, E)$, a subset $V' \subset V$ is called* **independent (stable) set** *if induced graph on V' is empty, i.e., a set of pairwise nonadjacent vertices.*

Example 1.18 (Example of independent set)**.** *If we take $\{v, z\} \subset V$ in figure 5, induced graph is empty since it has no edge.*

Definition 1.19 (Bipartite graph)**.** *A graph $G = (V, E)$ is* **bipartite graph** *if $V = X \cup Y$ such that $X \neq \emptyset, Y \neq \emptyset$, and X, Y are independent set. Note that $\cup$ is a notation for disjoint union. Also, a bipartite graph $G = (X \cup Y, E)$ is called* **complete bipartite graph** *if $\forall x \in X, y \in Y$, $\{x, y\} \in E$.*

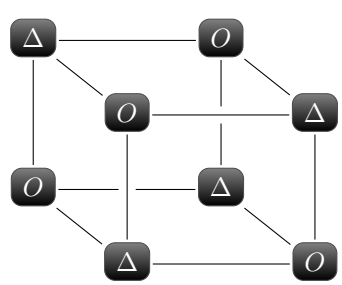


Figure 6: A cubical graph is indeed bipartite graph; think its vertices as disjoint union of Δ and O.

Definition 1.20 (Walk, trail, path, length, distance)**.** *A* **walk** *is a sequence of vertices $v_0 \to v_1 \to \cdots \to v_k$ such that $\{v_i, v_{i+1}\} \in E$. If $v_0 = v_k$, then it is called* **closed walk**. *A* **trail** *is a walk with distinct edges. A* **path** *is a walk with distinct vertices, except $v_0 = v_k$. If a path has a property that $v_0 = v_k$, it is called* **closed walk** *or* **cycle**. *And* **length** *of a walk, a trail or a path is the number of edges in a walk, a trail, or a path respectively. Thus, for $u, v \in V$, uv-walk, uv-trail, or uv-path denotes a walk, a trail or a path starting from u and ending at v. And,* **distance** *between u and v is defined as minimum length of all uv-path.*

Some books represent this concept by edges.

Proposition 1.21. *Let $u, v \in V$ for some graph $G = (V, E)$. Then,*

$$\exists uv\text{-walk} \iff \exists uv\text{-path}.$$

Proof. If uv-path exists, it is also uv-walk. If uv-walk exists, then note that it is finite; say $\{v_i\}_{i=1}^k$ Thus, if we find that $v_n = v_m$ for some distinct $n \leq m < k$, then make a new uv-walk $\{v_i'\}_{i=1}^{k-m+n}$ such that

$$v_i' = \begin{cases} v_i & \text{if } 1 \leq i \leq n \\ v_{m-n+i} & \text{if } n < i \leq k-m+n. \end{cases}.$$

Note that $\{v_i'\}$ is also a uv-walk, since $v_0 = u$, $v_{k-m+n} = v_k = v$, and

$$\{v_n', v_{n+1}'\} = \{v_n, v_{m+1}\} = \{v_m, v_{m+1}\} \in E.$$

Repeat this process until a new walk has no repeated vertices. Since $\{v_i\}_{i=1}^k$ are finite, it has only finitely many repeated vertices, thus it terminates in finite time. The final result is uv-path, since it contain no repeated vertices. □

Definition 1.22 (Girth)**.** *A* **girth** *of G is a length of minimum cycles of a graph G. If it has no cycle, the girth is infinite.*

Definition 1.23 (Connected graph)**.** *A graph $G = (V, E)$ is* **connected** *if $\forall u, v \in V, \exists uv$-walk.*

Definition 1.24 (cut-edge and cut-vertex)**.** *Let $G = (V, E)$ be a graph. If $\exists e \in E$ or $v \in V$ whose deletion will increase the number of connected components of G, then it is called* **cut-edge** *or* **cut-vertex**.

Proposition 1.25. *An edge is a cut-edge if and only if it belongs to no cycle.*

Proof. Let $e \in E$ such that $e = uv$. Assume that G is connected. (Otherwise, take its connected component containing $e = uv$.) Suppose e belongs to no cycle. Then, for any uv-path in G say $\{v_i\}_{i=1}^k$, it contains e; otherwise, let $\{u_i\}_{i=1}^{k+1}$ be a path such that $u_i = \begin{cases} v_i \text{ if } 1 \leq i \leq k \\ u \text{ if } i = k+1. \end{cases}$ Then, it is closed path, thus it is cycle, and it contains $e = uv = \{v_k v_{k+1}\}$, contradicting to the assumption. Thus, by deleting e, there is no uv-path in $G - \{e\}$, since every uv-path should contain e in G. Thus, there is no uv-walk in $G - \{e\}$. Hence, $G - \{e\}$ is disconnected, thus connected component of G increased by the deletion.

Conversely, let $e = uv$ is a cut edge. Also assume G is connected. Suppose there exists a cylce $\{v_i\}_{i=1}^k$ such that $v_1 = u$, $v_{k-1} = v, v_k = u$. Then, think about $G' = (V, E - \{e\})$. Pick $a, b \in V$ and let $\{u_i\}_{i=1}^m$ be ab-walk in G. If $\{u_i\}_{i=1}^m$ contains e, then we also have a new $ab-$walk by replacing e to $\{v_1\}_{i=1}^{k-1}$, say $\{u_i'\}$. Then $\{u_i'\}$ is a ab-walk in G'. Since a, b were arbitrarily chosen, G' is also connected, contradicting the assumption that e is a cut-edge. □

Note that deleting an edge from G increases the number of connected components at most 1. On the other hand, if we delete a vertex from G, the number of connected components may increases many.

Theorem 1.26 (König)**.** *G is bipartite $\iff$ G has no odd cycle.*

Proof. Suppose G is connected. (Otherwise, we can represent this as union of connected bipartite graphs, thus it suffices to show that each connected component is bipartite.)

Let $G = (X \cup Y, E)$ and $\{v_i\}_{i=1}^k$ be a cycle. Without loss of generality, let $v_1 \in X$. Then, $v_k = v_1 \in X$. Since every edges connects a vertex in X to a vertex in Y, $v_2 \in Y$, since $\{v_1 v_2\} \in E$ and $v_1 \in X$. Thus, if $v_i \in X$, then $v_{i+1} \in Y$. Hence, $v_{2l-1} \in X$ and $v_{2l} \in Y$ for any $1 \leq l \leq [\frac{k}{2}]$, where $[\frac{k}{2}]$ is the largest integer not exceeding $\frac{k}{2}$. Since $v_k \in X$, $k = 2l - 1$ for some l. Thus, k is odd, therefore the cycle is even since it contains $(2l-1) - 1 = 2l - 2 = 2(l-1)$ edges.

Conversely, Assume G has no odd cycle. Take $v \in G$. Define $X = \{u \in V : \exists uv\text{-walk which has even length }\}$ and $Y = \{u \in V : \exists uv\text{-walk which is odd length }\}$. Since G is connected, $X \cup Y = V$. Suppose $X \cap Y \neq \emptyset$. Then, $\exists w \in X \cap Y$, so it has two vw-walks, one is odd length and the other is even length. Hence,

their concatenation gives an odd walk in G and by the below lemma, it gives an odd cycle, contradiction. Thus, $X \cap Y = \emptyset$. Also, for any $x_1, x_2 \in X$, $x_1x_2 \in E$, then vx_1-walk is even, x_1x_2 is odd walk (since it is an edge), and x_2v-walk is even. Thus, $v \to x_1 \to x_2 \to v$ is an odd-walks, thus it gives an odd cycle by the lemma, contradiction. Similarly, for any $y_1, y_2 \in Y$, if $y_1y_2 \in E$, $v \to y_1 \to y_2 \to v$ gives an odd walks, thus it contains an odd cycle by the lemma, contradiction. Hence G is bipartite. □

Lemma 1.27. *If there exists an odd closed walk in G, then there exists an odd cycle.*

Proof. First of all, if the length of odd closed walk is 1, then it is loop, thus it is cycle. Suppose it holds for any $2k-1$ length of odd closed walk. Then, let $\{v_i\}_{i=1}^{2k+2}$ be the odd closed walk of length $2k+1$. If it has no repeated vertices, then it is a odd cycle itself, done. If it contains a cycle, say $\{v_i\}_{i=n}^{m}$ we have two closed walks, one is $\{v_i\}_{i=n}^{m}$ and the other is $\{u_i\}_{i=1}^{(2k+2)-m+n}$ such that

$$u_i = \begin{cases} v_i & \text{if } 1 \le i \le n \\ v_{m-n+i} & \text{if } n < i \le (2k+2) - m + n. \end{cases}.$$

Since $(m-n) + ((2k+2) - m + n - 1) = 2k+2$, there are two cases. If $m-n$ is even, then $\{v_i\}_{i=n}^{m}$ is an odd closed walk, thus it contains an odd cycle by the inductive hypothesis, done. If $m-n$ is odd, then $(2k+1) - (m-n) = $ (odd)-(odd) $=$ (even) thus $\{u_i\}_{i=1}^{(2k+2)-m+n}$ is an odd closed walk, thus it contains an odd cycle by the inductive hypothesis. In any case, $\{v_i\}_{i=1}^{2k+2}$ contains an odd cycle. □

1.1 Eulerian Circuit

Definition 1.28 (Eulerian Circuit)**.** *An* **Eulerian Circuit** *is a closed trail containing all edges.*

Usually, circuit denotes a cycle with specifying the starting vertex.

Definition 1.29. *A graph G is called Eulerian if it has an Eulerian circuit.*

Proposition 1.30. *G is Eulerian $\iff$ G is connected and $\forall v \in V$, $\deg(V)$ is even.*

Proof. To see (LHS) $\implies$ (RHS), let $\{v_0\}_{i=1}^{k}$ be the closed trail. Then, every a pair of vertices have a walk which is the part of the closed trail. Hence it is connected. Now suppose $v \in V$ be an arbitrary vertex. Then, it may have a loop, which count 2 when calculating the degree. Otherwise, it may have an edge which is not a loop, say $vu \in E$. In this case, it should have another edge which is not a loop, since the closed trail gives two walks from u to v, one is the edge uv, and the other is going in backward; which means, the complement of uv in the closed trail. In the complement, it should have an edge which is not a loop. Thus, for any *edge* containing v, there is another distinct edge containing v, to keep the closed trail exist. Therefore, $deg(v)$ is even, since it is sum of two times of the number of loops and the sum of all incident edges, which can be paired in each other.

To see (RHS) $\implies$ (LHS), suppose G has a size $m = |E(G)|$. If $m = 0$, then done, since the closed trail contain a one vertex. Suppose it holds for any graph with size less than equal to $m-1$. Now consider G has a size m. Since every vertex's degree is even, it has a cycle C, by the below lemma. Let $G' = G(V, E - C(E))$. Then, the size of any connected component of G' is less then $m-1$. Thus, it has the eulerian circuit. Now, we can get the eulerian circuit of G, by traversing C, and if we met a vertex of each connected component of G' at first, then we traverse the eulerian circuit of the component, and restart traversing C from the point we stopped to traversing the eulerian circuit of the component. This is a closed trail of G, and it contains every edge of G, since any edge is contained in C or G', and this trail contains eulerian circuit of every connected component of G'. Hence, G is Eulerian. □

Remark 1.31 (Proof strategy in Graph theory)**.** *First of all, think constructively. Second, go to extreme case.*

Definition 1.32 (Maximal Path)**.** *A maximal path in a graph G is a path that is not contained in a longer path.*

If G is finite graph, it always have a maximal path, since any path cannot be extended forever.

Lemma 1.33. *If every vertex of a graph G has degree at least 2, then G contains a cycle.*

Proof. Let P be a maximal path in G, and let $v \in P$ be an endpoint of P. If P is a cycle, done. Otherwise, since P is maximal, P contains any neighbors of v, otherwise we can add the edge between v and the neighbor to extend P, contradiction. However, since P is not a cycle, P contains only one edge containing v. However, since $deg(v) \geq 2$, there exists an edge containing v but not in P. Say this edge contains u. Then, u is the neighbor of v, thus it is contained in P. Hence, by adding uv in the P, P has a cycle starting from u to v along P, and go back to u by the added edge. □

Definition 1.34 (Eulerian Trail)**.** *An* **Eulerian trail** *is an open trail containing all the edges.*

Proposition 1.35. *$G = (V, E)$ has an open Eulerian trail start at u and end at v. $\iff$ G is connected and $\forall w \neq u, v$, $\deg(w)$ is even. $\deg(u)$ and $\deg(v)$ are odd.*

Proof. To see (LHS) $\implies$ (RHS) let T be the open trail starts at u and ends at v in G. This open trail gives a walk for any pair of vertices, thus G is connected. Then, $T \cup \{v \to u\} \subseteq G' = (V, E \cup \{vu\})$ gives a Euler circuit of G', thus G' is Eulerian, hence every vertices in G' has even degree. Thus, u and v has even degree in G' by the above proposition, hence it has odd degree in G since G is a graph reducing one edge containing u and v.

To see (RHS) $\implies$ (LHS), just think G' again, and this gives a Euler circuit T', then cutting vu in that circuit gives an open trail starting at u and ending at v. □

Remark 1.36.

1. *Counting each loops contributes 2 to the degree of the vertex.*
2. $\sum_{v \in V} \deg(v) = 2|E(G)|$, *since one edge, say uv, should be counted twice when we calculate* $\deg(u)$ *and* $\deg(v)$.
3. *If $G = (V, E)$ and the order $|V| = n$, the size $|E| = m$, then the average degree is $\frac{\sum_{v \in V} \deg(v)}{n} = \frac{2m}{n}$. Thus, if G is a k-regular graph on n-vertices, then $\sum_{v \in V} \deg(v) = kn = 2m$.*

1.2 Hypercube

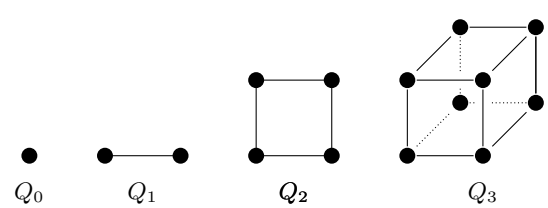


Figure 7: Examples of hyper cube.

Let $V_k = \{(\epsilon_1, \cdots, \epsilon_k) : \epsilon_i = 0 \text{ or } 1\}$. For $x, y \in V$, where $x = (x_1, \cdots, x_k), y = (y_1, \cdots, y_k)$, define $E_k = \{xy : x, y \in V_k \text{ such that } \sum_{i=1}^{k} |x_i - y_i| = 1\}$. Then,

Definition 1.37. *$Q_k = (V_k, E_k)$ is called a* **hypercube**.

Note that $dist(x, y) := \sum_{i=1}^{k} |x_i - y_i|$ is called the ***hamming*** distance.

Proposition 1.38. *Q_k is bipartite.*

Proof. Fix $v \in V_k$ Let $X = \{x \in V_k : dist(v, x) \text{ is even}\}$, $Y = \{x \in V_k : dist(v, x) \text{ is odd.}\}$ Then, for any $x, y \in X$, □

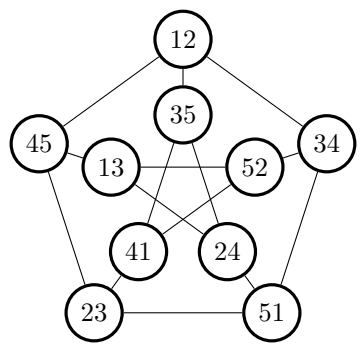


Figure 8: Peterson Graph

1.3 Peterson Graph

Let $S = \{1, 2, 3, 4, 5\}$. Then define

$$V = \{\text{ 2 subsets of } S\}, E = \{v_i \sim v_j : v_i, v_j \in V, v_i \cap v_j = \emptyset\}.$$

Then $G(V, E)$ is the Peterson graph, as shown in the above. Note that

1. G is 3-regular graph.
2. G is not bipartite, since it has an odd cycle.
3. G is Δ-free; there is no 3-cycle.
4. G has 5-cycles. How many 5-cycles?
5. $\forall u, v \in V$, either $u \sim v$ or $|N(u) \cap N(v)| = 1$; by condition of E, if $u \cap v \neq \emptyset$ but $u \neq v$, this implies $|u \cap v| = 1$. Thus, $|u \cup v| = 3$, therefore for $w := S \setminus (u \cup v) \in E$, and w is the only element in $N(u) \cap N(v)$.
6. Thus, the diameter, $\max(dist(u, v)) = 2$.

It has a lot of extremal properties.

1.4 Extremal Properties

1. If G is connected, $|V| = n \implies |E| \geq n - 1$. Actually, $|E(G)| + \#\text{connnected component} \geq n$.

   Proof. Use Induction. If $n = 2$, G is connected if and only if there exists at least an edge for two vertices. Suppose it holds for $n - 1$ and G is connected with n vertices. Then, G has a connected subgraph G' with $|V(G')| = n-1$. By the inductive hypothesis, $|E(G')| \geq n-2$. Let $v \in V(G) \setminus V(G')$. Since G is connected, there is an edge $e \in E(G)$ such that $v \in e$. Thus, $G' \cup e$ gives a connected subgraph of G containing every edges of G and $|E(G' \cup e)| \geq n - 1$.

   To see second proposition, let P be partition of G which consists of each connected components of G. Then, for each partition $p \in P$, $|E(p)| + 1 \geq |V(p)|$. Hence

   $$|E(G)| + \#\text{connnected component} = \sum_{p \in P} (|E(p)| + 1) \geq \sum_{p \in P} |V(p)| = V(G).$$

   □

2. Let $\delta(G)$ be minimum degree, and $\Delta(G)$ be maximum degree.

   Proposition 1.39. $\delta(G) \geq \frac{n-1}{2}$. *Then G is connected.*

Proof. Let $u, v \in V(G)$. Then, $\deg(u), \deg(v) \geq \delta(G) = \frac{n-1}{2}$; To get a contradiction, $N(u) \cap N(v) = \emptyset$. Then, since $|N(u)| = \deg(u), |N(v)| = \deg(v)$,

$$|N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \geq \frac{n-1}{2} + \frac{n-1}{2} = n - 1.$$

However, $N(u) \cap N(v) = \emptyset$ gives $N(u)$ has possibly $n-2$ candidates, excluding itself and v, and $N(v)$ also has possibly $n-2$, vice versa. Thus,

$$n - 1 \leq |N(u) \cup N(v)| \leq n - 2,$$

contradiction. □

3. Assume G has no loop. Then, G has a bipartite subgraph with at least $\frac{|E(G)|}{2}$ edges.

Constructive proof. Use the greedy method. Find an edge $\{u, v\}$. Put $u \in X, v \in Y$. Now add vertex as maximizing the number of crossing edges; for example, if w has more edges incident to vertices in X than incident to those in Y, then w should be in Y. This process must be terminated since this process increases number of edges on the bipartite subgraph of X. Thus, when this process terminated, the degree of each vertex in the bipartite graph is greater than or equal to half of the degree in G. Thus, the number of edges in the bipartite graph is greater than or equal to $\frac{|E(G)|}{2}$. □

Probabilistic proof. A scheme for probabilistic method is to construct probabilistic space reflecting problems. We will show that a desired property happens with positive probability.

For the given problem, first of all, do a random bipartition by building a random subset X in $V(G)$. In other words, the bipartition is $(X, V(G) - X)$. To build X, use random picking such as

$$\forall v \in V(G), P(v \in X) = \frac{1}{2},$$

which is independent between different vertices. Let N be the number of crossing edges from X to $V - X$. For every edge $e \in E(G)$, define

$$X_e = \begin{cases} 1 & \text{if } e \text{ is crossing} \\ 0 & \text{otherwise.} \end{cases}$$

Then, $N = \sum_{e \in E(G)} X_e$. Then,

$$Exp(N) = \sum_{e \in E(G)} Exp(X_e),$$

and since

$$Exp(X_e) = 1 \cdot P(e \text{ is crossing}) + 0 \cdot P(e \text{ is not crossing}) = 1 \cdot (\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}) = \frac{1}{2},$$

since $e = uv$ is crossing if and only if $u \in X, v \in Y$ or $u \in Y, v \in X$. Thus,

$$Exp(N) = \sum_{e \in E(G)} Exp(X_e) = \frac{|E(G)|}{2}.$$

Therefore, we can conclude that there is a structure for which $N = \frac{|E(G)|}{2}$. □

Definition 1.40 (H-free)**.** *A graph G is H-**free** if G has no induced subgraph isomorphic to H.*

Theorem 1.41 (Mantel's theorem)**.** *Let G is a simple graph of order n. If G is Δ-free, then $|E(G)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$.*

Proof. Assume the maximum degree of G is k on a vertex v. Then, $N(v)$ should be independent set, since it is Δ-free. Now we can partitioning the $E(G)$ as two sets; edge containing v and those not containing v. Thus, every edges has at least one end point in $\{v\} \cup (V \setminus N_G(X))$. Thus,

$$|E(G)| \leq \deg(v) + \sum_{y \in V \setminus (\{v\} \cup N(v))} \deg(y) \leq k + (n-k-1)k = k(n-k),$$

since every vertex in G has degree at most k. Thus, by first derivative test, we know that $k(n-k) \leq \frac{n^2}{4}$. And since $|E(G)|$ is an integer,

$$|E(G)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Note that if $|E(G)| = \deg(v) + \sum_{y \in V \setminus (\{v\} \cup N(v))} \deg(y)$, then $V \setminus (N(v) \cup \{v\})$ is also independent set. Also, if

$$\deg(v) + \sum_{y \in V \setminus (\{v\} \cup N(v))} \deg(y) = k(n-k),$$

then it is the complete bipartite graph; think that in the complete bipartite, $V - N(x)$ gives the set of one partition in the bipartite, excluding x. Actually, think $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. It is the case second inequality is actually an equality. If $|E(G)| = \left\lfloor \frac{n^2}{4} \right\rfloor$ in this graph, then n should be even. □

We can generalize this result; if a graph with n vertices is K_m-free, then we can divides the n vertices into $m-1$ equal parts.

1.5 Hamilton Path and cycles.

Definition 1.42 (Hamilton cycle and paths)**.** *The* **Hamilton cycle** *or* **Hamilton path** *is a spanning cycle or path such that every vertices in a graph is contained in the cycle exactly once.*

Definition 1.43 (Ore's property)**.** *Let $G = (V, E)$. If $\forall x, y \in V$, if $x \not\sim y$, then* $\deg(x) + \deg(y) \geq n$.

Theorem 1.44 (Ore's theorem)**.** *If G is simple and has Ore's property, then G has a Hamilton cycle.*

Proof. Note that G is connected; since either $x \sim y$ or $N(x) \cap N(y) \neq \emptyset$, by Ore's property, from the fact that $|N(x) \cup N(y) \setminus \{x, y\}| \leq n-2$. Now suppose we have the longest path with distinct vertices, $P : y_1 \to y_2 \to \cdots \to y_m$.

1. ***Case 1:*** Suppose $y_1 \sim y_m$. If $m = n = V(G)$, done. If $m < n$, there exists $x \notin V(P)$ and $y \in V(P)$ such that $x \sim y$; to see this, suppose x has no edges incident to $y' \in V(P)$. Then, by above argument used for proving connectedness of G, $N(x) \cap N(y') \neq \emptyset$; let $y \in N(x) \cap N(y')$. If $y \in V(P)$, done. Otherwise, $y \to y_1 \to y_2 \to \cdots \to y_m$ gives the longest path with distinct vertices, contradiction. Thus, there exists such y in $V(P)$, say $y = y_k$. In this case, we con construct a path such that

   $$x \to y_k \to y_{k-1} \to \cdots \to y_1 \to y_m \to y_{m-1} \to \cdots \to y_{k+1}.$$

   This is longer than P, contradiction.

2. ***Case 1:*** Suppose $y_1 \not\sim y_m$. Then, $N(y_1) \subseteq V(P)$, $N(y_m) \subseteq V(P)$, otherwise it is not maximal. Idea is this; if we have some y_k and y_{k+1} such that $y_1 \sim y_{k+1}$ and $y_k \sim y_n$, then we can have longer (closed) path such that

   $$y_k \to y_{k-1} \to \cdots \to y_1 \to y_{k+1} \to y_{k+2} \to \cdots \to y_m \to y_k.$$

   If we got this path, then it is contradiction, and also we can go back to case 1 to have the Hamilton cycle.

   How to make our dream come true? Suppose if not; then let $\deg(y_1) = r$, and $N(y) \cap V(P) = \{y_{i_1}, y_{i_2}, \cdots, y_{i_r}\}$. Then, $y_{i_j - 1} \notin N(y_m)$ by our assumption. Thus,

   $$\deg(y_m) = |N(y_m)| \leq m - 1 - r \leq n - 1 - r,$$

thus

$$\deg(y_m) + \deg(y_1) \leq n - 1 - r + r = n - 1,$$

contradicting to Ore's property.

□

Corollary 1.45. *If $\delta(G) \geq \frac{n}{2}$ then G has a Hamilton cycle.*

Proof. $\forall u, v \in V(G)$, $\deg(u) + \deg(v) \geq n$, thus G satisfies Ore's property. □

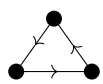
Corollary 1.46. *$\forall x, y \in V$, suppose $\deg(x) + \deg(y) > n - 1$ whenever $x \not\sim y$. Then G has the Hamilton path.*

Proof. In fact, this inequality supports the same argument on the proof of Ore's theorem, i.e., we can replace Ore's property with this condition. □

1.6 Tournament

Definition 1.47 (Tournament)**.** *A* **Tournament** *on n vertices is a complete graph with orientation.*

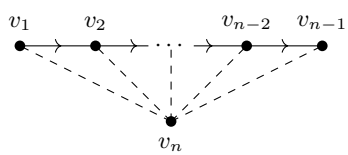
For example,



is an example of tournament. In this case, $i \to j$ can be represented as i ***beats*** j.

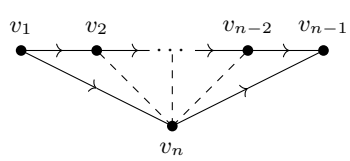
Theorem 1.48. *Let T be a tournament. Then, T has a Hamilton path.*

Proof. Let $|V(T)| = 2$. Then it has Hamilton path, which is T itself. Suppose it holds for $|V(T)| = n - 1$. Now we have T_{n-1}, which is subtournament of T having $n - 1$ vertices. This has a Hamilton path, say $v_1 \to v_2 \to \cdots v_{n-1}$, up to renumbering. Then, we can see as below picture.

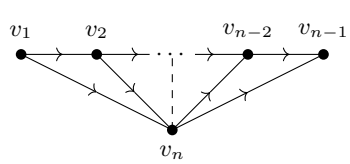


Note that dashed line represent an edge, since a tournament is completet graph, but we don't know it's orientation. There are two cases;

1. ***If*** $v_n \to v_1$ ***or*** $v_{n-1} \to v_1$ ***exists,*** then we are done.
2. ***Otherwise***, we have below pictures.



Now we have another cases. If $v_n \to v_2$, then $v_1 \to v_n \to v_2 \to \cdots \to v_{n-1}$ is a Hamilton path. Or if $v_{n-2} \to v_n$, we have $v_1 \to v_2 \to \cdots \to v_{n-2} \to v_n \to v_{n-1}$ as a Hamilton path. Otherwise, we have another picture;



Thus, by repeating the same argument, the only case left from the argument is $v_i \to v_n$ for $i \leq k$, and $v_n \to v_j$ for $j > k$. Then, take

$$v_1 \to v_2 \to \cdots \to v_k \to v_n \to v_{k+1} \to \cdots \to v_{n-1}$$

as our Hamilton path. Thus, in any case, T has a Hamilton path. □

Theorem 1.49. *The number of Hamilton path on a tournament T is less than $\frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}$*

To see this, we need the Stirling's formula;

Proposition 1.50 (Stirling approximation)**.**

$$n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$$

Proof is just from Gamma distribution approximated by Normal distribution. In this note, we just use it.

Proof of the theorem. Given a Hamilton path $x_1 \to x_2 \to \cdots \to x_n$, this contain arcs (edges in digraph) $e_i : x_{2i-1} \to x_{2i}$, from $i = 1$ to $t = \lfloor \frac{n}{2} \rfloor$. Different Hamilton paths will give different list of arc sequences. Note that it may be restricted to find connecting arc for chosen arc, thus we didn't include connecting arc in counting. In fact, any chosen two distinct arc should be path by using an arc connecting them; since by choosing distinct nonadjacent two arc, we can have subtournament consisting of four vertices, which has a Hamilton path by the above theorem. Thus, let N be the number of Hamilton paths on T. Then, it is below then the number of ways choosing disjoint nonadjacent arcs in T; thus

$$N \leq \binom{n}{2}\binom{n-2}{2}\cdots\binom{n-(2\cdot\lfloor\frac{n}{2}\rfloor)}{2} = \frac{n!}{2^{\lfloor \frac{n}{2} \rfloor}}.$$

□

Theorem 1.51. *There exists a tournament on n vertices with at least $\frac{n!}{2^{n-1}}$ Hamilton paths.*

Proof of the theorem. Take a random tournament. For all $i \neq j$, where $i, j \in [n]$, assume that

$$P(i \to j) = \frac{1}{2}.$$

For any ordering $O := x_1 x_2 \cdots x_n$ of vertices,

$$P(O \text{ is a Hamiltonian path}) = \frac{1}{2^{n-1}},$$

since for each consecutive pair (x_i, x_{i+1}), there are only two cases, where $x_i \to x_{i+1}$ or $x_{i+1} \to x_i$, and the above event is when $x_i \to x_{i+1}$ occurs. Let N be the number of Hamilton paths in T. Note that N is random variable in this case. Then,

$$N = \sum_{\sigma \text{ is ordering of } V} X_\sigma, \text{ where } X_\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is a Hamilton path} \\ 0 & \text{otherwise.} \end{cases}$$

Then, note that $E(X_\sigma) = 1 \cdot \frac{1}{2^{n-1}} + 0 \cdot (1 - \frac{1}{2^{n-1}}) = \frac{1}{2^{n-1}}$. Hence,

$$E(N) = \sum_{\sigma \text{ is ordering of } V} E(X_\sigma) = \frac{n!}{2^{n-1}},$$

from linearlity of the expectation operation. Thus, there exists a tournament T with $\frac{n!}{2^{n-1}}$ Hamilton paths. □

Remark 1.52. *The best known result for this problem is in [1]; it shows that $N \leq c \cdot n^{\frac{3}{2}} \cdot \frac{n!}{2^{n-1}}$.*

2 Trees

Let G be simple graph.

Definition 2.1 (Acyclic graph, forest, and tree)**.** *G is an* **acyclic** *if and only if G has no cycle. Denote an acyclic graph as* **forest***, and connected acyclic graph as* tree*.*

Theorem 2.2. *The followings are equivalent.*

(A) G is connected and acyclic.

(B) G is connected and $|E(G)| = n - 1$.

(C) G is acyclic, $|E(G)| = n - 1$.

(D) $\forall u, v \in V(G)$, $\exists! uv$-path.

Before prove this proposition, we need a lemma.

Lemma 2.3. *Every tree T with at least two vertices has at least two leaves. Deleting a leaves from an n-vertex tree produces a tree with $n - 1$ vertices.*

Proof of the lemma. Take a maximal path on the tree. Two endpoints of the maximal path has no other neighbors except in the path; otherwise, since T is acyclic, we can make the path longer by adding the other neighbor, contradiction. Hence, those two points are leaves in the tree. Also, by deleting a leaf v, it cannot generates a cycle. Also, since it is degree 1, every pair of vertices in $V(T) - \{v\}$ also has a path in $T - v$, since the edge e connecting v and an other point cannot be in the uv-path; if it is included in the uv-path, then it gives repeatition of using e, contradicting to the definition of path. Thus, $T - v$ is also connected and acyclic, therefore it is a tree with $n - 1$ vertices. □

Proof of the theorem.

1. $A \implies B, C$ Use induction. If $n = 1$, then it has no edge, done. Suppose it holds for graph with vertices fewer than n. G has a leaf by the lemma, and $G - v$ is also a tree. Thus, by inductive hypothesis, $E(G - v) = n - 2$, therefore, $E(G) = n - 1$.

2. $B \implies A, C$ It suffices to show that G is connected. Let G' be a subgraph of G deleting an edge from each cycle in G. Then, G' has no cycle. By the proposition 1.25, no edges in a cycle is cut-edge, G' is still connected, but acyclic. Thus $|E(G')| = n-1$, by (A), which is the same as $|E(G)|$. Hence $G = G'$, and G is acyclic.

3. $C \implies A, B$ Let $G_1, \cdots, G_k$ be connected components of G. Note that G_i are connected and acyclic, thus $|E(G_i)| = |V(G_i)| - 1$ by (A), for any $i \in [k]$. Thus,

$$n - 1 = |E(G)| = \sum_{i=1}^{k} |E(G_i)| = \sum_{i=1}^{k} |V(G_i)| - k = n - k$$

   implies $k = 1$.

4. $A \implies D$ Suppose there exists two uv-path, P and Q. If P and Q are distinct, there exists $k, k' \in \mathbb{N}$ such that $P(k) = Q(k')$ but the subpath $P(k) \to P(end)$ and $Q(k') \to Q(end)$ do not share any internal points except $P(end) = v = Q(end)$. This gives a cycle, contradiction. Thus, $P = Q$.

5. $D \implies A$ If G has a cycle, G has a two uv-path for some u, v in the cycle, contradiction.

□

Let T be a tree.

Corollary 2.4.

1. *Every edge is a cut-edge.*

2. *$e + T$ form exactly one cycle, called* **unicycle** *graph.*

3. *Every connected graph has spanning trees.*

Proof. By the proposition 1.25, an edge is cut-edge iff it belong to no cycle. Since T has no cycle, every edge is cut edge. Also, let $e = \{u, v\}$ for some $u, v \in V(T)$. Let uv-path in T is P. Since P is unique uv-path in T, $e + T$ gives a cycle $e + P$. If $e + T$ has more than one cycle, such as $P + e$ and $P' + e$, (note that P' is just any path.) then we can generate cycle in T by connecting endpoint of P to those of P', contradiction. To show last statement, let G' be a subgraph of G deleting an edge from each cycle in G. Then, G' has no cycle. By the proposition 1.25, no edges in a cycle is cut-edge, G' is still connected, but acyclic. Thus, G' is a spanning tree of G. □

Proposition 2.5. *Let G be a simple connected graph, and T be a spanning tree of G. Let $e' \in E(G) \setminus E(T)$. Then, $\exists e \in E(T)$ such that $T + e' - e$ is a spanning tree. Conversely, let $e \in T$ such that there exists a spanning tree T' such that $e \notin E(T')$, then $\exists e' \in T'$ such that $T' - e + e'$ is a spanning tree.*

Proof. Note that $T + e'$ gives exactly one cycle, by the above corollary. Thus, take any cut-edge e in the cycle and delete it. Then, it is an tree containing every vertex in G, since $T + e'$ is unicycle graph. Thus, such e exists.

Conversely, $e + T'$ is unicycle graph, thus there exists $e' \in T'$ such that $T' - e + e'$ is a spanning tree, by the same argument. □

Definition 2.6 (Diameter)**.** *A* **diameter** *of a simple graph G is*

$$diam(G) = \max\{d(u, v) : u, v \in V(G)\}.$$

Proposition 2.7. *G is a simple graph if $diam(G) \geq 3 \implies diam(\overline{G}) \leq 3$.*

Proof. $diam(G) \geq 3 \implies \exists u, v \in V(G)$ such that $dist(u, v) \geq 3$. Thus, $N_G(u) \cap N_G(v) = \emptyset$. Hence, for any $x, y \in V(G)$ x and y are not in at least one of $N_G(u)$ or $N_G(v)$. Thus,

$$(x \in N_{\overline{G}} \text{ or } x \in N_{\overline{G}}) \text{ or } (y \in N_{\overline{G}} \text{ or } y \in N_{\overline{G}})$$

Since $uv \in E(\overline{G})$, $dist_{\overline{G}}(x, y) = 2$ or 3. □

2.1 Chemical Graph Theory

This is a topological indices on Graph.

Definition 2.8 (Wiener index on connected graph G)**.** *The* **Wiener index on connected graph** *G, $D(G)$ is defined as*

$$D(G) := \sum_{u,v \in G} d_G(u, v),$$

where $d(u, v)$ is the distance between u and v, which is minimum length of uv-path.

Theorem 2.9. *Among all trees of n vertices, the minimum of Wiener index is achieved on Star, and the maximum of Wiener index is achieved on path.*

Proof. Note that $d(u, v) = 1$ if $u \sim v$, and $d(u, v) \geq 2$ if $u \not\sim v$. So

$$D(T) \geq n - 1 + 2\left(\binom{n}{2} - (n - 1)\right),$$

where first $n - 1$ comes from edges of T and the last $2\left(\binom{n}{2} - (n - 1)\right)$ comes from the minimal distance of each pair of nonadjacent vertices. And equality holds if every pair of nonadjacent vertices has distance 2, which is satisfied by Star. To show that no other trees achieves this, let u be a leaf of T, using the lemma, and v is the adjacent vertex of u. If all other vertices have distance 2 from u, it must be neighbor of v, thus T is star.

Also, we can prove that $D(T)$ is maximized when T is a path using induction. If $n = 1$, done. Suppose it holds for any number of vertices less than n. Let u be a leaf on T. Then, we know that for some leaf $u \in T$,

$$D(T) = \sum_{u \neq v} d(u, v) + D(T \setminus u).$$

By inductive hypothesis, $D(T \setminus u) \leq D(P_{n-1})$. where P_{n-1} is a path with $n-1$ vertices. Thus it suffices to show that $\sum_{u \neq v} d(u,v)$ is maximized when $T = P_n$. Note that maximum possible distance between u and some vertex is $n-1$, since T has only $n-1$ edges. If u has distance $n-1$ from some vertex, then u has every other distances, such as $1, 2, \cdots n-2$. This is when $T = P_n$. If u has no such vertex, then there exists a repetition of some number in $[n-2]$, and no $n-1$ distance, therefore sum of distances from u to every other vertex is less than that of P_n; thus $T = P_n$. $\square$

2.2 Enumeration of trees

Let $V = \{1, 2, \cdots, n\} := [n]$. Then,

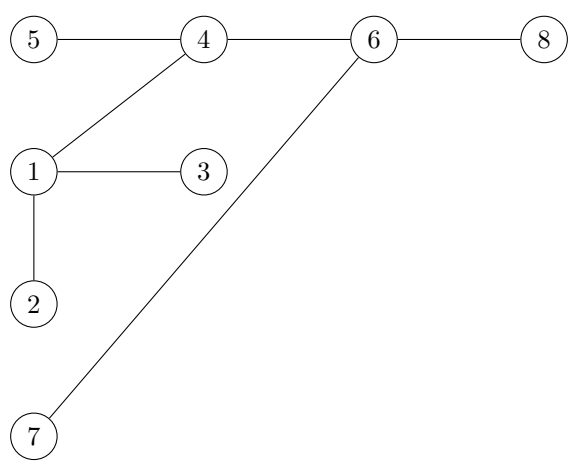
Theorem 2.10 (Cayley's theorem)**.** *The number of all possible trees on the vertices set $[n]$ is n^{n-2}.*

To prove this, we need some tools, called Prüfer code.

Definition 2.11 (Prüfer code)**. Prüfer code** *is an algotihm mapping a tree T to sequences of $[n]$ with length $n-2$. This algorithm is below;*

1. *Find a minimum leaf, say v, with respect to order in V.*
2. *Write down the neighbor of V*
3. *Delete v*
4. *Go to 1 until there are only 2 vertices left.*

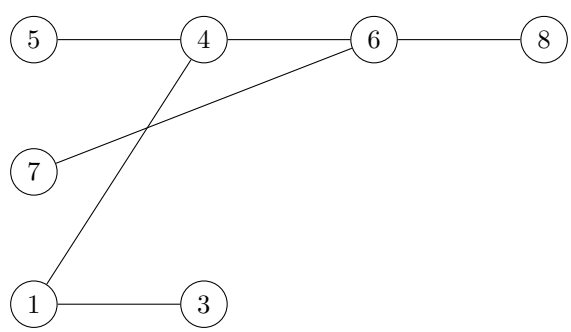
For example, let T be a tree in below.



Then, 2 is the minimum leaf. Thus write

$$a_1 = 1$$

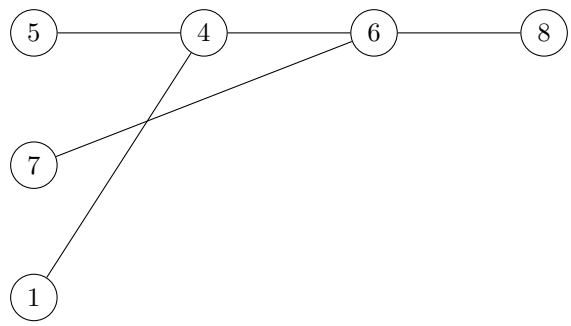
and delete 2.



The minimum leaf is 3. Thus write

$$a_2 = 1$$

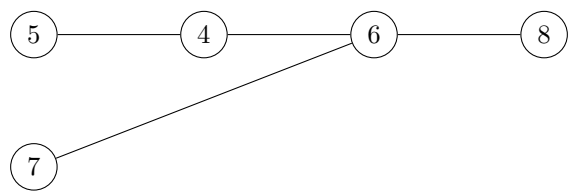
and delete 3.



The minimum leaf is 1. Thus write

$$a_3 = 4$$

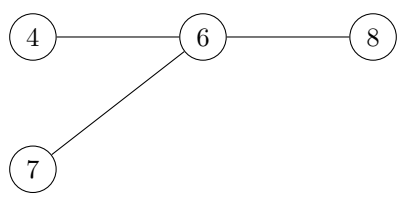
and delete 1.



The minimum leaf is 5. Thus write

$$a_4 = 4$$

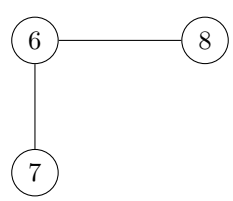
and delete 5.



The minimum leaf is 4. Thus write

$$a_5 = 6$$

and delete 4.



The minimum leaf is 7. Thus write

$$a_6 = 6$$

and delete 7.

Now our Prüfer code is 114466, done.

Note that the total number of Prüfer code is n^{n-2}. Note that leaves are not shown in the code. In general, for any $v \in T$, v appears on Prüfer code with $\deg(v) - 1$ times. Also, recovering a tree from given Prüfer code is given below.

First of all, given $(a_1, a_2, \cdots a_{n-2}) \in [n]^{n-2}$, find vertices not in a set $A = \{a_i\}_{i=1}^{n-2}$. This is our starting set S. Then, take the least number x on S. Now delete x and a_1 by making an edge xa_1. Update

$A = \{a_i\}_{i=2}^{n-2}$, and $S = \begin{cases} (S - \{x\}) \cup \{a_1\} & \text{if } a_1 \notin A \\ (S - \{x\}) & \text{otherwise.} \end{cases}$ Repeat this until A is emptyset. Note that for each step i, $n - i + 1 = |S| + |A|$ and $n - i - 1 = |a|$, thus $|S|$ has at least two elements for each step. The last two elements in S is the last edge added for tree. Then we are done. For example, from our code 114466,

$(a_1, a_2, \cdots, a_6)$	S	added edge in this step
114466	$2, 3, 5, 7, 8$	12 should be added
14466	$3, 5, 7, 8$	13 should be added
4466	$1, 5, 7, 8$	14 should be added
466	$5, 7, 8$	45 should be added
66	$4, 7, 8$	46 should be added
6	$7, 8$	67 should be added
	$6, 8$	68 should be added.

Let $d_1, d_2, \cdots d_n \in \mathbb{N}$ are sequence of degree of each vertex, i.e., $d_i = \deg(i)$ for all $i \in [n]$. Then, $\sum_{i=1}^{n} d_i = 2(n-2)$. From this and using the fact that the number of appearance of i in a Prüfer code is $\deg(i) - 1$, we have such corollary

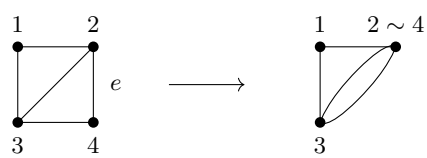
Corollary 2.12. *The number of labeled trees on* $[n]$ *such that* $\deg(i) = d_i$ *is*

$$\binom{n-2}{d_1 - 1, d_2 - 1, \cdots, d_n - 1}.$$

Proof. It is just find a sequence in $[n]^{n-2}$ which contains exactly $d_i - 1$ many i for each $i \in [n]$. □

2.3 Spanning Tree

Let G be a connected; G does not have to be simple. Let $\tau(G)$ be the number of spanning trees of G. Now, we have two operations. Let $e \in E(G)$. Then ***Deletion*** means $G - e$, and contraction means G/e, i.e., remove e and glue two endpoints of e together. For example, below picture is when we get G/e (right graph) from G (left graph).



Proposition 2.13 (Deletion-contraction principle)**.** *Let* $e \in E(G)$ *be not a loop. Then*

$$\tau(G) = \tau(G - e) + \tau(G/e).$$

Proof. Let S be set of all spanning trees of G. Then, E can be partitioned as two set; a spanning tree containing e and a spanning tree doesn't containing e. The number of spanning tree containing e is the same as spanning trees in G/e, since if T is a spanning tree of G, then T/e is a spanning tree of G/e, and vice versa. Also, if T is a spanning tree not having e, then it is a spanning tree of $G - e$, and vice versa. □

Remark 2.14.

1. *Note that empty graph and disconnected graph has* $\tau(G) = 0$. *If* G *is a tree with multi edges,* $\tau(G) = \prod_{i=1}^{n} m_i$, *where* m_i *is the number of multi edge in each* i*-th position of edges.*
2. $\tau(K_n) = n^{n-2}$. *Proof is required.*

2.3.1 Build a spanning tree

The main question is this; given a connected graph G, each edge has a weight $w(e) > 0$. We want a spanning tree with minimum total weight, i.e., $\min_{T \text{ is spanning tree}} \sum_{e \in T} w(e)$.

Definition 2.15 (Kruskal's algorithm). *1. Input $(V(G), E(G))$.*

2. Start at $F = \emptyset$.

3. While $\exists \alpha \in E(G)$ such that $\alpha \notin F$ and that $F \cup \{\alpha\}$ is acyclic, find such an edge with minimum weight. Join the minimum such edge to F.

4. Output (V, F)

Note that it is finite, since it terminates eventually in finite time, since G has only finitely many edges. Also, (V, F) is a tree, since it maintain acyclic property. It is needed to show that Kruskal's algorithm gives the minimum spanning tree.

Claim 2.16. *The output of Kruskal's algorithm is the minimum spanning tree.*

Proof. Let the edges of $T = (V, F)$ be indexed, i.e., $F = \{e_1, \cdots, e_{n-1}\}$, in the order they join F. Let T' be a minimum spanning tree which shares the most number of edges with T. It suffices to show that $T' = T$. If not, there exists $k \in [n-1]$ such that $e_k \notin T'$, such that $e_1 \cdots, e_{k-1} \in T \cup T'$, but $e_i \notin T'$ for $i \geq k$, up to renumbering. Let $\beta \in T' \setminus T$ such that $T'' = T' + e_k - \beta$ is a spanning tree. Such β exists by the theorem about tree. Now there are two cases. Let $w(G)$ be the total weight of a graph G.

Note that $w(T'') \geq w(T')$, since T' is the minimum spanning tree. Thus, $w(e_k) \geq w(\beta)$. Then, by the rule of the algorithm, $w(\beta) \geq w(e_k)$, otherwise β should be chosen in the k-th step of the algorithm. Thus, $w(e_k) = w(\beta)$. By repeating this argument, we can conclude that $w(T') = w(T'')$. Then T'' has more edges in common with T, contradiction. □

Now there is another algotirhm.

Definition 2.17 (Prim's algorithm). *1. Start at any vertex u. Let $V_1 = \{u\}, F_1 = \emptyset$.*

2. For each i-th step, let $\alpha_i = \{x, y\} \in E(G)$ such that $x \in V_i, y \notin V_i$ with minimal weight. Such α_i exists and we can find it, since there exists only finitely many edges.

3. Update $V_{i+1} = V_i \cup \{y\}$ and $F_{i+1} = F_i \cup \{\alpha_i\}$.

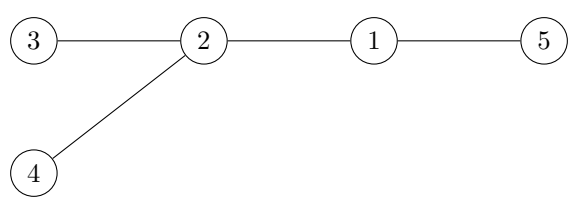
4. Repeat this n times.

5. Output is (V_n, F_n).

Now think about the special case of graph G, where G has vertices on $[n]$ and edges are ordered by the lexicographic order. Then, $Prim(G)$ is an well-defined tree on $[n]$. Thus, we can think of $Prim(\cdot)$ as a map such that

$$Prim : A = \{ \text{ connected graph on } [n]\} \mapsto B = \{ \text{ labeled trees on } [n]\}.$$

Note that $\forall T \in B$, there exist $G \in A$ such that $Prim(G) = T$; put $G = T$. We can ask what the inverse image of T is. Obviously, for any $G \in A$ such that $Prim(A) = T$, $T \subseteq G$. Thus $G = T \cup \{ \text{ other edges}\}$. Let $C = \{ \text{ other edges}\}$. Let's begin with example for some T, as below.



In this example, if $13 \in C$, then $Prim(G) \neq T$, since Prim algorithm should choose 13 instead of $23 \in T$. Thus $13 \notin C$. $14 \notin B$ for the same reason; otherwise, T should contain 14 instead of 24. However, $25, 34, 35, 45$ can be chosen for C, since even these edges are in B, Prim algortihm will choose $15, 24, 15, 15$, respectively. Thus, by this observation, we can conclude that

Observation 2.18. *If $Prim(G) = T$, then $Prim(G \cup e) = T$ if and only if e is the longest one (an edge having the maximum weight) among vertices in the cycle (which is unique) in $T \cup e$.*

From this, we can define some

Definition 2.19 (Externally active vertices)**.** *Let T be a tree on $[n]$ with $E(T) \subset \binom{n}{2}$, where $\binom{n}{2}$ is the set of all 2-subset of $[n]$, having a lexicographical order. Then $\forall e \in \binom{n}{2} \setminus E(T)$, if e is the longest edge in the cycle of $T \cup e$, then we say e is* **externally active** *with respect to T. Denote*

$$EA(T) := \{ \text{ externally active edges of } T\}, ea(T) := |EA(T)|.$$

Theorem 2.20. *$Prim^{-1}(T) = [T, T \cup EA(T)]$, where $[T, T \cup EA(T)] = \{G : T \subseteq G \subseteq T \cup EA(T)\}$ and $|Prim^{-1}(T)| = 2^{ea(T)}$.*

Proof. From the observation, we know that if e is not externally active, then there exists $e' \in T \cup e$ such that e' is in the cycle of $T \cup e$, and $e < e'$, thus e should be chosen by the Prim's algorithm before it chose e, contradiction. Thus, $Prim(G) = T$ iff $G \setminus T \subseteq EA(T)$, which implies $G \subseteq T \cup EA(T)$. And it is trivial to see that $T \subseteq G$. Also, there are $2^{ea(T)}$ choices to including e on construction of $G \in Prim^{-1}(T)$. □

In the corollary below, q is dummy variables to show combinatorial relationship.

Corollary 2.21. *Fix T be a tree on $[n]$. Then,*

$$\sum_{\substack{G \text{ on } [n] \\ Prim(G)=T}} q^{|E(G)|} = q^{n-1} \cdot \sum_{\substack{G=T\cup S \\ S\subseteq EA(T)}} q^{|S|} = q^{n-1} \cdot \left(\sum_{k=0}^{ea(T)} \binom{ea(T)}{2} q^k \right) = q^{n-1}(1+q)^{ea(T)}.$$

Note that the third equality can be derived from the fact that for each $k \in [ea(T)]$, there are $\binom{ea(T)}{1}$ sets having cardinal k. Therefore,

$$\sum_{\substack{G \text{ is connected,} \\ \text{simple on } [n]}} q^{|E(G)|} = q^{n-1} \cdot \sum_{\substack{T \text{ is spanning} \\ \text{tree on } [n]}} (1+q)^{ea(T)}.$$

Remark 2.22. *1. In the viewpoint of each connected simple graph G,*

$$\sum_{\substack{H\subseteq G \\ H \text{ is spanning} \\ \text{subgraph of } G}} q^{|E(H)|} = q^{n-1} \sum_{\substack{T \text{ is spanning} \\ \text{tree of } G}} (1+q)^{ea(T)}.$$

Note that each partition of H, which is in the same inverse image of $Prim(\cdot)$ gives a one element of the sum.

2. Given T, a spanning tree of G, fix a linear order on edge of G. Now define **internally active** *vertices as below;*

Definition 2.23 (Internally active vertices)**.** *Let T be a tree on $[n]$ with $E(T) \subset \binom{n}{2}$, where $\binom{n}{2}$ is the set of all 2-subset of $[n]$, having a lexicographical order. Then $\forall e \in E(T)$, $T - e$ is a forest with two connected components, say $T_{e,1} \cup T_{e,2}$. Now we can define the* **cut**$_e$ *as*

$$cut_e = \{e' \in \binom{n}{2} : e' \cap V(T_{e,1}) \neq \emptyset, e' \cap V(T_{e,2} \neq \emptyset\},$$

the set of edges connecting $T_{e,1}$ and $T_{e,2}$ if they added. From the order of $\binom{n}{2}$, we can take the minimal weight edges in cut_e, say $e_{\min}$. Then $e_{\min}$ is called **internally active** *with respect to T. Denote*

$$IA(T) := \{ \text{ internally active edges of } T\}, ia(T) := |IA(T)|.$$

And this internally active edges are used for define Tutte polynomial, which we will deal later.

Definition 2.24 (Tutte polynomial of 2 variables)**.**

$$T_G(X, Y) = \sum_{\substack{T \text{ is spanning} \\ \text{tree of } G}} X^{ia(T)} Y^{ea(T)}.$$

2.3.2 Depth first search and Breadth first search

DFS and BFS can be used to get a spanning tree.

Definition 2.25 (Depth First search)**.** *1. Input: G on $[n]$.*

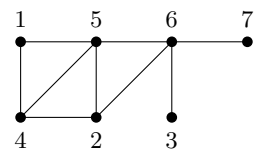
2. Start at the minimum vertex 1

3. Go to the great adjacent unvisited vertex if there is one.

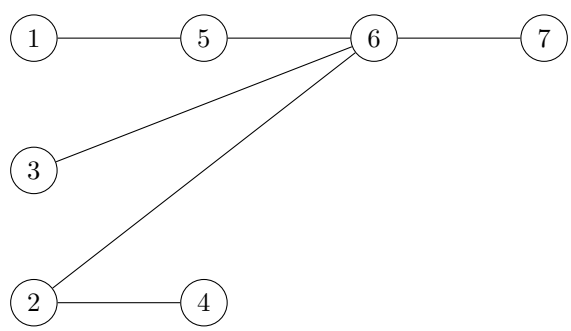
4. Otherwise back track and repeat the above.

5. Output: T on $[n]$.

For example, think about below graph



In this graph, DFS gives edges such that $1 \to 5, 5 \to 6, 6 \to 7$ first, and back track once, and gives $6 \to 3$, back track, $6 \to 2$, $2 \to 4$. Thus, the resulted tree is as below.



Thus, we can think DFS as a map such that

$$DFS : \{ \text{ connected graph on } [n]\} \twoheadrightarrow \mathcal{T}_n := \{ \text{ trees on } [n]\}.$$

Fix $T \in \mathcal{T}_n$. Then, if $DFS(H) = T$, then $T \subseteq H$. In the previous example, adding $12, 13, 14$ keep the result of DFS, however, neither does 1617. From this, we can observe that

Observation 2.26. *An edge $\{j, k\}$ with $j > k$, can be added back $\iff$ The unique path from j to k is contained in the path 1 to k and $i > k$ where i is the first vertex on the unique path from j to k.*

To formalize this observation, we need some definition.

Definition 2.27 (Inversion)**.** *Let T be a tree on $[n]$. Then, a pair (i, k) is an inversion of T if and only if i is on the path from 1 to k and $i > k$.*

Theorem 2.28 (Gessel, Wang [2])**.** *$DFS^{-1}(T) = [T, T \cup \mathcal{E}(T)]$ where $\mathcal{E}(T)$ is a set of edges which has 1-1 corresponding to set of inversions, such that $(i, k) \mapsto \{j, k\}$ where j is the predecessor of i in the path from 1 to k.*

Proof. Let $G \in [T, T \cup \mathcal{E}(T)]$. Then $G = S \cup T$ for some edges $S \subseteq \mathcal{E}(T)$. Then, $Prim(G) = T$; first of all, first vertex is 1 by the definition of algorithm. And suppose the algorithm choose $v_1, v_2, \cdots, v_k$, the first k vertices the same as T. Then, note that $\forall e \in S$ with $\{u_k v_k\} = e$, $u_k < v_{k+1}$; if $u_k < v_k$, then the unique path from v_k to u_k in T is contained in the unique path from 1 to u_k, and w_k is the first vertex of the path from v_k to u_k such that $v_k < w_k$, by the construction of $\mathcal{E}(T)$. (See observation.) Hence, $w_k = v_{k+1}$

since $v_k w_k \in E(T)$, and $u_k < v_{k+1}$. This implies that e cannot be chosen. If $u_k > v_k$, then the unique path from u_k to v_k in T is contained in the unique path from 1 to v_k, thus $u_k \in \{v_1 = 1, v_2, \cdots, v_k\}$, since we already construct the path from 1 to v_k in T. Hence, e cannot be chosen. Therefore, in any case, $u_k < v_{k+1}$. Conversely, if $v_1, v_2, \cdots, v_k$ are the first k vertices visited, then in order to go to v_{k+1} in the next step, v_k must not be connected to any unvisited vertex u such that $u > v_{k+1}$. Thus, v_{k+1} will be chosen. $\square$

Corollary 2.29. *Fix T. Then,*

$$\sum_{H \in DFS^{-1}(T)} q^{|E(H)|} = q^{n-1} \cdot \sum_{\substack{G=T\cup S \\ S\subseteq \mathcal{E}(T)}} q^{|S|} = q^{n-1} \cdot \left(\sum_{k=0}^{|\mathcal{E}(T)|} \binom{|\mathcal{E}(T)|}{2} q^k \right) = q^{n-1}(1+q)^{|\mathcal{E}(T)|}.$$

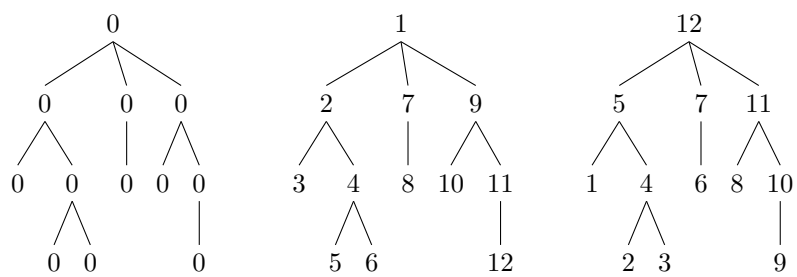
as the same reason in the case of Prim's algorithm. Thus,

$$\sum_{\substack{H \text{ is connected} \\ \text{component on } [n]}} q^{|E(H)|} = q^{n-1} \sum_{T \text{ is tree on } [n]} (1+q)^{|\mathcal{E}(T)|}.$$

DFS defines two orderings on the (unlabeled) vertices.

Definition 2.30 (Preorder and postorder)**.** **Preorder** *is the order induced from the vertices have been visited for the first time.* **Postorder** *is the order that the vertices have been visited for the last time.*

To understand ordering example, see below picture; we visited graph from left to right using DFS.

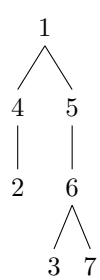


(a) Unlabeled graph (b) Preorder (c) Postorder

Definition 2.31 (Breadth-First search)**.** *Breadth-First search is an algorithm making a spanning tree as follow.*

1. *Input: G on $[n]$.*
2. *Start at $v_1 = 1$.*
3. *At each k-th time, adding all unvisited neighbors of v_{k-1} to Q, which is first in first out, adding from left to right. Then let v_k be the output of Q, and delete it from Q.*
4. *Stop when Q is empty.*

For example in the below tree,



BFS is as below;

n	v_i	Queue
0		1
1	1	$4, 5$
2	4	$5, 2$
3	5	$2, 6$
4	2	6
5	6	$3, 7$
6	3	7
7	7	$\emptyset$

We can also think BFS as the map such that

$$BFS : \{ \text{ connected graph on } [n]\} \to \mathcal{T}_n := \{ \text{ trees on } [n]\}.$$

Fix T and we can find the analogous statement for the inverse image of T.

Theorem 2.32.

$$BFS^{-1}(T) = [T, T \cup \mathcal{E}_B(T)]$$

where

$$\mathcal{E}_B(T) = \left\{ (i,j) \in \binom{n}{2} : i, j \textit{ have been in } Q \textit{ at the same time.} \right\}$$

Proof. Note that (i, j) is in $\mathcal{E}_B(T)$, then it means that i and j are chosen as a neighbor of other vertices. Thus, including (i, j) doesn't change the decision by BFS, since output of Q is the same even (i, j) exists or not. Conversely, if adding an edge (i, j) to T doesn't change the decision of BFS, then this edge fail to give new neighbors to the Q, which implies that Q has already a state that i, j are in Q. □

We can compute the distance of u, v using BFS.

Definition 2.33 (Algorithm for $d(u, v)$.)**.**

1. *Start at u. Let $d(u, u) = 0$.*
2. *Do BFS.*
3. *If a vertex v is introduced to the queue by a vertex w, then let $d(v) = d(w) + 1$.*

2.4 Dijkstra's algorithm

Consider a weighted graph G, where each edge e has a nonnegative weight. Can we find the minimal weight uv-path for any $u, v \in V(G)$? We can find the path using Dijkstra's algorithm.

Definition 2.34 (Dijkstra's algorithm)**.**

1. *Put $U = \{u\}, F = \emptyset, D_w(u, u) = 0$, where D_w is the weight distance.*
2. *While there exists edges connecting u and $V(G) - u$, determine an edge $\alpha = \{x, y\}$ with $x \in U, y \notin U$ and $D_w(x) + w(\alpha)$ minimal.*
3. *Then, update U as $U \cup \{y\}$, F as $F \cup \{\alpha\}$, $D_w(y)$ as $D_w(x) + w(\alpha)$.*

2.5 Plane graph

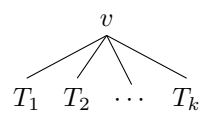
In the labeled graph, we can distinguish some graphs, even if it has the same as graph isomorphism; for example,$2 - 1 - 3$ and $3 - 1 - 2$ is the same tree, but it is distinguishable with $1 - 2 - 3$. To make this distinction clear, we can adopt the definition of plane tree. To define this precisely, we need some other concepts.

Definition 2.35 (Rooted tree and Plane tree, [3] p.101)**.** *A* **rooted tree** *is a tree with one vertex r chosen as* **root**. *For each vertex v, let $P(v)$ be the unique vr-path. The* **parent** *of v is its neighbor on $P(v)$; its* **children** *are its other neighbors. Its* **ancestors** *are the vertices of $P(v) - v$. Its* **descendants** *are the vertices u such that $P(u)$ contains v. The* **leaves** *are the vertices with no children. A* **rooted plane tree** *is a rooted tree such that ther e is a linear order on the children of each vertex.*

Note that from the definition of plane tree, $2-1-3$ is not the same as $3-1-2$, since children of 1 has distinct linear order. We can define plane tree using recurrence relation.

Definition 2.36 (A recurrence definition of plane tree)**.** *Define $\emptyset$ and a vertex without edge are in a plane tree. If $T_1, T_2, \cdots, T_k$ are plane tree such that $V(T_1), \cdots V(T_k)$ are pairwise disjoint, then $\bigcup_{i=1}^{k} T_i \cup e = \{x_i, v\}$, where $x_i \in V(T_i)$, $v \notin \bigcup_{i=1}^{k} V(T_i)$.*

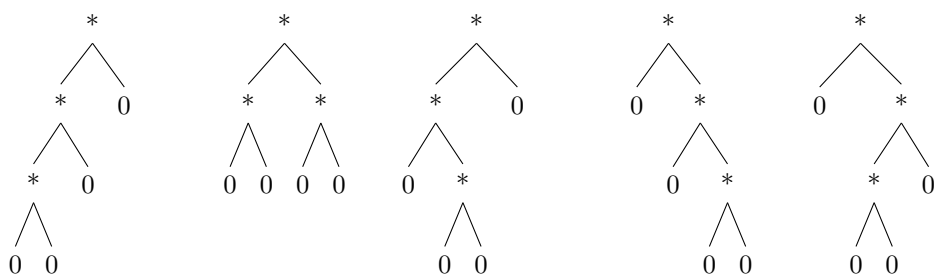
The last condition can be represented as the below picture;



And binary tree is the special kind of plane tree.

Definition 2.37 (Complete binary tree)**.** *A* **complete binary tree** *is a plane tree such that each vertex has 0 or 2 children. Then we can call vertices having degree at least 2, i.e., which are not in a set of leaves as* **internal vertices**.

Note that there are 5 types of complete binary tree with 3 internal number of vertices (represented as $*$).



Thus,

Proposition 2.38. *A complete binary tree has m internal vertices, then it has $2m+1$ vertices, and $m+1$ leaves.*

Proof. Every internal vertex has 2 children vertices, for a total of $2m$ vertices. Every vertex except the root has a parent, for a total of $m-1$ internal vertices with parents. These $m-1$ parented nodes are all children, and each takes up 1 child vertex. Thus,

$$\text{(Children vetices)} - \text{(used child pointers)} = \text{(unused child pointers)} \iff 2m - (m-1) = m+1$$

Thus, there are $m+1$ leaves. □

Proposition 2.39. *If given internal vertices as n, the number of complete binary trees with n internal vertices is*

$$c_n = \frac{1}{n+1}\binom{2n}{n},$$

where c_n is the Catalan number.

Proof. It can be derived by one-to-one correspondence of complete binary tree and well-parenthesized product. I will refill this part after the semester. □

Thus if $n = 3$, $\frac{1}{n+1}\binom{2n}{n} = \frac{1}{4}\frac{6\cdot5\cdot4}{3\cdot2\cdot1} = 5$. To see this proof, we need a proper labeling of the internal vertices. For binary tree with exactly n internal vertices, label the vertices by $[n]$, thus there are $n!$ labelings. However, we adopt ***increasing labeling***, which is a labeling satisfying a condition; for any vertex u in the tree, which is a parent of a vertex v, then label of u is less than the label of v. Then we can convert such $n!$ sequences from $[n]$ to the complete binary tree. To see this, for given a sequence $\{a_1\}_{i=1}^n$ from $[n]$, we can construct the rooted tree, which is bone of the complete binary tree as this algorithm.

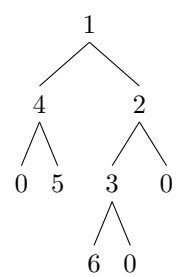
1. Put 1 on the middle point. Without loss of generalization, let $a_k = 1$, for some $k \in [n]$. Then, we can divide the sequence as $A_1 := \{a_1, \cdots a_{k-1}\}$ and $A_2 := \{a_{k+1}, \cdots a_n\}$.

2. For second step, take the smallest number from A_1 and A_2, say b_1, b_2, and let b_1 be a left child of 1, and b_2 be a right child of 1. If one of the set is empty, then just ignore. Again, partition A_1 and A_2 if it is nonempty, say A_{11}, A_{12} and A_{21}, A_{22}

3. Repeat above procedure; in the given partition, generated by the previous level's vertex, give the smallest number for each partition as a left child or right child of the previous level's vertex respectively. And partition again using the chosen smallest numbers.

4. If there are no remaining number, then gives a child for each leaves to make the complete binary tree.

For example, 451632 is



Now fix a complete binary tree with n internal vertices. How many increasing labelings are there?

Theorem 2.40. *The number of increasing labeling on T is $\frac{n!}{\prod_{v\in T} h_v}$, where h_v is the* **hook length of vertex** *v, means that the number of vertices in the subtree rooted at v.*

In the above example,$h_1 = 6, h_2 = 3, h_3 = 2, h_4 = 2, h_5 = 1, h_6 = 1$.

Proof of the theorem. There are $n!$ ways to label the vertices of compelete binary tree with n internal vertices from $[n]$. For any $v \in T$, where T is given complete binary tree with n internal vertices from $[n]$, let A_v be a event that v is the minimal in the subtree which has root at v. Then,

$$P(A_v) = \frac{1}{h_v},$$

since we can think of it as assigning every other vertices in the complement of the subtree first; then the rest vertex is h_v, now we can choose label of v as minimal number in h_v numbers. And note that A_v's are independent to each other; thus,

$$P(\text{labeling is an increasing}) = P(A_{v_1} \cdots A_{v_n}) = \prod_{i=1}^{n}(A_{v_i}) = \prod_{i=1}^{n}\frac{1}{h_{v_i}}.$$

Hence, the number of labeling is $\frac{n!}{\prod_{i=1}^n h_{v_i}}$. □

3 Matchings and Factors

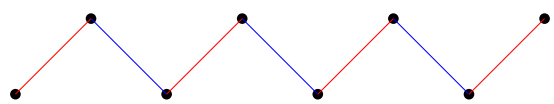
3.1 Basic matchings

Suppose G has no loops.

Definition 3.1 (Matching, saturated vertex)**.** *A* **matching** *M is a set of (non-loop) edges that are pairwise disjoint. If vertex v is incident to edge in M, say v is* **saturated** *by M. Also, M is called* **perfect matching** *if it saturates all vertices.*

Definition 3.2 (Maximal matching, maximum matching)**.** *A* **maximal matching** *M is a matching M such that $M \cup \{e\}$ is not a matching for any $e \in E(G)$. A* **maximum** *matching is a matching M in G of the largest size.*

For example, the red edges and blue edges in below figure forms maximal matchings respectively; however, the red edges are the maximum matching.



Definition 3.3. *Given a matching M, an M-***alternating path** *is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are unsaturated by M is an M-***augmenting** *path.*

Thus, if there exists a M-augmenting path P, then we can get new matching by replace edges in $M \cap P$ with edges in $M^c \cap P$; and this matching is consists of other edges in $M \cap P^c$, since every vertices in P except the endpoints are already saturated; Therefore, M is not maximum, if there exists a M-augmenting path. Our question is to find maximum matching in the graph.

Theorem 3.4. *M is maximum matching in G if and only if it has no M-augment path.*

Proof. only if part is already proved in previous paragraph. Thus, Suppose M is not maximum matching. Thus, there exists M' which is larger than M. Consider edges in $M \Delta M'$, which is symmetric difference on M and M'. Then by the below lemma, $M \Delta M'$ has only cycle or paths. Since M' is larger than M, M' should have more (unique) edges then those of M, thus $M \Delta M'$ should contain a path. This path is M-augmenting path. □

Lemma 3.5. *Any two matching M and M', $M \Delta M'$ consists of cycles and paths.*

Proof. Let $F = M \Delta M'$. Since M and M' are matching, every vertex in F has at most two edges. Thus, $\Delta(F) \leq 2$, therefore F is cycle or paths. Furthermore, if F has a path, then it is alternates between $M \setminus M'$ and $M' \setminus M$. Thus each cycle in F has even length. □

Then, we can ask how to find such maximum M by looking M-augmenting path. To see this, we need a definition of vertex cover

Definition 3.6 (Vertex cover)**.** *Let C be a subset of $V(G)$ such that $\forall e \in G$, e has at least one endpoint in C. Then C is called* **vertex cover***.*

For example, the star has singleton set as a vertex cover. So, we can define minimal vertex cover and minimum vertex cover as follow.

Definition 3.7 (Minimal vertex cover)**.** *A* **minimal vertex cover** *C is a vertex cover C such that $C \setminus \{v\}$ is not a vertex cover for any $v \in C$. A* **minimum** *vertex cover is a vertex cover C in G of the smallest size.*

Then we can observe below;

Observation 3.8. *For any vertex cover C, and any matching M, $|C| \geq |M|$*

Proof. Even if M is perfect matching, M covers two vertices in G by one edge. Thus if C is a vertex cover, then for any edge in M, a one endpoint should be in C. Also, since M is set of disjoint edges, such vertex is not repeated. Hence $|C| \geq |M|$. $\square$

Form this observation, we can conclude that

$$\min |(\text{vertex cover})| \geq \max |(\text{matching})|.$$

We can use this fact to find the augmenting path.

Definition 3.9 (Augmenting path algorithm)**.** *Input is (G, M), where G, a bipartite graph. and M, a matching M on G. Our goal: to find the M-augmenting path. Steps are below.*

(0) Begin by labeling with $(*)$ for all the vertices in X that is unsaturated. In this time, every vertices are not scanned. The meaning of scanned is in the step (2).

(1) If in the previous step no more label has been given to any vertex in X, then stop. Otherwise, go to (2)

(2) While $\exists$ labeled unscanned vertice in X, do

  (a) take such a vertex x_i,

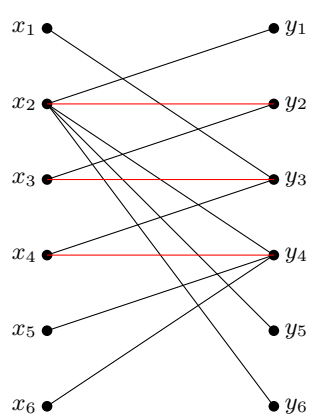
  (b) scan x_i by $\forall y \in Y$, if $x_i \sim y, \{x_i, y\} \notin M$ and y has no label. Then label it by $(x_i.)$.

(3) If in step (2), no new label has been given to anything in Y, stop. Otherwise, go to step (4).

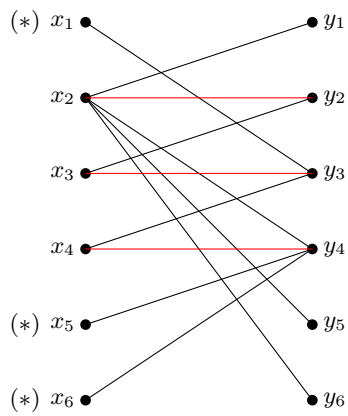
(4) While there exists a labeled unscanned vertices in Y, take such a y_j, scan it by, for all $x \in X, \{x, y\} \in M$ and x has no label. Then label this x by (y_j). If there are no labeled but unscanned vertices, then go to (1).

After completing this procedure, we can get the M-augmenting path by start from the minimal labeled but unscanned edge in Y (if you stopped at step (2)) or in X (if you stopped at step (4)), and follow the label.

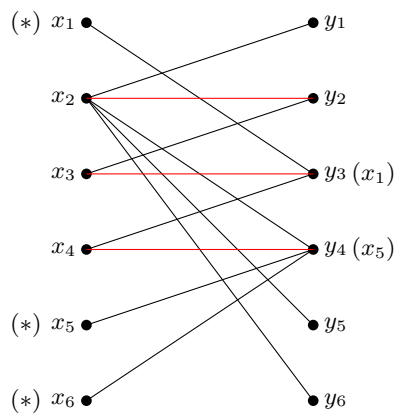
For example, let's think about the below example. In this example, a matching M is represented by the red line.



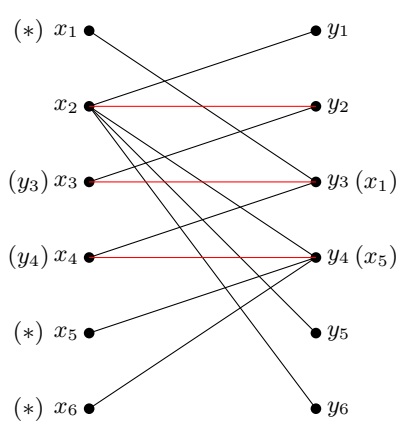
By the first step, we can label unsaturated vertices as $(*)$.



Note that definitely there are some vertices we can label it. Thus we can go to step (2). Note that the labeled unscanned vertices are x_1, x_5 and x_6. In this case, by scanning x_1, we can label y_3 by (x_1), and by scanning x_5, we can label y_4 as (x_5). x_6 has no such vertices in Y since x_5 labeled the possible vertex, y_4. Thus,

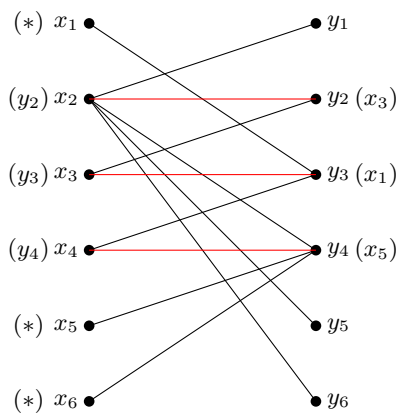


Now we can go through step 3 and step 4. In this case, y_3 and y_4 are the labeled unscanned vertices in Y. Thus, by scanning y_3, we can label x_3 as (y_3) and by scanning y_4, we can label x_4 as (y_4) since $\{x_3, y_3\} \in E(M), \{x_4, y_4\} \in E(M)$. Thus we get

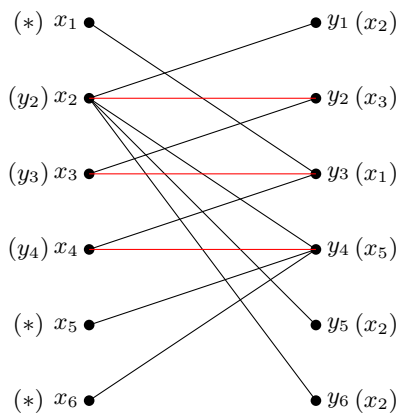


Now, go to step (2) and we know that the labeled unscanned vertices are x_3, x_4, thus by scanning them we

can label y_2 as (x_3). And from step 4, we can label x_2 as (y_2), as below.



Now the only labeled unscanned vertex is x_2, thus we can label y_1, y_5, y_6 as (x_2). This complete the procedure.

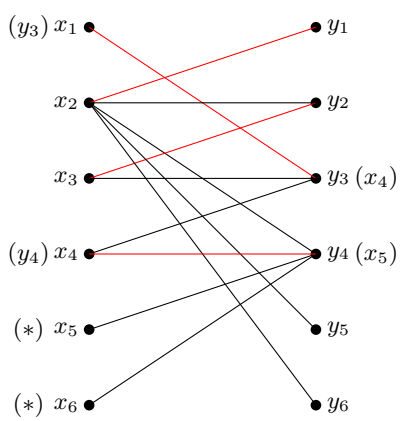


In this case, we stopped at step (2), thus and the minimal unscanned vertex in Y is y_1. By following the label, we have the path

$$y_1 \to x_2 \to y_2 \to x_3 \to y_3 \to x_1,$$

which is an M-augment path. We call this case as ***break-through*** case.

However, it would fail for some matching; for example, below is the same graph with different matching M', represented as blue. In this case, we cannot do anything after labeling x_1 as (y_3). We call this case as ***non-break-through*** case.



To analyze what happens in the above, let $S = X^{\text{unlabeled}} \cup Y^{\text{labeled}}$. In the above example, $S = \{x_2, x_3\} \cup \{y_3, y_4\}$. In the previous example, $S = Y$. Then,

Claim 3.10. *If this is break-through case or non-break-through cases, S is a vertex cover. If it is a non-break-through case, and if M is maximal matching, then $|S| \leq |E(M)|$.*

Proof. To see why the S is a vertex cover, it suffices to show that there are no edges between X^{labeled} and $Y^{\text{unlabeled}}$. Then, every edge should contain a point in $X^{\text{unlabeled}} \cup Y^{\text{labeled}}$, thus S is a vertex cover, by definition.

To get a contradiction, suppose not; there exists e such that $e \subseteq X^{\text{labeled}} \cup Y^{\text{unlabeled}}$, say $e = xy$ and $x \in X^{\text{labeled}}, y \in Y^{\text{unlabeled}}$. If a label of x is $(*)$, then $e \notin M$, thus y should receive a label by the step 2 of the procedure of finding augmenting path algorithm, contradiction. If a label of x is (y_i) for some $i \in \mathbb{N}$, then there are two cases; if $e \notin M$, then there are another $e' \in M$ such that $e' = xy_i$. In this case, the step 2 of the procedure makes y be labeled, contradiction. If $e \in M$, then $y_i = y$, thus y cannot be unlabeled, otherwise x cannot be labeled, contradiction. Thus, in any case, such e cannot exists.

To see $|S| \leq |M|$, note that for any $e \in M$, there are only three cases; e connects labeled x and labeled y; in this case, $|S|$ count only one vertex for e. The other case is e connects unlabeled x with unlabeled y; in this case, $|S|$ cannot count such e. The last case is when e connects unlabeled x with labeled y; however, in this case, since e is from M, the procedure labeled x, contradiction. Hence, in any case, $|S|$ counts less than $|M|$ for each edge. Also note that a labeled y from the $(*)$-labeled x should be a vertex in M; otherwise, we can have a new matching $M + xy$, contradicting to the maximality of M. Also, a unlabeled x should be a vertex in M, by step 1. Hence, we only consider a vertices in M to count $|S|$. Therefore, $|S| \leq |M|$. □

Theorem 3.11 (König)**.** *If G is bipartite, let $C(G) :=$ the cardinal of minimal vertex cover, and $\phi(G) :=$ the cardinal of maximal matching. Then, $C(G) = \phi(G)$.*

Proof. Let M be a maximum matching. It suffices to generate a vertex cover C having size of $|E(M)|$, from the above lemma that $|C| \geq |E(M)|$ for any vertex cover C. Let $G = (X \cup Y, E)$, i.e., $X - Y$ bipartite, where X is in left side, Y is in right side. Then, $U = X \setminus M$, $Z = \{v \in X \cup Y : v \in U$ or v is connected to a vertex in U with M-alternating path$\}$. Let

$$K = X \setminus Z \cup (Y \cap Z).$$

Now suppose $e \in E$. Then e may belong to an M-alternating paths from U or not. If it is belong to an M-alternating path from U, then the right endpoint of e, $e \cap Y$, is reachable from U by an M-alternating path, so it should be in $Y \cap Z$. Otherwise, if e is not in any M-alternating path connecting U, there are two cases occur; if $e \in M$, then its left endpoint (point in $e \cap X$) should not be in alternating path connecting U, otherwise such path plus e gives another alternating path connecting U containing e. Hence, the left end of e should be in $X \setminus Z$. Or if e is not in M, then still, e's left endpoint should not be in any M-alternating path connecting U, otherwise we can also extension of M-alternating path. (Note that in each case, actually M-alternating path occurs with direction, so be careful about deriving the conclusion.) Thus, in any case, the leftend of e should be in $X \setminus Z$. Hence, K is a vertex cover of G.

Now note that $X \setminus Z$ is contained in $X \setminus U$, so they are vertices in X. And, $Y \cap Z$ is also should be in M; to see this, suppose that there existed an M-alternating path from U to an unmatched vertex in $Y \cap Z$. Then changing the matching by removing the matched edges from this path and adding the unmatched edges in their place would increase the size of the matching, contradiction. However, as we shown above, no matched edge can have both ends staying in K, as we have shown above. So, K has a cardinality $|E(M)|$. □

Corollary 3.12. *If G is a bipartite and p-regular with $p \geq 1$, then G has a perfect matching.*

Proof of the corollary. Assume the bipartition is $X \uplus Y$. Then, $p|Y| = |E(G)| = p|X|$ implies $|X| = |Y|$. Say $|X| = |Y| = n$. Now assume that C is a vertex cover, then p-regularity implies $\forall v \in C$, and $\forall C$ vertex cover, $|E(G)| \leq p|C|$. Thus, $pn \leq p|C| \implies n \leq |C|$. Also, we know that there exists a C such that $|C| = n$ by the König theorem. Thus, by the theorem, there exists M such that $\min |C| = n = \max |M|$, hence the M is perfect. □

Let Y be a finite set. Let $\mathfrak{a} = \{A_1, A_2, \cdots, A_n\}$ is a family of subsets of Y, with $A_i \neq \emptyset$ for any $i \in [n]$. Then,

Definition 3.13 (System of distinct representatives)**.** *An ordered set of elements* $(e_1, e_2, \cdots, e_n), e_i \in Y$ *for any* $i \in [n]$*, is called* **System of distinct representatives** *of* $\mathfrak{a}$ *if all* e_i*'s are distinct and* $e_i \in A_i$ *for all* $i \in [n]$.

Note that the easy necessary condition to have SDR is below; for all subsets of $\mathfrak{a}$ of size k, the union is at least of size k, i.e.,

$$\forall k \in \mathbb{N} \cup \{0\}, |A_{i_1} \cup \cdots \cup A_{i_k}| \geq k.$$

In graph theory, let $G = (X \cup Y, E)$, where $X = \mathfrak{a}$ and $Y = Y$, and $E = \{(A_i, v) : A_i \in X, v \in Y, v \in A_i\}$. Then, there exists SDR if and only if there exists a matching that saturates X. (By definition of mathcing, SDR and gives such matching and vice versa.)

Then we call that ***Hall's condition*** for

$$\forall S \subseteq X, |N(S)| \geq |S|,$$

where $N(S)$ means the neighbor of S, and we can rewrite the Hall's marriage theorem as below;

Theorem 3.14 (Hall's marriage Theorem)**.** *A bipartite graph* $G = (X \cup Y, E)$ *has a matching that saturates* X*, if and only if* $\forall S \subseteq X, |N(S)| \geq |S|$.

if part. Suppose the Hall's condition holds. If we assume that $|X| = n$. Then, by taking $S = |X|$, $|Y| \geq n$ since $N(S) \subseteq Y$. Now it suffices to show that for any vertex cover C, $|C| \geq n$, from the Konig' theorem, we can create matching with size n, which implies it generates saturation of X. Note that C is a vertex cover of G if and only if $V(G) - C$ is an independent set. Now take any vertex cover C. Then let $X_1 = C \cap X, Y_1 = C \cap Y$, $X_2 = X \setminus X_1, Y_2 = Y \setminus Y_1$. Assume $|C| < n$, i.e., $|X_1| + |Y_1| < n$. We know no edge exists between X_2 and Y_2, since C is a vertex cover. Hence a subgraph from vertices $X_2 \cup Y_2$ has no edges, so

$$N(X_2) \subseteq Y_1 \implies |N(X_2)| \leq |Y_1| < n - |X_1| = |X_2|$$

where the last equality comes from $|X| = n$, which implies

$$|X_2| > |N(X_2)|$$

contradicting the Hall's condition. □

Only if part, from E. Easterfield, Halmos, and Vanghan. This is induction on $|X| = n$. If $n = 1$, then this is a graph without edge or graph with only one edge. So if there exists a matching that saturates X, then $|N(X)| \geq 1 = |X|$, so done.

Now assume that there exists saturation matching for all sub-bipartite graph of G having less than n elements of X, and let $|X| = n$ for given graph G.

- Case 1: $\forall S \subsetneq X$, $S \neq \emptyset$, suppose $|S| < |N(S)|$. Then, for any $x \in X$,

  $$|X \setminus \{x\}| < |N(X \setminus \{x\})| \implies |X| = |X \setminus \{x\}| + 1 \leq |N(X \setminus \{x\})| \leq |N(X)|$$

  since $N(X)$ contains $N(X \setminus \{x\})$, done. So every subset of S satisfy the Hall's condition.

- Case 2: If there exists a nonempty subset K of X such that $|K| = |N(K)|$, then $G|_{(K,N(K)}$ has a matching by induction. So look at $G_2 = G|_{(K^c,N(K)^c}$ where $\bullet^c$ is a complement of a set. Now we want to show that G_2 satisfy the Hall's condition, then from disjointness, $|X| = |K| + |K^c| \leq |N(K)| + |N(K^c)|$. Now note that for all $T \subseteq K^c$,

  $$|N_G(T \cup K)| \geq |T \cup K| = |T| + |K|$$

  by the inductive hypothesis, And note that

  $$N_{G_2}(T) = N_G(T \cup K) - N(K) = N_G(T) - N(K)$$

  as a setminus. So,

  $$|N_{G_2}(T)| \geq |N_G(T \cup K)| - |N_G(K)| \geq |T| + |K| - |N_G(K)| = |T|.$$

  So done.

□

Corollary 3.15. *If $\forall x \in X$, $\deg(x) \geq k$ then the number of matchings saturating X is greater than $k!$.*

Proof. Index $X \cong [n]$. Then if Case 1 in the above proof holds, then for $1 \in X$ it has k choices. Now choose one of them and delete 1 and its incidence vertex. Then the reamaining one has at least $k-1$ choices, so we can take an edge from it and delete it. Repeat this until having matching done, so we have at least $k!$ matchings.

If Case 2 in the above holds, then $\exists S \subseteq X$ such that $|S| = |N(S)|$. Now think about $G|_{S,N(S)}$. Then from the condition, $G|_{S,N(S)}$ still has $\min \deg(S) \geq k$ where $\deg(S) = \{\deg(x) : x \in S\}$. So by the induction, there exists $k!$ matchings. Then, if we denote $M(S)$ be a set of vertices correspond to a specfic matching, then $G|_{S^c,N(S^c)\setminus M(S)}$ has also a matching, and the union of the specific matching for S and S^c is a matching in G, so done. □

Corollary 3.16. *If the degree of vertices in X are $m_1 \leq m_2 \leq \cdots \leq m_n$, then the number of matchings saturating X is greater than or equal to $m_1(m_2-1)(m_3-2)\cdots(m_n-n+1)$.*

Proof. Actually, do the same thing using those m_i. □

Lemma 3.17 (Rado's lemma)**.** *Let $G = (X \cup Y, E)$ satisfies the Hall condition with respect to X. If $\{x, y_1\}, \{x, y_2\}$ are two edges of G, then at least one of $G_1 = G - \{x, y_1\}, G_2 = G - \{x, y_2\}$ still satisfies Hall's condition.*

If we have this lemma, then we can start with the assumption that every vertex has degree 1.

Proof. To get a contradiction, assume both G_1, G_2 do not satisfies Hall's condition. Then, $\exists Z \subseteq X \setminus \{x\}$ such that

$$|Z| \leq |N_G(Z)|, |Z| \leq |N_{G_1}(Z)| \text{ but } |Z \cup \{x\}| > |N_{G_1}(Z \cup \{x\})| = |N_{G_1}(Z) \cup N_{G_1}(x)|.$$

Thus,

$$|Z| + 1 = |Z \cup \{x\}| > |N_{G_1}(Z \cup \{x\})| \geq |N_{G_1}(Z)| \geq |Z| \implies |N_{G_1}(Z \cup \{x\})| = |N_{G_1}(Z)| = |Z|$$

This implies $y_2 \in N_G(Z), y_1 \notin N_G(Z)$. Similarly, in G_2 $\exists W \subseteq X \setminus \{x\}$ such that

$$|W| = |N_G(W)| = |N_{G_2}(W)|$$

and $y_1 \in N_G(W), y_2 \notin N_G(W)$. This implies

$$N_G(x) \setminus \{y_2\} \subseteq N_G(W), N_{G_2}(W \cup \{x\}) = N_G(W).$$

Now

$$|Z| + |W| = |N(Z)| + |N(W)| = |N_{G_1}(Z \cup \{x\})| + |N_{G_2}(Z \cup \{x\})|.$$

Note that $|Z| = |N(Z)|, |W| = |N(W)|$ derived from above inequalities, and the last equality comes from above equations. Let $T_1 = N_{G_1}(Z \cup \{x\}), T_2 = N_{G_2}(W \cup \{x\})$. Then,

$$T_1 \cup T_2 = N_G(Z \cup W \cup \{x\})$$

since $y_2 \in N_G(Z) = N_{G_1}(Z), y_1 \in N_G(W) = N_{G_2}(W)$, so they are actually stay in $N_G(Z \cup W \cup \{x\})$, and any other points in T_i is already in $N_G(Z \cup W \cup \{x\})$, so $T_1 \cup T_2 \subseteq N_G(Z \cup W \cup \{x\})$. For other direction, if $y \in N_G(Z \cup W \cup \{x\})$, then if it has an edge with Z, it is in T_1, if it has edge with W, it is in T_2, if it has edge with x, then if $y \neq y_1$ or y_2, it is still in $T_1 \cup T_2$, and if it is one of y_1, y_2, then by the same argument it is in the T_1 or T_2.

Also,

$$T_1 \cap T_2 = N_G(Z) \cap N_G(W) \subseteq N(Z \cap W)$$

from the equation $N_{G_1}(Z) = N_G(Z), N_{G_2}(W) = N_G(W)$ we derived above. Hence,

$$|T_1 \cap T_2| \geq |N(Z \cap Y)| \geq |Z \cap Y|$$

from the Hall's condition. And,

$$|T_1 \cup T_2| = |N_G(Z \cup W \cup \{x\})| \geq |Z \cup W \cup \{x\}| \geq |Z \cup Y \cup \{x\}| = |Z \cup Y| + 1$$

from the Hall's condition. Thus,

$$|Z| + |Y| = |T_1| + |T_2| = |T_1 \cup T_2| + |T_1 \cap T_2| \geq |Z \cup W| + 1|Z \cap W| = |Z| + |Y| + 1$$

contradiction. □

Corollary 3.18. *Let $\mathfrak{a} = \{A_1, \cdots, A_n\}$ be a set system, i.e., set of subsets of Y. Then, the largest number of sets in $\mathfrak{a}$ with an SDR is*

$$n + \min\{|\cup_{i \in S} A_i| - |S| : S \subseteq [n]\}.$$

And by the Hall' condition, $\min\{|\cup_{i \in S} A_i| - |S| : S \subseteq [n]\} = 0$.

Proof. Let

$$-d = \min\{|\cup_{i \in S} A_i| - |S| : S \subseteq [n]\}.$$

Let $G = (\mathfrak{a} \cup Y, E)$ be a graph generated in previous construction. Then, let G' be a graph adding d more points on Y, and adding edges from every point in $\mathfrak{a}$ to every newly added points. This gives for any $S \subseteq \mathfrak{a}$ with nonempty,

$$|N_{G'}(S)| = |N_G(S)| + d.$$

Thus, G' satisfies the Hall's condition, thus G' has an SDR of size n. Now remove points of SDR from newly added points, then it has at least an SDR with size $n - d$. □

Now we can prove the Konig's theorem, stating that if G is bipartite, let $C(G) :=$ the cardinal of minimal vertex cover, and $\phi(G) :=$ the cardinal of maximal matching. Then, $C(G) = \phi(G)$.

Proof. Suppose $G = (X \cup Y, E)$ bipartite. If Q is a vertex cover and M is a matching, then $|Q| \geq |M|$ since distinct vertices must be used for cover the edges of matching. (If one vertex cover two edges in M, then M is not a matching.) Given the Q with size $C(G)$, we always construct a matching having $C(G)$ as its cardinality. Let

$$R = Q \cap X, T = Q \cap Y, H = G|_{R \cup Y - T}, H' = G|_{X - R \cup T}.$$

Since $R \cup T$ is a vertex cover, G has no edge from $Y - T$ to $X - R$.

□

3.2 Independent sets and edge covers

Recall terminology we discussed;

$$\begin{aligned} \alpha'(G) = M(G) &:= \text{size of maximum matching} \\ \beta(G) = C_v(G) &:= \text{size of minimum vertex cover} \\ \alpha(G) &:= \text{size of maximum independent set} \\ \beta'(G) = C_e(G) &:= \text{size of minimum edge cover} \end{aligned}$$

Definition 3.19 (Edge cover)**.** *An edge coveer is $S \subseteq E(G)$ such that any vertex is incident to at least one edge in S.*

The existence of edge cover implies that G has no isolated vertices. Also, it is observed earlier in our discussion that

$$C_v(G) \geq M(G).$$

And it is tie in the bipartite group. And from the fact that C is a vetex cover if and only if $V(G) \setminus C$ is an independent set, we know

$$\alpha(G) + C_v(G) = n = |V(G)|.$$

Claim 3.20.

$$C_e(G) \geq \alpha(G)$$

for G is of isolated vertex free.

To show this claim, we need a theorem;

Theorem 3.21. *If G is isolated vertex free, then $M(G) + C_e(G) = n$.*

If we have this theorem then

$$n - M(G) \geq n - C_v(G) = \alpha(G) \implies n \geq M(G) + \alpha(G) = n - C_e(G) + \alpha(G) \implies C_e(G) \geq \alpha(G).$$

Proof. Let M be a maximum matching. It suffices to show that we can have an edge cover of size $n - |M|$. It proves $C_e(G) + M(G) \leq n$. Note that M already cover a $2|M|$ vertices, where $|M| := |E(M)|$. For the remaining $n - 2|M|$ vertices, pick one edge for each in remaining edges. Then, the number of chosen edges with edges in M is

$$n - 2|M| + |M| = n - |M|$$

and it covers all vertices, so done.

Conversely, let L be a minimum edge cover. We will construct a matching of size $n - |L|$. This shows $M(G) \geq n - |L| = n - C_e(G)$. From the minimality of L, we claim that a subgraph $G' = (V, L)$ does not contain any path of length 3. To see this, if it contain such path, say $(\bullet - \bullet - \bullet - \bullet)$, then by deleting the middle edge we have an edge cover which is smaller than the existing cover, contradicting minimality. So, G' is just disjoin union of star-shape graphs, i.e., a connected graph having maximum path length is 2. (So it has a center.) Now let P be the number of connected component of G'. Then,

$$|L| + P = n$$

since each component has k vertices with $k-1$ edges, and $|L|$ represents all number of edges in all component, so P represents all centers in each connected component. Thus, let M be a matching generated by choosing one edge from each component. Then, $|M| = P = n - |L|$, as we desired. □

Corollary 3.22. *If G is isolated vertex free, and bipartite, then $\alpha(G) = C_e(G)$.*

Proof. Apply König's theorem. □

Definition 3.23 (Dominating Set)**.** *Let $G = (V, E)$, $v \in V$. For all $X \subseteq V$, define*

$$N(X) = \{u : \exists x \in X \text{ s.t. } ux \in E\}, N[X] = X \cup N(X).$$

Then, we call a subset X of V is **dominating set** *if $N[X] = V$.*

Our goal is to find the most smaller dominating set. Let

$$\nu(G) := \textbf{\textit{domination number}} = \text{ minimum size of dominating set.}$$

If G has no isolated vertices, then any vertex cover is dominating, which implies $C_v(G) \geq \nu(G)$. Let δ be the minimal degree of a vertex in G, and Δ be the maximal degree of that in G. Then,

Theorem 3.24.

$$\frac{n}{\Delta + 1} \leq \nu(G) \leq \frac{n}{\delta + 1}(1 + ln(n + 1)).$$

For $\nu(G) \leq \frac{n}{\delta+1}(1 + ln(n + 1))$, we need a lemma.

Lemma 3.25. *Given $X \subseteq V, Y_X = V - N[X]$, there exists a vertex $y \notin X$ such that y dominates at least $|Y_X| \cdot \frac{\delta+1}{n}$, i.e., connected to at least $|Y_X| \cdot \frac{\delta+1}{n}$ many vertices. Especially, if $|Y_X| = k$, then*

$$|V - N[X \cup \{y\}]| \leq k\left(1 - \frac{\delta + 1}{n}\right)$$

Proof. For each $v \in Y_X$, list $N[v]$. Then clearly, $|N[v]| \geq \delta + 1$ by definition of δ, minimum degree of a vertex in G. Hence,

$$\sum_{v \in Y_X} |N[v]| \geq (\delta + 1)|Y_X|.$$

Now thinking about whole lists of $N[v]$, for each $v \in Y_X$ allowing repeatition. Then, those are all vertices in $V - X$. (If there exists a vertex in X, then $v \notin Y_X = V - N[X]$, contradiction.) Thus, by the Pigeonhole principle, there exists a vertex y who appears at least $\frac{(\delta+1)|Y_X|}{|V-X|}$ sets of $N[v]$, which implies

$$\deg(y) \geq \frac{(\delta + 1)|Y_X|}{|V - X|} \geq \frac{(\delta + 1)|Y_X|}{n}.$$

Thus, $|V - N[X \cup \{y\}]| \leq |Y_X| - \frac{(\delta+1)|Y_X|}{n} = k\left(1 - \frac{\delta+1}{n}\right)$. □

Proof of the theorem. For $\frac{n}{\delta+1} \leq \nu(G)$, just a form an independent set by taking a vertex, deleting every vertex of neighbors, and taking a vertex from remainings, and so on. For each selection, we delete at most $\Delta + 1$ vertices including chosen one. Thus, $|S|$ should be greater than $n/(\Delta + 1)$.

For $\nu(G) \leq \frac{n}{\delta+1}(1 + ln(n + 1))$, build a dominating set in a greedy way. Start with a vertex of maximum degree. Each round add a new vertex y that increases $N[X \cup \{y\}]$. Then, after r round, the reamining undominated vetices are at most

$$n\left(1 - \frac{\delta + 1}{n}\right)^k.$$

Thus, after $n\frac{\ln(\delta+1)}{\delta+1}$ the number of undominated vertices is at most

$$n\left(1 - \frac{\delta + 1}{n}\right)^{n\frac{\ln(\delta+1)}{\delta+1}} \leq n\left(e^{\frac{\delta+1}{n}}\right)^{n\frac{\ln(\delta+1)}{\delta+1}} = \frac{n}{\delta + 1}$$

from the inequality $(1 - \epsilon) \leq e^{-\epsilon}$ for all $\epsilon \in \mathbb{R}$; to see this note that

$$x + exp(-x) \geq 1.$$

This can be shown easily for $|x| \geq 1$. For $-1 < x < 0$, use the fact that $|\frac{de^{-x}}{dx}| > x' = 1$ and for $0 < x < 1$, use the fact $(x + e^{-x})' > 0$.

Now define a set D consisting of chosen vertices and remaining undominated vertices. Then, the number of chosen vertices are at most $n\frac{\ln(\delta+1)}{\delta+1}$, and the number of

$$|D| \leq n\frac{\ln(\delta + 1)}{\delta + 1}$$

□

There is a proof using probabilistic method.

Proof. Let X be picked randomly with $Pr(v \in X) = p$, independent and identically distributed for v. Let $Y_X = V - N[X]$. Then $X \cup Y_X$ is a dominating set. Thus,

$$|X \cup Y_X| = |X| + |Y_X| \implies E(|X\ cupY_X| = E(|X|) + E(|Y_X|)$$

then

$$E(|X|) = np$$

since if we let $X = \sum_{v \in V(G)} X_v$ where X_v is a probabilistic variable having 1 if $v \in X, 0$ if $v \notin X$, then

$$E(X_v) = p \implies E(|X|) = np.$$

Similarly,

$$E(|Y_X|) = \sum_{v \in V(G)} Pr(v \notin N[X]) = \sum_{v \in V(G)} (1 - p)^{\deg(v)} \leq n(1 - p)^{\delta+1}.$$

This is because $v \notin N[X] \implies v \notin X$ and any neighbor of v is not in X. Thus,

$$E(|X\ cupY_X| = E(|X|) + E(|Y_X|) = n(p + (1-p)^{\delta+1}).$$

Now choose nice number $p \in [0,1]$ so that make the expectation as small as possible. Now we know that from the inequality $(1-\epsilon) \leq e^{-\epsilon}$,

$$E(|X\ cupY_X| = E(|X|) + E(|Y_X|) = n(p + (1-p)^{\delta+1}) \leq n(p + (e^{-p})^{\delta+1})$$

so take $p = \frac{\ln(\delta+1)}{\delta+1}$, then

$$E(|X\ cupY_X| \leq n\left(\frac{\ln(\delta+1)}{\delta+1} + \frac{1}{\delta+1}\right)$$

as desired. □

Theorem 3.26. *Let $G = (V, E)$ be a simple graph with $|V| = n$, $|E| = \frac{nd}{2}$ for some $d \geq 1$. Then,*

$$\alpha(G) \geq \frac{n}{2d}.$$

Proof. Let X be a random subset of V by $Pr(v \in X) = p$. For all edge in X, i.e., 2-subsets of X, remove one vertex from X. Now define Y_X be the number of edges in X. Then we will have an independent set of size $|X| - Y_X$. Then,

$$E(|X| - Y_X) \geq E(|X|) - E(Y_X) = np - p^2\frac{nd}{2}$$

since $E(Y_X)$ is determined by the probability that for each edge $e \in E$. $Pr(e \subseteq X) = p^2$ from iid condition, thus $E(Y_X) = p^2\frac{nd}{2}$. Now taking $p = \frac{1}{d}$, we get

$$E(|X| - Y_X) \geq \frac{2n}{2d} - \frac{n}{2d} = \frac{n}{2d}.$$

□

3.3 Maximum Bipartite matching

Suppose $G = (X \cup Y, E)$ be a $X - Y$ bipartite graph and each edge has a weight $w(e) \geq 0$. Now we want a matching of maximum total weight

$$w(M) = \sum_{e \in M} w(e).$$

For example, if $|X| \leq |Y|$, then add $|X| - |Y|$ vertices in X and give zero weighted edges from those newly added vertices in X to every point in Y, we can assume that $|X| = |Y| = n$. So it suffices to deal with complete bipartite graph case, which has a perfect matching.

Remark 3.27. *if we want minimum weight matching, then just reweight the graph, i.e., $w(e) \mapsto M - w(e)$ for some M greater than $w(e')$ for all $e' \in E$.*

Now we can make a representation matrix, where j-th column denote $y_j \in Y, i$-th row denote $x_i \in X$, and (i, j) position denotes $w(x_i, y_j)$. Then a perfect matching can be represented by n entries in the representation matrix which are all different rows and columns.

For special case, if G is unweighted, then the matrix is 0-1 matrix, and a matching gives vertex cover. Now to deal with this problem, we should introduce weighted vertex cover.

Definition 3.28. *An* **weighed vertex cover** *$C = (\bar{u}, \bar{v})$ is a pair of tuples $\bar{u} = (u_1, \cdots, u_n), \bar{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$ such that*

$$u_i + v_j \geq w(x_i, y_j).$$

Cost of the cover *$(\bar{u}, \bar{v})$ is defined as*

$$Cost(C) = \sum_i u_i + \sum_j v_j.$$

This definition is generalization of vertex cover, since we can represent the vertex cover C of unweighted graph as

$$u_i = \begin{cases} 1 & x_i \in C \\ 0 & \text{o.w.} \end{cases}, v_j = \begin{cases} 1 & y_j \in C \\ 0 & \text{o.w.} \end{cases}.$$

Proposition 3.29. *If $C = (\bar{u}, \bar{v})$ is a weighed vertex cover and M is a perfect matching, then*

$$Cost(C) \geq w(M).$$

Proof. Note that each vertex in G covered by only one edge in M. So take a pair of vertices (x_i, y_j) which is an edge of M. Then,

$$w(M) = \sum_{(x_i,\, y_j) \text{ is pair in } M} w(x_i, y_j) \leq \sum_{(x_i,\, y_j) \text{ is pair in } M} u_i + v_j = \sum_{i=1}^{n} u_i + v_i = Cost(C).$$

□

Definition 3.30 (The equality subgraph)**.** *For given an weighted vertex cover $C = (\bar{u}, \bar{v})$, the* **equality subgraph** *$G(\bar{u}, \bar{v})$ is a spanning subgraph of $K_{n,n}$ where $e \in G(\bar{u}, \bar{v})$ if and only if $w(e) = u_i + v_j$.*

In the equality subgraph $G(\bar{u}, \bar{v})$, any perfect matching of $G(\bar{u}, \bar{v})$ is a perfect matching of $K_{n,n}$, since it is spanning subgraph. Also, for any perfect matching M of $G(\bar{u}, \bar{v})$, $w(M) = Cost(\bar{u}, \bar{v})$ by construction of $G(\bar{u}, \bar{v})$.

Using this equality subgraph, we can find a maximum weighted matching in $K_{n,n}$, called the ***Hungarian Algorithm*** (Kuhn). Input is just weighted $K_{n,n}$ with $w(e) \geq 0$ for all $e \in E(K_{n,n})$.

1. Start by setting $(\bar{u}, \bar{v})$ such that $u_i = \max\{w_{ij}\}, v_j = 0$ for all $i, j \in [n]$.
2. Leg $G(\bar{u}, \bar{v})$ be the equality subgraph with respect to $\bar{u}, \bar{v}$. Find a maximum matching M in $G(\bar{u}, \bar{v})$. (Actually, this can be done by excessive matrix as we describe below in the example.) If $|M| = n$, i.e. perfect, then stop. Output is M Otherwise, let C be a vertex cover of size $|M|$ in $G(\bar{u}, \bar{v})$. Let $X_1 = C \cap X, Y_1 = C \cap Y$. Now define

   $$\epsilon := \min\{u_i + v_j - w_{ij} : x_i \in X \setminus X_1, y_j \in Y \setminus Y_1\}.$$

   Now decrease u_i by ϵ for all $x_i \in X \setminus X_1$, and increase v_j by ϵ for $y_i \in Y_1$.
3. Repeat step 2.

For example, think about $K_{5,5}$ with weighted matrix

$$\begin{pmatrix} 4 & 1 & 6 & 2 & 3 \\ 5 & 0 & 3 & 7 & 6 \\ 2 & 3 & 4 & 5 & 8 \\ 3 & 4 & 6 & 3 & 4 \\ 4 & 6 & 5 & 8 & 6 \end{pmatrix}.$$

Note that each column with index j denotes y_j, each row with index i denote x_i. Then start with an weighted vertex cover

$$\bar{u} = (6, 7, 8, 6, 8), \bar{v} = (0, 0, 0, 0, 0)$$

as in the initialization. Then we can calculate ϵ_{ij} for each i, j so that make an ***excess matrix***; for eamxple, in the above,

$$\epsilon_{ij} = \begin{pmatrix} 2 & 5 & 0 & 4 & 3 \\ 2 & 7 & 4 & 0 & 1 \\ 6 & 5 & 4 & 3 & 0 \\ 3 & 2 & 0 & 3 & 2 \\ 4 & 2 & 3 & 0 & 2 \end{pmatrix}.$$

So $G(\bar{u}, \bar{v})$ contains an edge $x_i y_j$ whose $\epsilon_{ij} = 0$. So we can take a maximum matching by picking those 0 edges; in the above, take $M = \{x_1y_3, x_2y_4, x_3y_5\}$. Then, since it is not perfect, take an arbitrary vertex cover of M, say $C = \{x_3, y_3, y_4\}$. Then, $Y_1 = \{y_3, y_4\}, X_1 = \{x_3\}$, thus

$$\epsilon = \min \begin{pmatrix} 2 & 5 & 3 \\ 2 & 7 & 1 \\ & & \\ 3 & 2 & 2 \\ 4 & 2 & 2 \end{pmatrix} = 1.$$

By update, we have an weighted vertex cover

$$\bar{u} = (5, 6, 7, 5, 7), \bar{v} = (0, 0, 1, 1, 0)$$

Then, the excess matrix for this cover is

$$\epsilon_{ij} = \begin{pmatrix} 1 & 4 & 0 & 4 & 2 \\ 1 & 6 & 4 & 0 & 0 \\ 6 & 5 & 5 & 4 & 0 \\ 2 & 1 & 0 & 3 & 1 \\ 3 & 1 & 3 & 0 & 1 \end{pmatrix}.$$

So, in this case, take M as the same above, and take C, in this case just $\{y_3, y_4, y_5\}$, then

$$\epsilon = \min \begin{pmatrix} 1 & 4 & \\ 1 & 6 & \\ 6 & 5 & \\ 2 & 1 & \\ 3 & 1 & \end{pmatrix} = 1$$

So by updating, we get the cover

$$\bar{u} = (4, 5, 7, 4, 6), \bar{v} = (0, 0, 2, 2, 1)$$

and the excess matrix

$$\begin{pmatrix} 0 & 3 & 0 & 4 & 2 \\ 0 & 5 & 4 & 0 & 0 \\ 5 & 4 & 5 & 4 & - \\ 1 & 0 & 0 & 3 & 1 \\ 2 & 0 & 3 & 0 & 1 \end{pmatrix}.$$

So, we can take a perfect maximum matching from this excessive matrix by taking $M = \{x_1y_3, x_2, y_1, x_3y_5, x_4y_2, x_5y_4\}$. Then, $w(M) = Cost(\bar{u}, \bar{v}) = 26 + 5 = 31$.

There are several remarks for this algorithm.

1. $G(\bar{u}, \bar{v})$ is needed to find matching of maximal size, not maximal weight.
2. $u_i, v_j \in \mathbb{R}$ are not necessarily nonnegative.
3. In each round, $G(\bar{u}, \bar{v})$ might lose edges but the size of maximum matching never drops down.
4. Cost of $G(\bar{u}, \bar{v})$ drops! Overall change of cost is

$$-\epsilon(|X - X_1|) + \epsilon(Y_1) = -\epsilon(n - |X_1|) + \epsilon(|Y_1|) = \epsilon(|Y_1| + |X_1| - n).$$

5. The algorithm will stop only when $|M| = n$ in $G = (\bar{u}, \bar{v})$. In that case, M and $(\bar{u}, \bar{v})$ are optimal.

3.4 Stable matching

Suppose there are n women and n men. Each womem linearly ranks men and vice versa. Now take a matching between women and men. Then we define

Definition 3.31 (Stable matching)**.** *M is* **unstable** *if there are two couples $A-a, B-b$ in matching but A prefer b than a and b prefer A than B. Thus, M is stable if there is no unstable pair of edges.*

For example, say w_1 has preference $m_1 > m_2$, w_2 has preference $m_2 > m_1$, m_1 has preference $w_2 > w_1$ and m_2 has preference $w_2 > w_1$. We can denote it as a ***preference ranking matrix*** as

$$\begin{pmatrix} (1,2) & (2,2) \\ (2,1) & (1,1) \end{pmatrix}$$

where each column i denotes m_i, each row j denotes w_j, so each elements in (i,j) position is just (w_i's preference rank of m_j , m_j's preference rank of w_i).

then a matching

$$M = \{w_1m_1, w_2m_2\}$$

is stable, but

$$M' = \{w_1m_2, w_2m_1\}$$

is unstable, since both w_1 and m_1 prefer other partner than current partner.

Now our question is this;

- Given any ranking matrix, are there always stable matching?
- How to find stable matching if it exists?

Theorem 3.32 (Gale-Shapley Proposal Algorithm (or Deferred Acceptance Algorithm))**.** *For all preference ranking matrix, there exists a stable matching.*

The algorithm can be described as below.

- Begin with every women marked as 'rejected'
- While there exists a rejected woman, do the following things.
  1. For each woman marked as rejected, propose the man when she ranked highest and not yet rejected by her.
  2. Each men pick out the women he ranked highest among those who proposed to him. Remove her "rejected status," and reject other proposals.

For example, let A, B, C, D are women, a, b, c, d are men and they have a preference matrix

$$\begin{pmatrix} (1,2) & (2,1) & (3,2) & (4,1) \\ (2,4) & (1,2) & (3,1) & (4,2) \\ (2,1) & (3,3) & (4,3) & (1,4) \\ (1,3) & (4,4) & (3,4) & (2,3) \end{pmatrix}.$$

where each column index $1,2,3,4$ denotes men a, b, c, d and each row index $1,2,3,4$ denotes women A, B, C, D respectively. For example, c's rank toward women A, B, C, D is $2,1,3,4$ respectively, and an woman C has a rank toward men a, b, c, d is $2,3,4,1$. (Rank 1 is better than rank 2, obviously.) Also, we denote $Z \to z$ as an woman Z propose to z.

Then in the first round, the proposals are

$$A \to a, B \to b, C \to d, D \to a.$$

Then, a rejects D, since he ranks A and D as 2 and 3, respectively.

In the second round, since D's proposal is rejected, D proposes the second highest rank, d. Then, d reject C since he prefer D to C.

In the third round, C choose a, and a reject A since a prefer C. In the fourth round, A propose b. Then b reject B, and in the fifth round, B propose a, and a reject B, and in the sixth round, B propose to c, and no rejection occur. So done. So the result is

$$a - C, b - A, c - B, d - D.$$

- Why this algorithm halts? Because any women can be rejected at most $n-1$ times. And men rejects only when he has a better offer. So number of possible proposal is less than n^2.
- Why is it stable? Assume not. Because of (A, b), i.e., there exists two couple $A-a, B-b$ in the matching such that A prefer b than a and b prefer A than B. Then, in this algorithm, A must proposed to b earlier than A proposed to a. Thus, such matching happen only when b rejected A. This means b has a better choice at the moment than A. So the partner of b must rank higher than A, but B is not higher than A, contradiction.

Definition 3.33 (Optimal matching)**.** *A stable matching is* **optimal** *for a women A if A's partner in this matching, say a is the optimal one for her; namely, in every possible stable matching, A's partner is a or someone she ranked lower than a. And a stable matching is* **women-optimal** *if it is an optimal matching for every women.*

Theorem 3.34. *The stable matching obtained from Deferred acceptance algorithm proposed by women is women-optimal. Also, for any man he gets the lowsest ranked partner among all feasible ones.*

Proof. We will show below claim; given any women say A, if A is rejected by a man a in the algorithm then a is not feasible for A. We can show it by the induction. If an women A is rejected by a man a in round k of the algorithm, then a is not feasible to A.

In the round 1, suppose $A \to a, B \to a$. If a rejects A, then a prefer B than A. So, if there exists a stable matching in which $A \to a$ and $B \to$someone else, then (B, a) is unstable; since a is the highest rank choice of B, and B is higher rank choice of a than A.

Now assume it true for k. In the next $k+1$-th round, assume A is rejected by a. We need to show that a is not feasible for A. Assume $A - a$ is in some other stable matching M'. Then a rejects A because a has a proposed from B where a prefer B than A. So, in the matching M',

$$B - b, A - a$$

for some man b. Since b is feasible for B, it means b has not reject B in the first k rounds of DAA. This implies that at round $k+1$, B already proposed to boy a but not proposed to b yet, otherwise b does not reject B so that a having proposal from B cannot happen; Thus, B prefers a than b. Hence in M', (B, a) is unstable pair, contradiction. □

3.5 Matchings in general graph

Definition 3.35 (k-factor)**.** *A k-factor is a spanning k-regular graph of a graph G.*

Note that if G has perfect matching, then it is 1-factor of G, since 1-regular graph is just a set of disjoint edges. Also note that if G has a 1-factor, then $|V(G)|$ is even. Note that 2-regular graph is just a set of disjoint cycles. Now define

$O(G) :=$ # of conencted component of G that have an odd order, i.e., the number of vertices in the component is odd.

Now let S be a subset of $V(G)$. Then, we can divides $G - S$ as a set of connected components. If G has a 1-factor, then every odd component in $G - S$ has a vertex connected to S; to see this, from the oddness, the intersection of a 1-factor and the connected component, say C, gives a graph of $|V(C)| - 1$ vertices, so one left vertex should be connected to outside of the component, but since this component is closed by connection on $G - S$, this vertex should be connected to vertex in S. We can summarize it as

Definition 3.36 (Tutte's condition)**.**

$$\forall S \subseteq V(G), o(G - S) \leq |S|.$$

From this, if we take $S = \emptyset$, then $|V(G)|$ is even, as we mentioned earlier. Actually, this condition is sufficient to make G have 1-factor.

Theorem 3.37 (Tutte)**.** *A simple graph G has an 1-factor if and only if G satisfies Tutte's condition.*

Note that adding edges increases the chance to have 1-factor Also, if G satisfies the Tutte's condition, then so is $G + e$, i.e., the graph generated by adding an edge on G.

Proof. Necessary part is already shown. For the sufficiency part, suppose the Tutte's condition holds. If there are an counterexample, then such G satisfies Tutte's condition but no 1-factor of G exists. Also, adding edge on G doesn't increase $o(G - S)$ since it preserves or reduces connected component. Also, if $G + e$ has no 1 factor, then G has no 1-factor. Thus assume that G satisfies Tutte's condition but G has no one factor, however, $G + e$ has a 1-factor for any missing edge of G.

Let $U = \{v \in V(G) : \deg(v) = n - 1\}$, where $n = |V(G)|$.

- $G - U$ consists of disjoint union of complete graphs. Still, $o(G - U) \leq |U|$, so there are even complete subgraphs (which are disjoint with other subgraphs) and odd complete subgraphs which are connected to U. Then we can find a 1-factor for each even complete subgraph. And for odd complete subgraphs, from the property that $\deg(v) = n - 1$ if $v \in U$, we can assign each edge from distinct vertices of U to distinct vertices of odd component. Now the remaining vertices are in U. And from $\deg(v) = n - 1$ for all $v \in U$, those are also form a clique, a complete graph. Hence it suffices to show that the number of remaining vertices of U is even. Actually, by taking $S = \emptyset$, we know that n is even. And also, our construction of matching for even complete subgraph and odd components takes only even number of vertices, so the number of remaining vertices in U is still even, thus we can take a matching from the clique, which is even complete graph, done.

- $G - U$ is not a disjoint union of complete graphs. In this case, $\exists x, z \in G - U$ such that $dist(x, z) = 2$ in $G - U$; otherwise, for all $x, z \in G - U$, $dist(x, z) = 0, 1\infty$, which implies $G - U$ consists of distinct edges and distinct vertices, implies that $G - U$ consists of disjoint complete graph, contradiction. Thus we can suppose $x - y - z$ is in $G - U$. Also, note that y is connected to every point in U in the sense of G, but from $y \notin U$, there exists a point $w \in G - U$ such that yw is not an edge in G. So y and w are in the distinct connected component in $G - U$. Then, from the assumption of G, both $G + xz$ and $G + yw$ have a 1-factor, say M_1 and M_2. Then $xz \in M_1, yw \in M_2$. Now think about the graph $F = (V(G), M_1 \Delta M_2)$. Then, $xz, yw \in E(F)$. Since every vertex of G has a degree 1 in M_1 or M_2, every vertex of F has a degree 0 or 2. So F is disjoint union of cycles and isolated points. Since $M_1 \neq M_2$, there exists a cycle, each has even length since the edges in the cycle are alternating M_1/M_2 edges. Look at the cycle C containing edge xz. Then,

  - If $yw \notin C$, then we can construct 1-factor by taking all edges in M_1 not in C, and take all edges in $M_2 \cap C$, and this 1-factor clearly do not contain xz, yw, so it is 1-factor of G, contradiction.
  - If $yw \in C$, then we can take 1- factor of C without using xz and yw; to get such a factor, think about edges in $M_2 \cap C$. Then it clearly do not contain xz but contain yw. Now let P_1 be a path in C starting from x, ending at y without containing z. Since C is a cycle, P_1 is well defined, and the first edge in P_1 containing x should be in M_2, since C is just M_1/M_2 alternating cycle. Also note that P_1' last edge, say $a - y$ cannot be in M_2, otherwise $a - y - w$ is in M_2, contradicting the definition of matching. Thus, $a - y \in M_1$. Then, take $M_2 \cap P_1$. This captures all points in P_1 except y. Now, since P_1 start with an edge in M_2 and ending with an edge in M_1, it has even number of edges. Thus, $P_2 = C - P_1 - xz$ has odd number of edges, and starting from y with $yw \in M_2$ and ending at z with an edge in M_2. Now by taking $M_1 \cap P_2$, it captures all points in P_2 except y and z. Then, by taking yz which is not in M_1 or M_2, we can make an 1-factor of C, without using xz or yw, which implies we can take 1-factor of G, contradiction.

□

Corollary 3.38 (Berge, Tutte)**.** *The largest number of vertices saturated by a matching in G is*

$$n - \max_{S \subseteq V(G)} d(S) \text{ where } d(S) = o(G - S) - |S|.$$

Proof. Given $S \subseteq V(G)$, at most $|S|$ edges can match vertices of S to vertices in odd component $o(G-S)$. so every matching has at least $d(S) = o(G-S) - |S|$ unsaturated vertices. So the largest number of vertices saturated by a matching is at most $\min\{n - d(S)\} = n - \max_{S \subseteq V(G)} d(S)$.

Conversely, let $k = \max(o(G-S) - |S|)$. Let $G' = G \vee K_k$, a ***join*** of two graph, which is generated by adding edges $\{xy : x \in V(G), y \in V(K_k)\}$ on the disjoint union of G and K_k. Now if G' has a 1-factor, say M', then M' contains at most k edges connecting G and K_k, which implies that deleting those k edges from M' is a matching in G, and this matching saturated at least $n-k$ vertices in G. To see this, we need to check that G' satisfy the Tutte's condition.

- Case 1: $S = \emptyset$. It suffices to show $n(G')$ is even. Note that

  $$n = |S| + \# \text{ vertices in the odd component of } G - S + \# \text{ vertices in the even component of } G - S.$$

  Then,

  $$n \equiv |S| + o(G-S) \bmod 2 \equiv |S| - o(G-S) \bmod 2$$

  for any $S \subseteq V(G)$. Thus, by taking S which gives k,

  $$|V(G')| = n + k \equiv |S| - o(G-S) + o(G-S) - |S| \bmod 2 = 0 \bmod 2.$$

  Thus, Tutte's condition holds when $S = \emptyset$.

- Case 2: $S \subseteq V(G')$ and If $\exists v \in K_k$ such that $v \notin S$. In this case, $G' - S$ is connected since this v has an edge with every point on G and K_k, thus $o(G'-S) \leq 1 \leq |S|$ when $|S| \neq \emptyset$.

- Case 3: $V(K_k) \subseteq S$. Then let $S_1 = S \setminus K_k$. Then, $G' - S = G - S_1$, so

  $$o(G'-S) - |S_1| = o(G - S_1) - |S_1| \leq k \implies o(G'-S) \leq k + |S_1| = |S|.$$

  where the first inequality comes from the maximality of k.

□

Corollary 3.39. *Let G be a 3-regular graph with no cut-edge, i.e., any cutting of edge cannot make G be a separated graph. Then G has a 1-factor.*

Proof. We show G satisfied the Tutte's condition. Let $S \subseteq V(G)$. Then let k be the number of edges between odd components and S. Then,

$$k \leq 3|S|$$

from the 3-regular condition.

We want to show that if there exists at least 3 edges between each odd component in $G - S$ and S. Then, from the 3-regular condition,

$$3o(G-S) \leq 3|S|.$$

To see this, think about $o(G-S) = k$. If $k = 1, 2, 3$, then from 3-regular, the feasible $|S|$ is 3. For any $k = 3l + j$, $j = 0, 1, 2$, the minimal feasible $|S|$ is $3(l+1)$, so $3o(G-S) \leq 3|S|$.

To show there exists at least 3 edges, let H be an arbitrary odd component of $G - S$. Let $E(H, S)$ be a set of edges between H and S. If $|E(H,S)| = 1$, then it is a bridge, contradicting the condition that G has no cut-edge. From 3-regular condition, we know that each vertex in H has a degree 3. So the sum of the vertex degress of H is $3|V(H)| - |E(H,S)|$. Since H is also a graph, this value should be even. Since $|V(H)|$ is odd, $|E(H,S)|$ should be odd. And we already know that $|E(H,S)| \neq 1$, so $|E(H,S)| \geq 3$. Thus, there are at least 3 edges connecting H to S for any odd component H. □

4 Connectivity and Path

4.1 Some facts about digraph

Remark 4.1. *Remind that* **trail** *is a walk with no repeated edges.*

Definition 4.2. **Digraph** *is a graph $D = (V, E)$ where adjacent matrix of G need not be symmetric. In this graph, we call edge as* **arc**, *and $uv = (u, v) \in E$ means there is an arch from u to v. Thus, $uv \neq vu$. For an arbitrary vertex $u \in V$,* **Outdegree** *and* **Indegree** *of u is defined as follow;*

$$Outdeg(u) := \#\{v \in V : (u, v) \in E\}, Indeg(u) = \#\{v \in V : (v, u) \in E\}.$$

So the number of edges of D is

$$|E(D)| = |E| = \sum_{v \in V} indeg(v) = \sum_{v \in V} outdeg(v).$$

We say a digraph D is ***connected*** if the underlying undirected graph is connected, and ***strongly connected*** if $\forall x, y \in V(D)$, $x \neq y$, $\exists$ a directed path from x to y.

Proposition 4.3. *A nontrivial connected D has a closed eulerian trail if and only if $\forall v \in V(D)$, $outdeg(v) = indeg(v)$.*

Note that D has a closed eulerian implies D is connected. (Note that although D is connected but D has no Eulerian circuit; $\bullet \to \bullet \leftarrow \bullet$.)

Proof. Suppose D is Eulerian. but there exist a vertex v with $outdeg(v) \neq indeg(v)$. If we departing at v to follow the Eulerian path, then we must go back to v since it is closed trail. This implies that this eulerian trail should contain the same number of outflow edges and inflow edges, since edge is not repeated in trail, done.

Conversely, suppose every $v \in V(D)$ has the same outdegree and indegree. Now let C be the longest trail of D starting at v and ending at u for some $v, u \in V(D)$. We claim that it is actually a circuit, i.e., $v = u$. Suppose not. then since this trail does not return to v, there exists an arc from some point $w \in V(D)$ to v which is not contained in C. So by starting at w, goes to v and following C, we can make a trail which is longer than C, contradiction.

Now we can do the induction. If $|E(D)| = 0$, then done. Suppose it holds for any digraph D with $|E(D)| < m$, and now assume $|E(D)| = m$. Then, from the above argument, we can find a longest circuit C, and think about $D' = (V, E \setminus E(C))$. Note that for each connected component of D' has Eulerian circuit by the inductive hypothesis, and the fact that C contains the same number of outgoing arc and ingoing arcs for each vertex $v \in V(C)$; note that the latter is from the definition of circuit. Now, from the connectedness assumption, each Eulerian trail of connected component of D' should have a point shared with C; (otherwise, from the connectedness assumption, there exists an arc from a point of the connected component of D' to point of C which is not belong to C, (otherwise the Eulerian trail of the connected component of D' should have a point in C.) But, from the definition of Eulerian trail, this arc should be contained in the Eulerian trail, so the point in C should be in the Eulerian trail, contradiction.) Then, we can make the Eulerian circuit C' of D, traversing C and if we met a point in D', then traversing the Eulerian circuit of the connected component of D'. Since C' contains C, so it is longer or at least of the same length with C. From the maximality of C, $C' = C$, done. □

Theorem 4.4 (Ghouila-Houri)**.** *Let D be a strongly connected digraph without loops. If for any $v \in V(D)$, $indeg(v) + outdeg(v) \geq n$, where $n = |V(D)|$, then D has a directed Hamiltonian cycle.*

See [3][p. 420] for argument about this theorem and hamiltonian.

Proposition 4.5. *Let $G = (V, E)$ be a connected graph. Then G has a strongly connected orientation if and only if G has no bridges, i.e., cut-edge.*

Proof. If G has a bridge, say e, then $G - e$ has a 2 connected component, $U - V$ saying U and V are those two connected component. Then, in any case of orientation, we cannot make both paths from U to V or V to U simultaneously, since only $U \to V$ or $V \to U$ occur in any orientation, so no orientation make the digraph strongly connected.

Conversely, if G has no bridge, let $U = \emptyset$, and let C be an arbitrary cycle in G. Note that if C does not exists, then G is a tree, so it has a cut edge, contradiction. Now make orientation for C. Now, add all the vertices to U while $U \neq V(G)$. If $U \neq V(G)$, then there exists an edge $\alpha = xy$ which is not contained in C but $x \in V(C)$, so $y \notin V(C)$. Now take $C(\alpha)$ in G, a cycle containing α. (Also, this cycle must exists, otherwise α is a cut-edge, contradiction.) Now take a path in $C(\alpha)$ from x to z, where z is the first vertex in $C(\alpha)$ which is in U when we walk from x to y along $C(\alpha)$. Then orient edge α from x to z along this path (i.e., $x \to x_1 \to \cdots \to z$.) Then keep those vertices in the path from x to z in U. Repeat this process again. Since G is finite, thus this process is always terminated when $U = V(G)$. Now to show that there exists a direct path $u, v \in V(G)$ for this orientation, think about the generating process of the orientation; if u, v are added in the same step, then there exists a directed path, done. Otherwise, without loss of generality, assume u is added first and v is added in some later step. If v is added in U in the first step after u is added in U, then there exists a path from x to v, which are added in the same step with v, and x is added in the step when u belongs to U, so we have a directed path from u to v, done. If v is added in the n-step after u is added, and we have a path from u to any vertex added in the $1, 2, \cdots, n-1$-th step, then there exists a vertex x which has a directed path to v by construction, but x is added in the previous step, so we have a directed path from u to x, thus by joining those two paths, we have a directed path from $u \to x \to v$, done. So G under constructed orientation is strongly connected. □

Example 4.6 (2-trading problem)**.** *Suppose that n traders are here, and each one has an item for selling to the market, and that they can trade freely. At the end, each person gets one item back.*

This situation is just an allocation of items to each persons, so we can represent it by permutation of trading goods. So let $l(t_i) = t_j$ if trader t_i gets the item from t_j. Also suppose that every one has a linear ranking on all the items. (Note that l is a bijection on $[n]$, so we can regard it as permutation]

Definition 4.7 (Core allocation)**.** *l is a* **core allocation** *if after l permutes items, then there is no subset S of $\{t_i\}_{i=1}^n$ such that by trading among themselves, each receives an item that he ranks higher.*

Is there a core allocation exists? We can use the below lemma to find core allocation.

Lemma 4.8. *In a digraph in which every vertex has outdegree ≥ 1, then there is a directed cycles.*

Proof. Think only connected component first. Choose any point v_1 and walk into another vertex, say v_2. If v_2 has outflow to point which is not visited before, then go to there, otherwise we can make a cycle, done. Do this until 1) we have a cycle or 2) we visit every point in G. If we visit every point in G, it means that we didn't make a cycle; then, the last point, say v_n, also has an outflow, and this should connect with one of $v_1, \cdots, v_{n-1}$, say v_i. so we can make a cycle $v_i \to \cdots \to v_n \to v_i$, done.

For nonconnected case, we can find a cycle on the connected component, done. □

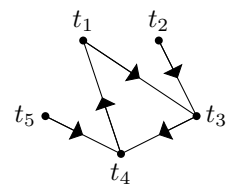
To use this, let's make a digraph from the linear rankings first. For example, say below matrix

trader \item	t_1	t_2	t_3	t_4	t_5
t_1	4	3	1	2	5
t_2	4	3	1	2	5
t_3	4	3	5	1	2
t_4	1	4	3	5	2
t_5	4	5	2	1	3

Build a digraph on n vertices by adding arc

$$t_i \to t_j \text{ if } t_i \text{ ranks the item of } t_j\text{'s the highest.}$$

For example, in the above ranking system, we have



Now find a directed cycle by the lemma. Remove these traders and items. In this case, we can get rid of t_1, t_3, t_4. Now we can generate a ranking matrix between t_2 and t_5, which is

trader \item	t_2	t_5
t_2	1	2
t_5	2	1

So the digraph is just $t_2 \to t_2, t_5 \to t_5$, hence we have a core allocation, which is

$$l = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ t_3 & t_2 & t_4 & t_1 & t_5 \end{pmatrix}.$$

Now we can prove the above process always gives a core allocation; each round, we have a cycle $C_1, C_2, \cdots, C_r$. If the result is not core allocation, then there exists nonempty $S \subseteq V$ such that make a trade on themselves so that they can have the item higher than the current item they have. Now find the minimum indexed cycle which contains a trader x in S. Then, $x \in C_i \cap S$ for some i. By this i-th round, x is already get the best possible item. So he cannot improve his item by trading in $y \in C_i \cup C_{i+1} \cup \cdots \cup C_r$. But, $S \subseteq C_i \cup C_{i+1} \cup \cdots \cup C_r$ by minimality of i, so x cannot make his item be better by trading with someone in S, so $x \notin S$, contradiction.

4.2 Network

Definition 4.9. *A* **network** *is defined as $N = (V, A, s, t, c)$ where V, A are denoting a set of vertices (called* **nodes***), and a set of arcs, and $s, t \in V$ denote the* **source** *and* **sink** *respectively, and $c : A \to \mathbb{R}^+$ is called capacity, and denote an weight associated to each arc.*

Definition 4.10. *A flow f on N assigns a value $f(e)$ for each edge e. We denote $f^+(v)$ for the total flow on edges leaving v, saying* **outlfow of** *v and $f^-(v)$ for the total flow on edges entering v, saying* **inflow of** *v. A flow is* **feasible** *if it satisfies the capacity constraints $0 \le f(e) \le c(e)$ for each edge $e \in A$, and the* **conservation constraints** *if $f^+(v) = f^-(v)$ for each node $v \notin \{s, t\}$, i.e.,*

$$\forall x \in V \setminus \{s, t\}, \sum_{e \text{ departs at } x} f(e) = \sum_{e \text{ arrives at } x} f(e)$$

After that, we only care about feasible flow, so 'flow' denotes just a feasible flow in later.

Also, we denote ***flow out of the source*** as

$$f^+(s) - f^-(s)$$

and ***flow entering the sink*** as

$$f^-(t) - f^+(t).$$

Let $U \subseteq V$, then define

$$\vec{U} := \{\alpha \in A : init(\alpha) \in U, term(\alpha) \notin U\}$$

where $init(e)$ is the initial point of e and $term(e)$ is the terminal point of e for any $e \in A$. Similarly,

$$\overleftarrow{U} := \{\alpha \in A : init(\alpha) \notin U, term(\alpha) \in U\}.$$

Lemma 4.11. *Let f be a flow on a network. Let U be a subset of V that contains s but not t. Then,*

$$\sum_{\alpha \in \vec{U}} f(\alpha) - \sum_{\alpha \in \overleftarrow{U}} f(\alpha) = f^+(s) - f^-(s).$$

Proof.

$$f^+(S) - f^-(S) = \sum_{x \in U} (f^+(x) - f^-(x)) = \sum_{x \in U} f^+(x) - \sum_{x \in U} f^-(x) = \sum_{\alpha \in \vec{U}} f^+(x) - \sum_{\alpha \in \overleftarrow{U}} f^-(x),$$

where the first equality comes from the fact that s is the only one vertex in U who may not satisfy the conservation law, and the last equality comes from the fact that if $x, y \in U$, we can cancel out those flow so that we only care about a flow out from U to U^c or a flow in from U^c to U. □

Now take $U = V \setminus \{t\}$. Then $\sum_{\alpha \in \vec{U}} f(\alpha) - \sum_{\alpha \in \overleftarrow{U}} f(\alpha) = f^-(t) - f^+(t)$, thus

$$f^+(s) - f^-(s) = f^-(t) - f^+(t)$$

by the above lemma. This number $f^+(s) - f^-(s) = f^-(t) - f^+(t)$ is called the ***value of*** f, denoted by $val(f)$.

Now one may question that the maximum value of N among all feasible flows.

Definition 4.12 (Cut)**.** *A cut C in a network N is a set of arcs such that after removing C, there is no directed path from s to t.*

Now define the ***capacity*** of a cut C as

$$cap(C) := \sum_{\alpha \in C} c(\alpha).$$

We say C is ***minimum*** if it has the smallest capacity among all cuts.

For example, if $U \subseteq V$ such that $s \in U, t \notin U$, then $\vec{U}$ is a cut.

Lemma 4.13. *Let C be a minimum cut, and $U := \{x \in V : \exists$ a path from s to x in $N - C.\}$ Then, $C = \vec{U}$.*

Proof. To see $\vec{U} \subseteq C$, Let $\alpha = (x, y) \in \vec{U}$. If $\alpha \notin C$, then $s \to x \to y$ is a path in $N - C$, so C is not a cut, contradiction. Hence $\alpha \in C$. Conversley, if $\alpha = (x, y) \in C$. Then, if we add α back to $N - C$, then from the minimum condition, C is not a cut, so $\exists$ a path from s to t containing α. Say this path A. This implies that A contains a path from s to x in $N - C$, so $x \in U$. Now it suffices to show that $y \in U^c$. If $y \in U$, then there exists a path $s \to y$ in $N - C$. Then since $y \to t$ is in A which is in $N - C$, so there exists a path $s \to t$ in $N - C$, contradicting the construction of C, so $y \in U^c$. □

Theorem 4.14. *Let $U \subseteq V$ containing s, not t. Then for all flow f on N,*

$$cap(\vec{U}) \geq val(f).$$

Hence, for minimum cut C,

$$cap(C) \geq val(f).$$

Proof.

$$val(f) = \sum_{\alpha \in \vec{U}} f(\alpha) - \sum_{\alpha \in \overleftarrow{U}} f(\alpha) \leq \sum_{\alpha \in \vec{U}} f(\alpha) \leq \sum_{\alpha \in \vec{U}} c(\alpha) = cap(\vec{U})$$

where the first equality comes from the above lemma, and the rest is just definition of capacity. The last statement comes from the fact that U is arbitrarily chosen, and one of U gives a minimum cut C. □

So, how to increase $val(f)$ by changing $f(e)$ for each $e \in A$?

Definition 4.15. f**-augmenting path** *is a path P from s to t in the underlying graph of N such that for each arc, say α, on the path, if direction of arc agrees with that of P, then $f(\alpha) < C(\alpha)$, and if the direction of α doesn't agree with that of P then $f(\alpha) > 0$.*

Let $\epsilon(e) = \begin{cases} c(e) - f(e) & \text{if direction of } e \text{ agrees with } P \\ f(e) & \text{otherwise.} \end{cases}$ *Then,* **tolerance** *of P is $\epsilon(P) = \min\{\epsilon(e) : e \in P\}$.*

Note that f-augmenting path may not be alternating; for example $\bullet \to \bullet \leftarrow \bullet$ and $\bullet \to \bullet \to \bullet$ can be f augmenting path if they satisfy the condition of augmenting path.

Theorem 4.16 (Ford-Fulkerson)**.** *In every network,*

$$\max_{f \text{ is flow}} val(f) = \min_{c \text{ is a cut}} cap(c).$$

Actually this is the result of algorithm described as below.

Proof. Input is $N = (V, A, s, t, c)$.

- Start with any flow (at least 0-flow, i,e, $f(e) = 0$ for any $e \in A$.). Set $U = \{s\}$.
- While $\exists \alpha = (x, y)$ such that either a) $\alpha \in \vec{U}$ and $f(\alpha) < c(\alpha)$ or b) $\alpha \in \overleftarrow{U}$ and $f(\alpha) > 0$,
  - In case a), add y to U
  - in case b), add x to U.
- Output:U.

Now we can check the outputs. If $t \in U$, then there exists a f-augmenting path Q say

$$Q = x_0 = s \to x_1 \to \cdots \to x_m = t.$$

Then, $\epsilon(Q) \geq 0$.

- If $\epsilon(Q) > 0$, then update f by

$$f(\alpha) = \begin{cases} f(\alpha) + \epsilon & \text{if direction of } \alpha \text{ agrees with } Q \\ f(\alpha) - \epsilon & \text{if direction of } \alpha \text{ disagrees with } Q \\ f(\alpha) & \text{if } \alpha \notin Q \end{cases}.$$

  Then this updated flow, say f^{new} is a flow with value $val(f) + \epsilon$.

- If $t \notin U$, then $\vec{U}$ is a cut. Note that for any arc in $\beta \in \vec{U}$, $f(\beta) = c(\beta)$ otherwise $term(\beta) \in U$ by the algorithm. Also, for any arc in $\beta \in \overleftarrow{U}$, $f(\beta) = 0$, otherwise, $init(\beta) \in U$.

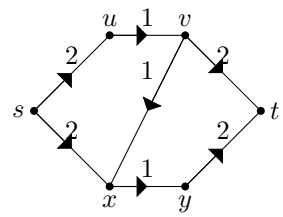
So

$$val(f) = \sum_{\alpha \in \vec{U}} f(\alpha) - \sum_{\alpha \in \overleftarrow{U}} f(\alpha) = \sum_{\alpha \in \vec{U}} f(\alpha) = cap(\vec{U}).$$

where the first equality comes from the lemma above, and the rest equality comes from the property of this specific U.

This equation shows that capacity of minimum cut is just the maximum value of flow, since the algorithm generates both the maximum flow and a cut having $val(f)$ as capacity simultaneously. □

For example, think about the below network



Start with $f(e) = 0$ for all edges. Then in the first round of the algorithm we get $U = \{s, u, x, v, t\}$. Then we have a f-augmenting path $s \to u \to v \to t$, and $\epsilon = \min(2, 1, 2) = 1$. Thus update it so that

$$f(s \to u) = f(u \to v) = f(v \to t) = 1, f(e) = 0 \text{ for other } e.$$

In the second round, we have $U = \{s, u, x, y, t\}$, and $Q = s \to x \to y \to t$, $\epsilon = \min\{2, 1, 2\} = 1$, so we update it then we have

$$f(s \to u) = f(u \to v) = f(v \to t) = f(s \to x) = f(x \to y) = f(y \to t) = 1, f(e) = 0 \text{ for other } e.$$

Then, in the third round, $U = \{s, u, x\}$ hence f reached maximum flow, so $val(f) = 2, cap(\vec{U}) = 2$.

Theorem 4.17 (Integrality theorem)**.** *If all capacities in a network are in $\mathbb{Z}$, then there is a maximum flow with $f(\alpha) \in \mathbb{Z}$. Furthermore, some maximum flow can be partitioned into flows of unit value along paths from s to t.*

Proof. This is because ϵ is determined from the capcity, and updating f is determined by ϵ. □

Note that in the proof of the Ford-Fulkerson algorithm, we implicitly use the fact that ϵ is rational; otherwise, this generating process of f-augment path may not end. Edmond and karp gives an algorithm using at most $(n^3 - n)/4$ augmentations to get the maximum flow for all real capacities.

Definition 4.18 (Set up for Menger's theorem)**.** *In a digraph $D = (V, A)$, let $s, t \in V$ with $s \neq t$. A s-t separating set of arcs is defined as a set where every path from s to t uses at least one arc in S .*

So, there are several s,t separating set, for example, A itself.

Theorem 4.19 (Menger's theorem)**.** *The minimum number of arcs in a s-t separating set of arcs S is equal to the maximum number of pairwise arc-disjoint paths from s to t.*

Proof. Make a network $N = (V(D), E(D), s, t, c)$ by setting $c(\alpha) := 1, \forall \alpha \in A(D)$. Then from the Ford-Fulkerson algorithm, minimum capacity of cut is equal to maximum value of flow. Say the maximum value as p. Also, from the integrality theorem, this flow is union of unit-value flow along path from s to t. Then, they are arc-disjoint because $0 < f(\alpha) = c(\alpha)$; if two path from distinct flow share the edge, then capacity of that edge should be at least 2, since its union is also a feasible flow. □

There are several applications.

- For a particular team X want to know whether it can win the championship by the end of the season. We know $X_1, \cdots, X_n$ other team's winning data up to now. We know the number of games still remain between any 2 team. Then we can use the network;

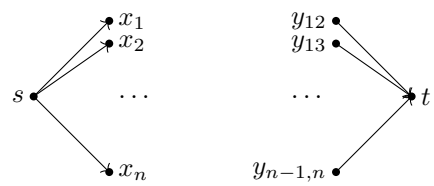
  Assume X can win all its remaining games and reach W wins. hope other teams don't win over W games. Let

  $$W_i = \# \text{ of games win by team } X_t.$$

  To win the championship, we need condition that

  $$X_i \leq W - W_i$$

i.e., the number of total winning games of team i is less than that of my team. Then make a network



where $x_i \to y_{ij}$ and $x_j \to y_{ij}$ for any $i < j \in [n]$ with capacity as below;

Arcs	(s, x_i)	(y_{ij}, t)	(x_i, y_{ij})	(x_j, y_{ij})
Capacity	$W - W_i$	a_{ij}	∞	∞

where a_{ij} is the number of games remaining between teams X_i and X_j. Then we can denote a game in which X_i beats X_j as a flow f with $f(e) = 1$ for $e \in s \to X_i \to y_{ij} \to t$ and $f(e) = 0$ otherwise. So, X can win the championship if and only if there exists a flow on the network which saturates the arcs (y_{ij}, t), i.e., $f((y_{ij}, t)) = a_{ij}$ for all $i, j \in [n]$, i.e., there exists a possibility that no team wins more than W. (Note that flow not saturating some (y_{ij}, t) means nothing since it cannot assure whether X can win or not if we count all possible remaining games of other teams. Also, if there exists a flow having maximum value greater than $\sum_{i<j} a_{ij}$, then it contradicts the fact that $cap(E($

Observe that $\{(y_{ij}, t), \forall i < j\}$ is a cut so if the desired f exists, then every cut on N has capacity greater than or equal to $\sum_{i<j} a_{ij}$. So we need to show that $\{(y_{ij}, t), i < j\}$ is a minimum cut. So we need to show that for all $U \subseteq V$ s.t. $s \in U, t \notin U$,

$$cap(\vec{U}) \geq \sum_{i<j} a_{ij}.$$

To get the minimum $cap(\vec{U})$, for any $x_i \in U$, assume $y_{ij}, y_{ji} \in U$; otherwise $x_i \to y_{ij}$ or x_i to y_{ji} are in $\vec{U}$ so that $cap(\vec{U}) = \infty$. Also, for any $x_k, x_l \notin U$, delete $y_{kl} \notin U$. (Since they affect capacity with a_{ij}.)Then we can represent U as

$$U = \{x_i : i \in K\} \cup \{y_{ij} : i \text{ or } j \in K\}$$

for some $K \subseteq [n]$. Then, $\{(y_{ij}, t), \forall i < j\}$ is a minimum cut if and only if

$$cap(\vec{U}) = \sum_{i \notin K} (W - W_i) + \sum_{i \text{ or } j \in K} a_{ij} \geq \sum_{i<j \in [n]} a_{ij}, \forall K \subseteq [n]$$

i.e.

$$\sum_{i \notin K} (W - W_i) \geq \sum_{i \text{ and } j \notin K} a_{ij}, \forall K \subseteq [n]$$

By taking $L = [n] - K$, we can say

$$\sum_{i \in L} (W - W_i) \geq \sum_{i \text{ and } j \in L} a_{ij}, \forall L \subseteq [n]$$

And this is reasonable condition; The remaining winnings for team i to reach W is greater than their total remaining games.

- General network model is a network with multiple sources x_i with supply $\sigma(x_i)$ and multiple sinks y_j with demand $\partial(y_j)$. In addition to capacity and conservertive constraints, there exists a transportation constraints such that

$$\text{Limited Source:} \forall x_i, f^+(x_i) - f^-(x_i) \leq \sigma(x_i)$$
$$\text{Meet Demand:} \forall y_j, f^-(y_j) - f^+(y_j) \leq \partial(y_j)$$

For a set of sources $X = \{x_j\}_{j=1}^{i}$ and a set of sinks $Y = \{y_j\}_{j=1}^{l}$, are there any feasible flow? To answer this we need a notation; for all $U \subseteq V(N)$, define supply as

$$\sigma(U \cap X) = \sum_{x_i \in U} \sigma(x_i)$$

and demand as

$$\partial(U \cap Y) = \sum_{y_i \in U} \partial(y_i)$$

For any 2 disjoint subset S, T of vertices, define

$$cap(S, T) := \sum_{\alpha : S \to T} cap(\alpha).$$

Then,

$$cap(\vec{U}) = cap(U, U^c), cap(\overleftarrow{U}) = cap(U^c, U).$$

Then the neccesary condition for feasible flow is that

$$\forall T \subseteq V(N), cap(T^c, T) \geq \partial(T \cap Y) - \sigma(T \cap X),$$

i.e., capacity betwen any subset of sinks and any subset of sources should be greater than the deficiency, which is difference of demand and supply of those sinks and sources respectively. This is called ***Gale's condition***.

Theorem 4.20 (Gale)**.** *there exists a feasible flow iff the Gale's condition holds.*

Proof. Sufficient part; construct a network N' by adding a supersource s and supersink t to N, with new arcs (s, x_i) with capacity $\sigma(x_i)$ and (y_j, t) with capacity $\partial(y_j)$. If N' has a flow saturates all arcs (y_j, t), then it is a feasible flow on N. Also note that $\{(y_j, t)\}_{j=1}^{l}$ is a cut in N'. Hence by the Menger's theorem, there exists a flow saturating $\{(y_j, t)\}_{j=1}^{l}$ if and only if $C = \{(y_j, t)\}_{j=1}^{l}$ is a minimum cut of N'. So all we need to do is to check that the Gale's condition implies C is a minimum cut.

Let U be any subset of $V(N)$, Then, we can calculate $cap(U \cup \{s\}, U^c \cup \{t\})$ in N'. Note that this value counts capacity of

1. $s \to$ source in U^c, which gives $\sigma(U^c)$
2. sinks in U to t, which gives $\partial(U)$
3. Arcs in N from U to U^c, which gives $cap(U, U^c)$ in N.

Thus,

$$cap(U \cup \{s\}, U^c \cup \{t\}) \geq \partial(Y) \iff cap(U, U^c) + \sigma(U^c) + \partial(U) \geq \partial(Y) \iff cap(U, U^c) \geq \partial(U^c) - \sigma(U^c)$$

so, by letting $T = U^c$, this is converted into

$$cap(T^c, T) \geq \partial(T \cap Y) - \sigma(T \cap X)$$

thus the condition from Menger's theorem is just equivalent to the Gale's condition since we choose U arbitrarily, done. □

Definition 4.21 (Bigraphic)**.** *Given $\overline{p} = (p_1, \cdots, p_m) \in (\mathbb{N} \cup \{0\})^m \overline{q} = (q_1, \cdots, q_n) \in (\mathbb{N} \cup \{0\})^n$, we say $(\overline{p}, \overline{q})$ is* **bigraphic** *if $\exists$ a simple bipartite graph $(X \cup Y, E)$ s.t. $|X| = m, |Y| = n$ and the degree sequence of X is $\overline{p}$ and that of Y is $\overline{q}$.*

For example, if $\bar{p} = (1, 2, 3), \bar{q} = (3, 3, 0)$ is not bigraphic; if it is, then a simple bipartite graph generated by $X \cup Y$ satisfy a condition that x_3 must have three edges from x_3 to three points in Y, however, only two points in Y has the degree greater than 0.

In contrast, $\bar{p} = (1, 2, 3), \bar{q} = (2, 2, 2)$ is bigraphic; make edges such as $x_1 \to y_1, x_2 \to y_2, x_2 \to y_3, x_3 \to y_1, x_3 \to y_2, x_3 \to y_2$.

Theorem 4.22. *Assume $p_1 \geq \cdots \geq p_m$ and $q_1 \geq \cdots \geq q_m$. Then $(\bar{p}, \bar{q})$ is bigraphic if and only if 1) $\sum_i p_i = \sum_j q_j$ and 2) $\forall k \in [n]$, $\sum_{i=1}^m \min(p_i, k) \geq \sum_{j=1}^k q_j$.*

Proof. Note that the first condition is not holds, then it is not bigraphic; since bipartite graph should have $\deg(X) = \deg(Y)$. So suppose it holds. Let G be a simple bipartite graph realizing $(\bar{p}, \bar{q})$. Take $T = \{t_1, \cdots, t_k\} \subseteq Y$. Then, the number of edges incident to T is

$$\deg(t_1) + \cdots + \deg(t_k) \leq q_1 + \cdots + q_k$$

since degrees are sorted as descending order. Also, note that in a viewpoint of x_i, x_i has at most $\min(p_i, k)$ neighbors in T. Thus

$$\sum_{i=1}^m \min(p_i, k)$$

is an upper bound on the number of edges incident to T with $|T| = k$. Thus we need a condition $\forall k \in [n]$, $\sum_{i=1}^m \min(p_i, k) \geq \sum_{j=1}^k q_j$.

To show a sufficiency, build a supply-demand network, by letting X be a set of sources with $\sigma(x_i) = p_i$ and Y be a set of sinks with $\partial(y_j) = q_j$. Consider the complete bipartite graph $X \cup Y$, and assign 1 for each edge's capacity. Then, since unit capacity preserves multiple edges, $(\bar{p}, \bar{q})$ is bigraphc if and only if there exists a feasible flow. And this is equivalent to say that this network satisfy the Gale's condition, by Gale's theorem.

Let $T \subseteq V(N) = X \cup Y$. Then, define $X_T = T \cap X, Y_T = T \cap Y, X_S = T^c \cap X, Y_S = T^c \cap Y$. Then, Gale's condition says

$$cap(T^c, T) + \sigma(T) \geq \partial(T).$$

In this network,

$$cap(T^c, T) = \# \text{ of arcs from } X_S \text{ to } Y_T = |X_s| \cdot |Y_T|$$

since underlying graph is complete bipartite. And

$$\sigma(T) = \sum_{x_i \in X_T} p_i, \partial(T) = \sum_{y_j \in Y_T} q_j.$$

Let's assume $|Y_T| = k$ Then,

$$cap(T^c, T) = |X_S|k = \sum_{x \in X_s} k \geq \sum_{x_i \in X_S} \min(p_i, k).$$

And

$$\sigma(T) = \sum_{x_i \in X_T} p_i \geq \sum_{x_i \in X_T} \min(p_i, k)$$

Hence

$$cap(T^c, T) + \sigma(T) \geq \sum_{i=1}^m \min(p_i, k) \geq q_1 + \cdots + q_n = \sum_{y_j \in Y_T} q_j$$

where the first inequality comes from the given condition, and the second inequality comes from the fact that $\max_{T \cap Y \subseteq Y} q_j \leq \sum_{i=1}^n q_j$, done. □

4.3 General Connectivity

Let $G = (V, E)$. Then

Definition 4.23. **Vertex separating set** *$S \subseteq V$ is a set such that $G - S$, a graph removing all vertices of S and all edges incident to S is not connected. Then we define* **connectivity** *as*

$$\kappa(G) := \min\{|S| : G - S \text{ is disconnected or } G - S \text{ has only 1 vertex}\}.$$

*Also, we say G is k-***connected** *if $\kappa(G) \geq k$.*

For example, the complete graph K_n has $\kappa(K_n) = n - 1$. The complete bipartite graph $K_{m,n}$ has $\kappa(K_{m,n}) = \min(m, n)$. Also note that

$$\kappa(G) \leq \delta(G)$$

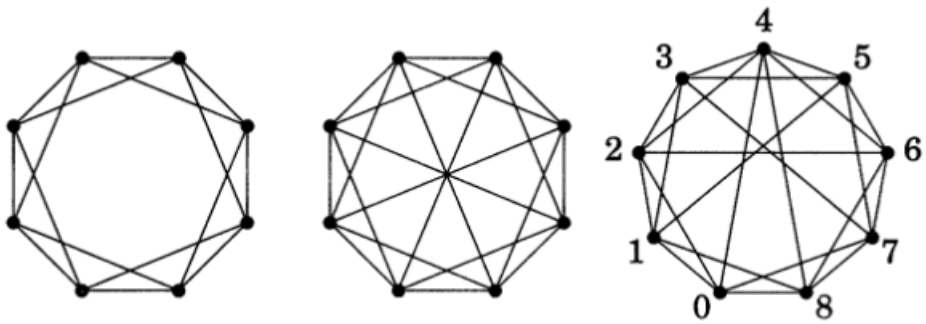
where $\delta(G)$ is the minimum vertex degree of G, since for a vertex $v \in G$ having degree $\delta(G)$, we can take $S = N(v)$.

Also, if G has connectivity k, then by above inequality, $\delta(G) \geq k$, hence G has at least $\lceil \frac{kn}{2} \rceil$ edges. And the bound is best possible whenever $k < n$ shown by Harary graph $H_{k,n}$.

Definition 4.24. *$H_{n,k}$ with $k < n$ is a graph generated by following step; place n vertices around a circle, equally spaced. If k is even, then form $H_{n,k}$ by making each vertex adjacent to the nearest $k/2$ vertices. If k is odd and n is even, then form $H_{n,k}$ by making each vertex adjacent to the nearest $\frac{k-1}{2}$ vertices on both sides (right and left) and to the diametrically opposite vertex. In each case, $H_{k,n}$ is k-regular.*

When k, n are both odd, then $H_{k,n} = H_{k-1,n} + \{(i, i + \frac{n-1}{2}) : 0 \leq i \leq \frac{n-1}{2}\}$.

Below graphs are $H_{4,8}, H_{5,8}$, and $H_{5,9}$ respectively, from the left to the right.



Claim 4.25. *If k is even, then $H_{k,n}$ are k-connected.*

Proof. If one can remove S vertices where $|S| < k$, and the resulted graph is disconnected, then for all $x \in V(G) \setminus S$, $\exists y_1 \in V(G) \setminus S$ the first vertex not reachable from x when we walk from x to y_1 in clockwise direction. Similarly, let y_2 be first vertex which is not reachable from x when we walk from x to y_2 in counterclockwise direction. (Note that $y_1 = y_2$ is possible.) So all neighbors of y_1 between x and y must in S, which is $k/2$ many vertices. by the same reason, all neighbors of y_2 vertices between x and y_2 must belong to S, this is also $k/2$ many vertices. Since those vertices are distinct from each other, so $|S| \geq k$, contradiction. □

Definition 4.26 (Disconnecting set)**.** *Let $F \subseteq E(G)$ is a* **disconnecting set** *of edge if $G - F$ has at least 2 connected components. We say G is k-***edge connected** *if every F has size $\geq k$. Then the* **edge-connectivity** *is defined as*

$$\kappa'(G) = \min\{|F| : F \text{ is diconnecting set of } G\}.$$

For any $S \subsetneq V(G)$ with $S \neq \emptyset$, an **edge cut** *is defined as*

$$E(S, S^c) = \{ \text{ edges connecting } S \text{ to } S^c\}$$

Note that every minimal disconnecting set of edges is an edge cut when $n(G) > 1$. To see this, if $G - F$ has more than one component for some $F \subseteq E(G)$, then for some component H of $G - F$, we have deleted all edges with exactly one endpoint in H. Hence F contains the edge cut $E(V(H), V(H)^c)$ and F is not minimal disconnecting set unless $F = E(V(H), V(H)^c)$.

Obviously, $\kappa'(G) \leq \delta(G)$, since taking F be a set of all edges connected to the vertex having degree $\delta(G)$ implies F is disconnecting set.

Theorem 4.27 (Whitney, 1932)**.** $\kappa(G) \leq \kappa'(G)$.

Proof. Let F be the minimal edge disconnecting set of size $\kappa'(G)$. Then by the above argument, $F = E(S, S^c)$ for some $\emptyset \subsetneq S \subsetneq V(G)$. Now observe edges between S and S^c. Since the upperbound of $\kappa(G)$ is $n(G) - 1$, if every vertex in S is adjacent to every vertex in S^c, then

$$k' = |E(S, S^c)| = |S||S^c| \geq n(G) - 1 \geq \kappa(G),$$

since $\min |S||S^c|$ is just $\min_{i \in [n]}\{i \cdot n - i\}$.

Otherwise, $\exists x \in S, y \in \S^c$ but xy is not an edge in G. Let $T = N(x) \cap S^c \cup \{s \in S : s \neq x \text{ and } N(s) \cap S^c \neq \emptyset\}$. Then, x and y stays in the different connected components in $G - T$ (to see this, note that $N(y) \subseteq T$, so y and x has no path in $G - T$.) Thus T is separating set. Now each $u \in N(x) \cap S^c$ corresponds to the edge $xu \in E(S, S^c)$ and each $s \in \{s \in S : s \neq x \text{ and } N(s) \cap S^c \neq \emptyset\}$ corresponds to the at least one edge from s to some point $s^c \in N(s) \subseteq S^c$. Thus

$$|T| \leq |E(S, S^c)|$$

and from the fact that T is a separating set and $E(S, S^c)$ is the minimal edge disconnecting set, we have

$$\kappa(G) \leq |T| \leq |E(S, S^c)| = \kappa'(G).$$

done. □

Example 4.28. *If G is 3-regular, then $\kappa(G) = \kappa'(G) \leq 3$. To see this, assume G is connected. Let S be a vertex cut of size $\kappa(G) = k$. It suffices to find a separating set of edges of size k. Let H_1, H_2 are connected components of $G - S$. Then check that each vertex s in S, s must be connected to both H_1 and H_2 by the minimality of S. There are three cases happen.*

- *Case 1: if $\#E(s, H_1) = 1$ and $\#E(s, H_2) = 2$.*
- *Case 2: if $\#E(s, H_1) = 2$ and $\#E(s, H_2) = 1$.*
- *Case 3: if $\#E(s, H_1) = 1$ and $\#E(s, H_2) = 1$,*

Now construct a set S' as follow; for each $v \in S$, if v is

- *Case 1, then take $s \to H_1$ be in S'. (Since this edge is uniquely determined, such notation is well-defined.)*
- *Case 2, then take $s \to H_2$ be in S'. (Since this edge is uniquely determined, such notation is well-defined.)*
- *Case 3, then take $s \to H_1$ be in S'. (Since this edge is uniquely determined, such notation is well-defined.)*

By this operation, $|S'| = |S|$ and this gives two connnected component containing H_1 and H_2, respectively. Say those as H_1', H_2' respectively. To see why this is a connected component, note that $v \in H_1'$ if and only if $v \in H_1$ or $v \in S$ with case 2. If $v \in H_1$, then there are four cases happen;

- *Case 1: if v is connected with $s, s' \in S$, then if s or s' is case 1 or 3, it implies v is separated with s or s' by S'. Or if one of them is case 2, then s or s' is in H_1'. So v has no edge connecting to H_2'.*
- *Case 2: if v is connected with s only, then by the same argument above, v has no edge connecting to H_2'*
- *Case 3: if v is connected with vertices in H_1 only, then trivially it has no edge connecting to H_2'.*

Also if $v \in S$, then by construction, v is case 2, so still no edge exists connecting v to H_2'. We can do the similar argument for H_2' so that H_1' and H_2' has no vertices having edge connecting to each other, done.

Now we can rewrite the Menger's theorem in digraph edge version;

Theorem 4.29. *Given $s, t \in V(D)$, a set $F \subseteq E(D)$ is called s, t* **disconnecting set** *if $D - F$ has no s, t edge disjoint path. Then,*

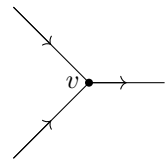
$$\max \# \text{ of edge disjoint (s,t) path in } D = \min \text{ size of s-t disconnecting set of edges s.}$$

4.4 Network capacity constraints on vertex

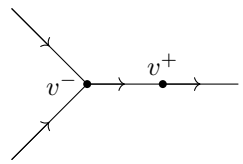
What if the capacity constraints are on vertices? That is, for all $v \in V(N)$, $v \neq s, t$, a feasible flow should satisfy

$$f^+(v) = f^-(v) \leq c(v).$$

Actually, this can be converted into another network; for example, if we have a vertex like below;



we can rethink the constraints from the vertex capacity as a constraints of edge capacity in a new network as below;



where new vertex v^+ and new arc $v^- \to v^+$ are inserted in the original graph, and the edge capacity of $v^- \to v^+$ is the vertex capacity.

Now assume N has only vertex capacity constraints. (i.e., edge capacity is ∞.) let N' be a new network generated for translating vertex capacity condition, and $E(N')$ be a set of edges in N'. In $E(N')$ we still have max flow = min cut. So to be minimum, it has to contain only edges of the form $v^- \to v^+$. So we can turn it into the set of vertices on N.

So, for given a digraph D, or an undirected graph G, with $s \neq t$, with st is not an edge or arc, make a network N with vertex capacity $C(v) = 1$ for all $v \neq s, t$. And let $c(e) = \infty$ for all edges. Then by above construction of N' we can get the Menger's theorem in vertex version as below;

Theorem 4.30 (Menger's theorem in vertex version)**.**

$$\min \# \textit{ of vetices separating } s \textit{ and } t = \max \# \textit{ of independent paths from } s \textit{ to } t$$

where ***independent path*** denotes a vertex disjoint path except s, t. (Two paths are independent if their vertices are disjoint to each other except s and t.)

Corollary 4.31. *For $k \geq 2$, a graph is k-connected if and only if G has at least two vertices and any two vertices can be joined by k independent paths.*

Similarly, G is k-edge-connected if and only if G has at least 2 vertices and any 2 vertices can be joined by k-edge-disjoint set.

Proof. First one and the second one is just rewritten Menger's theorem in a version of vertex and edges, respectively. □

5 Coloring and Ramsey Theorem

Definition 5.1. *Let G be a simple graph, and $f : V \to S$ be a* **vertex coloring** *where S is a set of colors. We call f is* **proper coloring** *if and only if adjacent vertices receive different colors. Then we say G is* **k-colorable** *if G has a proper k-coloring. Then we define* **chromatic number of** *G as*

$$\chi(G) := \min\{k : \exists \textit{ a proper } k\textit{-coloring in } G\}.$$

For example, $\chi(C_{2n+1}) = 3, \chi(C_{2n}) = 2, \chi(K_n) = n$ where C_n is a cycle with n vertices. There are several basic properties for coloring.

- **Definition 5.2** (Clique)**.** *A* **clique** *of G is $S \subseteq V(G)$ such that $G|_S = K_{|S|}$, a complete graph with $|S|$ vertices. Then let $w(G)$ be the number of maximum size of clique in G.*

  Obviously, $\chi(G) \geq w(G)$ since any proper coloring of G should give a proper coloring of a clique with size $w(G)$, which implies it contains at least $w(G)$ colors.

- For a proper coloring, each class of color is an independent set. And G is k-coloroble if and only if $V(G)$ is union of k-independent set, which is equivalent to say that G is k-partite graph. Note that definition of k-partite graph is a graph with k independent set.

- Now recall the independent number $\alpha(G) := \max$ cardinality among all independent set of $G\}$. Then,

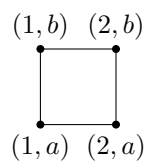
$$\sum_{t=1}^{k} \text{size of each color class } t = n(G)$$

  Since size of each color class should be less than $\alpha(G)$, we can get

$$n(G) \leq \chi(G)\alpha(G) \implies \chi(G) \geq \frac{n}{\alpha(G)}.$$

- $\chi(G_1 \cup G_2) = \max(\chi(G_1), \chi(G_2))$. This is from setting one of color sets for G_1 and G_2 are contained into the other.

- Let $G\square H$ be a cartesian product of G and H, i.e., $G\square H = (V, E)$ where $V = V(G) \times V(H)$ and $E = \{(u, v) \sim (u', v') : u = u', v \sim v' \text{ or } u \sim u', v = v'\}$. For example, let $P_1 = 1 - 2, P_2 = a - b$ a graph with one edge. Then, $P_1 \square P_2$ is



  which is Q_2, a hypercube. Actually,

$$Q_k = \underbrace{P_2 \square P_2 \square \cdots \square P_2}_{k \text{ times}}$$

  So, $P_m \square P_n$ gives a grid from $(0, 0)$ to (m, n), and $C_m \square P_n$ is a grid identifying $x = m$ and $x = 0$, so on.

  Lemma 5.3.

$$\chi(G\square H) = \max(\chi(G), \chi(H)).$$

  Proof. Note that $\chi(G\square H) \geq \max(\chi(G), \chi(H))$ since G, H are subgroup of $G\square H$. Now let $k = \max(\chi(G), \chi(H))$. Then both G and H have a k-proper coloring f, g respectively. Then color $(u, v) \in G\square H$ by

$$c(u, v) = f(u) + g(v) \bmod k.$$

  Now it suffices to show that c is a proper coloring of $G\square H$. If (u, v) and (u', v') are adjacent, then either $f(u) \neq f(u')$ or $g(v) \neq g(v')$, but not both holds. In any case $|f(u) + g(v) - f(u') - g(v')| < k$, thus $c(u, v) \neq c(u', v')$, done. $\square$

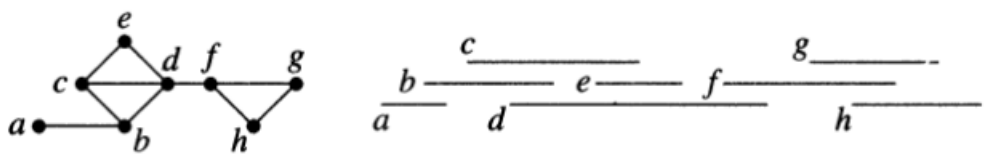
- $$\chi(G) \leq \Delta(G) + 1$$

  where $\Delta(G)$ is the maximum degree of vertices in G. To see this, list vertices in a row. Color 1 by 1 with greedy method; in the worst case, $v_1, \cdots, v_{i-1}$ are adjacent to v_i where $\deg(v_i) = \Delta(G)$. So $v_1, \cdots, v_i$ should have distinct color, which implies that at least $\Delta(G) + 1$ number is needed.

- A slight improvement is list the vertices by their degree large to small, i.e., $\deg(v_1) \geq \deg(v_2) \geq \cdots \geq \deg(v_n)$. Then,
$$\chi(G) \leq 1 + \max_i \left(\min(\deg(v_i), i-1)\right).$$
To see this, note that if we color v_i, then it has at most $\min(\deg(v_i), i-1)$ neighbors in $\{v_1, \cdots, v_{i-1}\}$, so at most this many color appeared previously. So v_i may be colored with $1 + \min(\deg(v_i), i-1)$ index, which implies the above result.

- Register allocation and Interval graph. Suppose there are a number of rooms available. Two variables can be in the same room if and only if they are never used simultaneously. To use the minimal rooms for saving those variables, let's make a graph $G = (V, E)$. V is a vertex set, which is just a set of variables. And two vertices are connected by an edge if those are called simultaneously. Then, a room is independent set in G; hence the minimum number of room needed is just $\chi(G)$.

  Define an ***interval representation of*** G be a family of intervals assigned to vertices so that vertices are adjacent if and only if the corresponding interval intersect. Below figure is the example of such graph.



  we call G is ***interval graph*** if G has such representation.

  Claim 5.4. *For an interval graph G, $\chi(G) = w(G)$.*

  Proof. Order the vertices by the position of left endpoint of each interval representing vertices. Then there exists v_i which has $\chi(G) = k$-th color. This implies that there exist vertices $v_{i_1}, \cdots, v_{i_{k-1}}$ having $1, 2, \cdots, k-1$ and they are connected to v_i. Then, by the same argument, $v_{i_{k-1}}$ is connected to $v_{i_1} \cdots, v_{i_{k-2}}$, and so on. Thus $\{v_{i_1}, \cdots, v_{i_k}\}$ forms a clique on G, so $w(G) \geq \chi(G)$. Since $\chi(G) \geq w(G)$ always, we have the desired result. □

Theorem 5.5 (Brook)**.** *Suppose G is connected and $G \neq K_m$ or C_n with $m \in \mathbb{N}, n \in 2\mathbb{N} - 1$. Then, $\chi(G) \leq \Delta(G)$.*

Proof. Let $k = \Delta(G)$. We may assume $k \geq 3$, since $k \leq 1$ implies G is a complete graph, $k = 2$ implies that G is an odd cycle or is bipartite. (Note that even case is contained in the bipartite case.)

If G is not a k-regular, then there exists a vertex of degree less than k, then say this is v_n. Then find a spanning tree J with root v_n. Then list vertices of J in a children first. Then for all $v_i \neq v_n$, its parents appears after v_i. Then, note that v_i has at most $k-1$ neighbors in $\{v_1, \cdots, v_{i-1}\}$ since k is the maximum degree and it already has a path to v_n along some vertices in the higher index. Thus, by applyging the greedy algorithm, $\chi(G) \leq k - 1 + 1 = k$.

If G is k-regular, then there are two cases;

- If there is a cut vertex x, i.e., deleting x gives distinct connectd component, then each $G_i \cup \{x\}$ is connected and maximum degree less than $\Delta(G)$, and $\deg(x) < k$ in $G_i \cup \{x\}$, so $G_i \cup \{x\}$ is k-Colorable. Then we can just let color of x is 1 for all $G_i \cup \{x\}$, so that G is also k-colorable.

- If there is no such cut vertex, then $\kappa(G) \geq 2$. Suppose the case that in a given vertex ordering, $v_1 v_2$ is not an edge in G, but $v_1, v_2 \in N(v_n)$, and $G - \{v_1, v_2\}$ is connected. Then, find a spanning tree in $G - \{v_1, v_2\}$ such that label $3, 4, \cdots, n$ increases along paths to the root v_n. In this case, by the same argument, every vertex $v_i \neq v_n$ has at most $k-1$ neighbors before i, so by the greedy algorithm we can generate a k-coloring of G, by filling $k-1$ colors for $v_3, \cdots, v_{n-1}$, and from v_n has neighbors at most $k-2$, it has $k-1$th color, then we can color v_1 and v_2 by k-th color since they are not connected by edge.

- Now think about the case where v_1, v_2, v_n gives a clique and $\kappa(G) \geq 2$. Now choose $x \in V(G)$. If $\kappa(G-x) \geq 2$, then let $v_1 = x$, v_2 be a vertex which is distance 2 from v_1, and v_n be the middle point of the distance 2 path from v_1 to v_2. Such vertex v_2 exists since G is regular, and not a complete graph. Then we can apply the above case 2 argument.

- Now we should deal with the bad case; G is k-regular, connected and $\kappa(G) = 2$, and $\forall x \in V(G)$, $\kappa(G-x) = 1$. Then, we need a concept of block.

  Definition 5.6. *A* **block** *of G is a maximal connected subgraph of G that has no cut-vertex.*

  Now from $\kappa(G-x) = 1$, $G-x$ has a cut vertex, so it has at least 2 blocks. And since G is 2-connected, x has an edge with each leaf blocks connected to not a cut vertex. Note that "leaf blocks" is a name for blocks in G where G itself is not a block. So take v_1, v_2 from each block which are connected to x by an edge. Then, v_1, v_2 are nonadjacent, and from $k \geq 3$, x has another adjacent vertex, so $G - \{v_1, v_2\}$ is still connected. Hence we can apply the above argument.

□

Basic properties of block are as below;

- If two blocks B_1, B_2 are exists, then their intersection has at most one vertex. (Otherwise, $B_1 \cup B_2$ is also a block, since it has no cut vertex, contradicting the maximality of blocks.)
- And if intersection point exists, then it is a cut vertex. (Otherwise, $B_1 \cup B_2$ is a block, contradiction.)
- We can make a graph $B_G = (Blocks, cutvertex)$.

5.1 Chromatic polynomial

Now we can investigate the enumerative aspect of $\chi(G)$. Let $\chi(G, k)$ be the number of proper colorings $f : V(G) \to [k]$. For example,

1. if G is just n isolated vertices, then $\chi(G, k) = k^n$.
2. if $G = K_n$, then $\chi(G, n) = k(k-1)\cdots(k-n+1) = k^{\underline{n}}$,a falling factorial.
3. if G is a tree on n vertices, then $k(k-1)^{n-1}$. (Use greedy algorithm)
4. if G consists of two connected component G_1, G_2 then

$$\chi(G, k) = \chi(G_1, k)\chi(G_2, k).$$

Claim 5.7. *$\chi(G, k)$ is a polynomial of k.*

Proof. Count the number of proper coloring of G using r colors of $[k]$. Then,

1. Partition $V(G)$ into r disjoint nonempty blocks of independent set. Say this number $P_r(G)$, since it depends on G only.
2. Then, give each independent set a disjoint color $r!$.

Then,

$$\chi(G, k) = \sum_{r=1}^{n} \binom{k}{r} P_r(G) r! = \sum_{r=1}^{n} k^{\underline{r}} P_r(G)$$

which is a polynomial of k. □

We say $\chi(G, k)$ is the ***chromatic polynomial***.

Theorem 5.8. *Let G be a simple graph and $e \in E(G)$. Then,*

$$\chi(G,k) = \chi(G-e,k) - \chi(G/e,k).$$

where G/e is an edge contraction of G with respect to e.

Proof. Note that above equation can be rewritten as

$$\chi(G,k) + \chi(G/e,k) = \chi(G-e,k)$$

and we can think $\chi(G/e,k)$ as the case when $init(e)$ and $term(e)$ has the same color, and $\chi(G,k)$ as the case when $init(e)$ and $term(e)$ has different color. And any proper k coloring of $G-e$ is divided for those two cases, done. □

For example,

$$\begin{aligned}\chi(C_5,k) &= \chi(\bullet - \bullet - \bullet - \bullet, k) - \chi(C_4,k)\\ &= k(k-1)^4 - (\chi(\bullet - \bullet - \bullet, k) - \chi(C_3,k))\\ &= k(k-1)^4 - \big(k(k-1)^3 - k(k-1)(k-2)\big)\\ &= k(k-1)\left((k-1)^3 - (k-1)^2 + k - 2\right).\end{aligned}$$

Corollary 5.9. *If $|V(G)| = n$, then $\deg(\chi(G,k)) = n$ and $\chi(G,k) = k^n - a_1k^{n-1} + a_2k^{n-2} + \cdots$ with $a_i \geq 0$, $a_1 = e(G)$*

Proof. We can do the induction. If $|E(G)| = 0$, then $\chi(G,k) = k^n$, so done. For induction, let $|E(G)| \geq 1$. Then, $G-e, G/e$ has $|E(G)|-1$ edges, so by inductive hypothesis, we can let

$$\chi(G-e,k) = \sum_{i=0}^{n}(-1)^i a_i k^{n-i}, \chi(G/e,k) = \sum_{i=0}^{n-1}(-1)^i b_i k^{n-1-i}.$$

Thus, by letting $b_{-1} = 0$, we have

$$\chi(G,k) = k^n + \sum_{i=0}^{n-1}(-1)^i(a_i + b_{i-1})k^{n-i}$$

Then $a_0 = 1$, $a_1 = a_1 + b_0 = |E(G)| - 1 + 1 = |E(G)|$. □

Proposition 5.10 (Whitney's formula)**.** *Let $C(G) = \#$ of connected component of G. For any $S \subseteq E(G)$, let $G(S) = (V(G), S)$. Then,*

$$\chi(G,k) = \sum_{S \subseteq E(G)} (-1)^{|S|} k^{C(G(S))}.$$

Proof. Idea is from inclusion exclusion principle. Assume edges of G are $e_1, \cdots, e_m$. Let $e_i = x_iy_i$ for some $x_i, y_i \in V(G)$. Then, let F be a set of all colorings of $V(G)$ by $[k]$. Let $A_i = \{f \in F : f(x_i) = f(y_i)\}$. Then f is not a proper coloring on G if and only if $f \notin A_1 \cup \cdots \cup A_m$. And from the inclusion exclusion principle, we have

$$|A_1 \cup \cdots \cup A_m| = \sum_{\emptyset \neq J \subseteq [n]} (-1)^{|J|-1} \left| \bigcap_{j \in J} A_j \right|.$$

Now note that

$$\left| \bigcap_{j \in J} A_j \right| = k^{c(G(S))}$$

where $S = \{e_i : i \in J\}$, since by condition of A_j, we should fill any two vertices in the same connected component as the same color, so there are only $k^{c(G(S))}$ possible way of colorings. Thus,

$$|A_1 \cup \cdots \cup A_m| = \sum_{\emptyset \neq S \subseteq E(G)} (-1)^{|S|-1} k^{c(G(S))}.$$

Then the number of proper coloring is just difference between all colorings and improper coloring. The number of all coloring is $k^n = k^{c(G(\emptyset))}$, so

$$\chi(G,k) = k^{c(G(\emptyset))} - |A_1 \cup \cdots \cup A_m| = k^{c(G(\emptyset))} - \sum_{\emptyset \neq S \subseteq E(G)} (-1)^{|S|-1} k^{c(G(S))} = \sum_{S \subseteq E(G)} (-1)^{|S|} k^{c(G(S))}$$

as desired. □

5.2 Ramsey Theory

Theorem 5.11 (Erdos-Szekeres 1935)**.** *Let $n \in \mathbb{N}$, and $\{a_i\}_{i=1}^{n^2+1}$ be a sequence of distinct numbers. Then, there is a monotone subsequence of length at least $n+1$*

Proof. For each a_i, label (x_i, y_i) where x_i is the longest increasing subsequence of $a_1, \cdots, a_i$ ending at a_i, and y_i is the longest decreasing subsequence of $a_1, \cdots, a_i$ ending at a_i. If the sequence has no monotone sublist of length $n+1$ or longer, then $x_i, y_i \leq n$,so there are only n^2 possibilities for (x_i, y_i).

Then, from the Pigeonhol Principle with the fact that length of original sequence is n^2+1, two $(x_i, y_i), (x_j, y_j)$ has the same value. However, from the distinctness, it is impossible; if $a_i < a_j$, then $x_j > x_i$ by adding a_j on the longest increasing subsequence in $\{a_1, \cdots, a_i\}$. If $a_i > a_j$, then $y_j > y_i$ by adding a_j on the longest decreasing subsequence in $\{a_1, \cdots, a_i\}$. □

Definition 5.12. *Let S be a set, and $\binom{S}{r}$ be a collection of all r element subsets of S. Then k-**coloring** of $\binom{S}{r}$ is a function from $\binom{S}{r}$ to $[k]$. A subset $T \subseteq S$ is **monochrome** or **homogeneous** under a coloring of $\binom{S}{r}$ if all r elements subset of T receives the same color; it is i-monochrome if that color is $i \in [k]$.*

Let $r, p_1, \cdots, p_k$ be positive integers. If $\exists N \in \mathbb{N}$ such that every k-coloring of $\binom{[N]}{r}$ yields an i-monochrome set of size p_i for some i, then the smallest such integer is called the **Ramsey number** *$R(p_1, \cdots, p_k; r)$, i.e.,*

$$R(p_1, \cdots, p_k; r) := \min\{N \in \mathbb{N} : \text{ every } k\text{-coloring of } \binom{[N]}{r} \text{ yields an } i\text{-monochrome set of size } p_i \text{ for some } i.\}$$

Theorem 5.13 (Ramsey, 1930)**.** *$R(p_1, \cdots, p_k; r)$ exists for given positive integers $p_1, \cdots, p_k, r$.*

Proof. If $r = 1$, then let $N = \left(\sum_{i=1}^k p_i\right) - k + 1$. Then any coloring of N is just classifying singletones into k classes, i.e., partitioning N with k classes. If every k box do not meet criteria, i.e., the number of elements in i-th class is less than p_i, then the maximum number of cardinality of each class is $p_i - 1$, so only $\left(\sum_{i=1}^k p_i\right) - k$ elements can be classified in such way. However since N is the least upper bound of it, so there exists at least one class which has p_i elements, so N is the Ramsey number as desired.

If $p_i < r$, then any set of size p_i is monochromatic, since $\binom{p_i}{r}$ is empty, so there exists only one function $f : \emptyset \to [k]$, which gives monochromatic set vacuously. So,

$$R(p_1, \cdots, p_k; r) = \min\{p_1, \cdots, p_k\}.$$

if $\min\{p_1, \cdots, p_k\} < r$.

Now we use double induction; suppose $r > 1$, and assume that statement holds for k-colorings of the $r-1$ subsets of a set, no matter what thresolds (p_is) are. Also, we assume that statement holds for any $(p'_1, \cdots, p'_k) < (p_1, \cdots, p_n)$ coordinatewise.

Then let $q_i = R(p_1, \cdots, p_{i-1}, p_i - 1, p_{i+1}, \cdots, p_k; r), N = 1 + R(q_1, \cdots, q_n, r-1)$. By the inductive hypothesis on p_is, q_is exist and by the inductive hypothesis on r, N also exists. Let S be a set of N elements, and choose $x \in S$. Consider k-coloring f of $\binom{S}{r}$. We need to show that f has i-monochromatic q_i-set for some i.

Let f' be a k-coloring of $r-1$ sets of $S - x$ induced by following way; for each $(r-1)$-set in $S - x$ assign color i if union of the set with x has color i by f. Since $|S - x| = N - 1 = R(q_1, \cdots, q_n, r-1)$, f' has j-monochromatic subset of size q_j. Say T. Then, all $(r-1)$-subsets of T has the same color j by f'. Since $|T| = q_j = R(p_1, \cdots, p_{j-1}, p_j - 1, p_{j+1}, \cdots, p_k; r)$, so in S under f, T has i-monochromatic subset of size p_i if $i \neq j$, or $p_j - 1$ if $i = j$. If $i \neq j$, then we're done since f has i-monochromatic set with size p_i. If $i = j$,

then let P be a j-monochromatic $(p_j - 1)$-subset of T. Then let $P' = P \cup \{x\}$. by definition of T, every $(r-1)$-subset, say Q of P, which is also subset of T has color j since $Q \cup \{x\}$ has color j. Thus, if r-subset Q of P' is contained in P, then it has color j by definition of P. Otherwise, it contains x, thus $Q - x$ is $(r-1)$ subset of P, which is subset of T, which is color j under f', so Q is color j under f by construction of f'. Hence, in any case, every r-subset of P' is color j. Hence P' is j-monochromatic subset with size p_j, done.

(Note that since we already show that ramsey number exists for $r = 1$ and $r \in \mathbb{N}$ with $\min\{p_i\}_{i=1}^k < r$, above induction works well.) □

If $r = 2$, then we can think coloring of subsets as coloring of graph by below construction; for example, let $G = (S, \binom{S}{2}) = K_{|S|}$. Then k-coloring of $\binom{S}{2}$ is just k-coloring of edges. Then, a monochromatic subset of size t is just a clique K_t with same color.

For example, we can show $R(3, 3; 2) \leq 6$ using graph argument. Let G be a graph with 6 vertices having only red edges, and $\bar{G}$ be a graph with 6 vetices having only blue edges. Definitely, $G \cup \bar{G} = K_6$. Then, for any vertex x, $\deg(x)_G + \deg(x)_{\bar{G}} = 5$. Thus by the Pigeonhole principle, x has degree at least 3 for one of G or $\bar{G}$. Without loss of generality, $\deg(x)_G \geq 3$. Then, there are two cases; at least one pair of neighbors are adjacent, or no pair of neighbors are adjacent. If first case occur then G has a triangle with red edges. Otherwise, $\bar{G}$ has a triangle with blue edges. In any case it has triangle, done.

Theorem 5.14 (Erdos-Szekeres, 1935)**.** *$\forall m > 0, \exists N(m)$ such that every set of at least $N(m)$ points in plane in general position contains an m-set whose convex hull is a m-gon.*

General position means there is no triple of points forming collinear position. We need two lemmas to prove it.

Lemma 5.15. *$\forall$ 5 points in plane in general position, there exists a convex quadrilateral, i.e., a polygon with four edges and four vertices.*

Proof. Think about convex hull of 5 points. If it is pentagon or quadrilateral, then done. If it is triangle, then two points are inside of triangle. Now draw a line connecting inside points. Then vertices are not in the line, by constraints of general position. Thus, apply pigeonhole principle for three vertices of triangle to get two points in the same side with respect to line of inside points. Then, convex hull of these chosen two points and inside two points gives us quadrilateral. □

Lemma 5.16. *If every 4-subset of m points in the plane forms a convex quadrilateral, then the m points form a convex m-gon.*

Proof. Otherwise, convex hull of m points is t-gon for some $t < m$. So the remaining points lie insides in t-gon. Now triangulate t-gon. Then at least one point in the inside is also inside in the triangle (if all inside points lie in the edge of some triangle, then they are not in general position.) Then, the triangle containng inside point is just convex hull of four points (vertices of triangle + inside point), which is not quadrilateral, contradiction. □

Proof of the theorem. Let $N(m) = R(m, 5; 4)$. Let S be any set of $N(m)$ points in a plane with general position. For each 4-subset T, color it red if $conv(T)$ is convex quadrilateral; otherwise, color it blue. Then by the Ramsey's theorem, this coloring should have either red monochromatic subset of size m or blue monochromatic subset of size 5. If first case occur, then by the second lemma, this m points form a convex m-gon. If the latter case occur, then any four points in the subset of size 5 should not form quadrilateral, but this contradicts the first lemma; so the latter case cannot occur. □

Now we can investigate some bounds on $R(k, l; 2)$. Let $R(k, l; 2) = R(k, l)$ for notational efficiency. If we want to show $R(k, l) \leq M$, then we need to show that $\forall$2-coloring of edges of K_m, $\exists$ a red K_k or a blue K_l. Conversely, if we want to show $R(k, l) > m$, then we need to show that for K_m there exists a coloring such that there is no red K_k nor blue K_l.

Lemma 5.17. $R(p, q) \leq R(p-1, q) + R(p, q-1)$.

Proof. Let $N = R(p-1,q) + R(p,q-1)$, think about 2-coloring on edges on K_N. Then for each vertex x, it has $R(p-1,q)+R(p,q-1)-1$ neighbors. Then, by the pigeonhole principle, x has $R(p-1,q)$ incident edges with red or $R(p,q-1)$ incident edges with blue. Without loss of generality, suppose x has $R(p-1,q)$ incident edges with red. Then take a set consisting of neighbor of x connected by red edges. Then, cardinality of the set is $R(p-1,q)$, so there exists a clique of size $p-1$ which are red or clique of size q which are blue. If the latter case occur, done. If the former case occur, then adding x on the clique with red edges form a clique of size p with all red edges, done. □

Lemma 5.18. $R(2,n) = n = R(n,2)$

Proof. Think about K_{n-1} with all blue color. Then, there is no n-clique, so there is no blue monochromatic set with size n. Also, there is no size 2 red 2-clique, i.e., red edge, hence $R(2,n) > n-1$. For K_n, if at least one of edge is red, then this coloring has s2-clique with all edges are red. Otherwise, the whole K_n is just size n clique of all blue colored edges, so any coloring gives desired monochromatic sets, done. □

From above two lemma,

$$R(p,q) \leq \binom{p+q-2}{p-1}.$$

To see this, we use an induction on $p+q$. If $\min(p,q) = 2$, then

$$R(p,2) = p = \binom{p+2-2}{p-1} = \binom{p}{1} = p.$$

Otherwise,

$$R(p,q) \leq R(p-1,q) + R(p,q-1) \leq \binom{p+q-3}{p-1} + \binom{p+q-3}{p-2} = \binom{p+q-2}{p-1}$$

where first inequality comes from inductive hypothesis, and the last inequality comes from Pascal's identity.

Thus, if we fix p, then

$$R(p,3) \leq \binom{p+1}{2} \sim \frac{k^2}{2!}, R(p,4) \leq \binom{p+2}{3} \sim \frac{k^3}{3!} \implies R(k,k) \leq \binom{2k-2}{k-1} \sim c4^k/\sqrt{k}.$$

For the lower bound, original idea is from Erdos, using probabilistic method. For given K_n, flip a coin fo each edge to color it red or blue with probability

$$Pr(e = red) = Pr(e = blue) = 1/2.$$

Let X be the number of monochromatic k-subset of $[n]$. Then we want to compute $E(X)$. If $E(X) < 1$, then it gives you a graph without any monochromatic subset. Note that for any t-subset of vertices, say A,

$$Pr(A \text{ is monochromatic}) = 2^{1-\binom{k}{2}}$$

since there are only two cases, all $\binom{k}{2}$ edges are red or blue, and the probability of each one is $2^{-\binom{k}{2}}$. Thus,

$$E(X) = \sum_{A \subseteq [n], |A|=k} 1 * Pr(A \text{ is clique}) = \binom{n}{k} 2^{1-\binom{k}{2}}.$$

So if $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ then $R(k,k) \geq n$.To find the best n, note that

$$\binom{n}{k} \leq \frac{n^k}{k!}$$

So

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leq \frac{n^k}{k!} 2^{-\frac{k^2}{2}} \cdot 2^{1+\frac{k}{2}} \leq \left(\frac{n}{k/e}\right)^k \frac{1}{\sqrt{2\pi}} \cdot k^{-1/2} 2^{-\frac{k^2}{2}} \cdot 2^{1+\frac{k}{2}}$$

Thus if we take n satisfying $\left(\frac{n}{k/e}\right)^k \frac{1}{\sqrt{2\pi}} \cdot k^{-1/2} 2^{-\frac{k^2}{2}} \cdot 2^{1+\frac{k}{2}} < 1$, then done. Take k-th root for each side, we get

$$\frac{n}{k/e} C \cdot k^{-1/2k} 2^{-\frac{k}{2}} < 1$$

for some $C \geq \sqrt{2}\frac{2}{\sqrt{2\pi}}^{1/k}$ for all $k \in \mathbb{N}$. We claim $C = 1$, since $k \in \mathbb{N}$ and the maximum value of $\sqrt{2}\frac{2}{\sqrt{2\pi}}^{1/k}$ is when $k = 1$, so $C = \frac{2}{\pi} = 0.6366..$ satisfies the condition. Also, by numerical investigation, we know that $k^{-1/2k} < 1.3$, thus $C \cdot k^{-1/2k} \leq 0.8276.. < 1$. Thus we can regard $C \cdot k^{-1/2k} = 1$ to get a desired bound. Thus

$$\frac{n}{k/e} 2^{-\frac{k}{2}} < 1$$

which implies

$$n < \frac{k}{e} 2^{k/2}$$

Hence $n \leq e^{k/2}$ gives lower bound (Erdos' conclusion on the original paper). If we do more rigorous analysis, we can get

$$n < \frac{1}{\sqrt{2}} \frac{k}{e} 2^{k/2}$$

since actually $k^{-1/2k} < 1/\sqrt{2}$ if $k > 1$. Hence

$$R(k,k) > \frac{k}{e\sqrt{2}} 2^{k/2}(1 + o(1)).$$

In general, if $\exists p \in [0,1]$ such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{k} (1-p)^{\binom{l}{2}} < 1$$

then $R(k,l) > n$, since this is the expectation of K_n having size k monochromatic red subset or size l monochromatic blue subset. For example, if $k = 4$, we want to make both summand less than $1/2$. Thus,

$$\binom{n}{k} p^{\binom{k}{2}} = \binom{n}{4} p^6 = \frac{n^4}{24} p^6 < 1/2 \implies n^4 p^6 < 12$$

So take $p = \epsilon \cdot n^{-2/3}$. for some small $\epsilon > 0$ satisfying $\epsilon^6 < 12$. This makes the above inequality hold. Then, for the other part, note that $1 - p \sim e^{-p}$ which is from well known inequality, so

$$\binom{n}{e} (1-p)^{\binom{l}{2}} \sim \frac{n^l}{l!} e^{-p\frac{l^2}{2}}$$

Thus, $\frac{n^l}{l!} e^{-p\frac{l^2}{2}} < \frac{1}{2}$ satisfies the desired condition that $\binom{n}{k}(1-p)^{\binom{l}{2}} < 1/2$. And this is equivalent to saying that

$$\frac{2}{l!} n^l < e^{p\frac{l^2}{2}}$$

and for any l satisfying below inequality

$$n^l < e^{p\frac{l^2}{2}}$$

satisfy $\frac{2}{l!} n^l < e^{p\frac{l^2}{2}}$ since $2/l! \leq 1$ for any $l \geq 2$. By taking ln on the last inequality we get

$$l \ln n < p\frac{l^2}{2} \implies 2 \ln n < pl.$$

So by letting $p = \epsilon \cdot n^{-2/3}$, we get

$$l > \epsilon^{-1} n^{2/3} \ln n.$$

Hence,

$$R(4, \epsilon^{-1} n^{2/3} \ln n) > n$$

Thus,

$$R(4, l) > C \frac{l^{3/2}}{(\ln l)^{3/2}}$$

by some constant (which is a little bit tedious.), hence

$$R(4, l) \geq C l^{\frac{3}{2}(1+o(1))}.$$

There are another method to find the lower bound, called the ***alteration method.*** Idea is to start with a bicoloring with too many red K_k or blue K_l. Then remove one vertex from each cliques to get rid of such monochromatic set. Then, if we follow the probabilistic process, then the remaining vertice is

$$n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l} (1-p)^{\binom{l}{2}}$$

and there are no monochromatic cliques, which implies that $R(k, l)$ should be greater than this number. This can be summarized as below proposition.

Proposition 5.19. *$R(k, l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l} (1-p)^{\binom{l}{2}}$ for any $p \in [0, 1]$.*

If $l = k$, then

$$R(k, k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}.$$

To make sense of this inequality, we need

$$n >> \binom{n}{k} 2^{1-\binom{k}{2}}.$$

i.e., $\binom{n}{k} 2^{1-\binom{k}{2}} = o(n)$, which means $\lim_{n\to\infty} \frac{\binom{n}{k} 2^{1-\binom{k}{2}}}{n} = 0$, thus

$$1 >> \binom{n}{k} 2^{1-\binom{k}{2}}.$$

By the same procedure we did for Erdos' analysis, we get

$$n 2^{-k/2} < k/e$$

which implies $n \sim \frac{k}{e} \geq k/2$.

Lemma 5.20 (Lovasz Local Lemma(LLL), simplified version)**.** *Let $A_1, \cdots, A_m$ be events. If*

1. *$Pr(A_i) \leq p$ for all i*
2. *A_i is mutually independent of all but at most d events,*
3. *$ep(d+1) < 1$, where e is Euler's constant.*

Then, $Pr(\cap A_i^c) > 0$, i.e., probability of no event occur is nonzero.

Proof is omitted, since we just use it to calculate Ramsey number. For $R(k, k)$, think about coloring each edge of K_n by red or blue, with probability 1/2 for both. Then, we can think of bad case when k-clique is all red or l-clique is all blue For $S \subseteq [n]$ with $|S| = k$, let K_S be a clique generated by S. Then, the probability of K_s is monochromatic is

$$Pr(K_S \text{ is monochromatic}) = 2^{1-\binom{k}{2}}.$$

Say $p = 2^{1-\binom{k}{2}}$. And note that the event of K_S is monochromatic and that of $K_{S'}$ are dependent if and only if $|S \cap S'| \geq 2$. So fix S. Then,

$$\#\{S' \subseteq [n] : |S'| = k, |S \cap S'| \geq 2\} \leq \binom{k}{2}\binom{n}{k-2}$$

since it is just choosing two points from S and the other points from $k-2$. Let $d := \binom{k}{2}\binom{n}{k-2}$. Clearly, d is overcounting but asymptotically enough. Then we need

$$ep(d+1) < 1$$

to apply the LLL. If it is true, the by the LLL, the probability of no monochromatic subset of k exists is nonzero. Actually, by investigating numerically, we can show $ep(d+1) < 1$ when $n \geq \frac{\sqrt{2}}{e}2^{k/2}$. Thus,

$$R(k,k) \geq \frac{\sqrt{2}}{e}2^{k/2}.$$

For your information, by applying LLL to $R(3,k)$ and $R(4,k)$ we get

$$R(3,k) \geq C\frac{k^2}{\log^2 k} \text{[Ajtai-Komlos-Szeremederi, 1980s}$$
$$R(4,k) \geq Ck^{5/2+o(1)}$$

and we know

$$C\frac{k^2}{\ln k} < R(3,k)$$

by [Kim,1995]

6 Spectrum of a graph

In this section, all graph is simple with n vertices. ***Adjacency matrix*** of a graph G is $n \times n$ matrix defined by

$$(A_G)_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{o.w.} \end{cases}$$

So, in the undirected simple graph, A_G is symmetric. Thus, from the linear algebra, we know that all eigenvalues of A are real.

Definition 6.1. *A* **characateristic polynomial of graph** G *is defined as that of* A_G*, i.e.,*

$$ch(G,\lambda) := \det(\lambda I - A_G).$$

Then, **spectrum of** G *is all roots of* $ch(G,\lambda)$ *with order* $\lambda_1 > \lambda_2 > \cdots$ *with multiplicity* $m_1, \cdots$ *of those roots.*

For example, K_2 has $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So $ch(K_2,\lambda) = \lambda^2 - 1$, hence

$$spec(K_2) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

where the first row denotes roots and the second row denotes multiplicities. In general,

$$A(K_n) = J - I$$

where J is a matrix whose element is 1 for all position. So think about roots of $ch(K_n,\lambda) = \det(\lambda I - A(K_n))$. If $\lambda_1, \cdots, \lambda_n$ are roots, then $\lambda_1 + c, \cdots, \lambda_n + c$ are roots of $\det(\lambda I - A(K_n) - cI)$. And we know that

$rank(J) = 1$, thus null space of J has dimension $n-1$. Thus 0 is an eigenvalue of J with multiplicity $n-1$. And $tr(J) = n$ Hence, since trace is just sum of all eigenvalues, we know that J has an eigenvalues $n, 0, \cdots, 0$, which implies that $J - I$ has an eigenvalues $n-1, -1, \cdots, -1$ by above argument. So,

$$Spec(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

From linear algebra, we know the meaning of coefficients in $ch(G, \lambda)$, namely

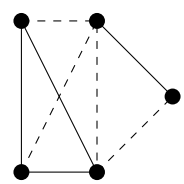
Proposition 6.2. *If $ch(G, \lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$, then*

$$(-1)^i c_i = \sum \text{ all principal minors of } A_G \text{ of size } i \times i.$$

Thus, if G is simple, $c_1 = 0$, since $c_1 = trace(A_G) = 0$ (To get the trace, note that this characteristic polynomial split with $(x - \lambda_i)$s.) $c_2 = -|E(G)|$ since only possible nonzero principal minor of 2×2 is $\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which corresponds to edges of G. $c_3 = 2\#$ of triangles in G, since only possible nonzero principal 3×3 minor of A_G is $\det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 2$.

Definition 6.3 (Elementary Graph)**.** *An* **elementary graph** *is a simple graph such that each connected component is 1-regular or 2-regular, i.e. single edge or cycle. So no isolated vertices appear in any elementary graph. A* **spanning elementary graph of** *G is an elementary graph which is subgraph of G but containing every vertices of G.*

For example,



The non-dashed graph is an elementary spanning graph. Let Λ be an elementary spanning tree of G. We define some property of Λ as follow;

- $c(\Lambda) = \#$ of connected component of Λ.
- $r(\Lambda) = n - c(\Lambda)$.
- $s(\Lambda) = m - r(\Lambda)$ where m stands for the number of edges of Λ. Thus, $s(\Lambda) = \#$ of cycles in Λ. To see this, note that for any cycle in Λ, the number of edges of cycle minus the number of vertices is just 0. And for each single edge, the number of edges of an edge minus the number of vertices is just -1. Hence, $m - n$ is minus number of single edges. So if we let $c(\Lambda) = a + b$ where a stands for number of single edges, and b stands for the number of cycles, then

$$s(\Lambda) = m - n + a + b = -a + a + b = b.$$

Proposition 6.4 (Harary)**.** *For a given simple graph G, and its adjacency matrix A,*

$$\det(A) = \sum_{\Lambda:\ spaning\ elementary\ subgraph} (-1)^{r(\Lambda)} 2^{s(\Lambda)}.$$

Proof. By definition,

$$\det(A) = \sum_{\pi \in S_n} sgn(\pi) a_{1\pi(1)} \cdots a_{n\pi(n)}.$$

Then, $a_{1\pi(1)} \cdots a_{n\pi(n)} = 1$ if and only if $(i, \pi(i)$ is an edge. (Also, if $a_{1\pi(1)} \cdots a_{n\pi(n)} \neq 1$, then $a_{1\pi(1)} \cdots a_{n\pi(n)} = 0$, since A is 0-1 matrix. Since G has no loop, $\Lambda = (V(G), \{i, \pi(i)\}_{i=1}^n)$ is an elementary spanning graph. Hence,

$$sgn(\pi) = (-1)^{\#\text{ of components having even number of vertices.}}$$

To see that # of components having even number of vertices is equal to $r(\Lambda)$, supposet π consists of C_l cycles of length l for each $l = 1, 2, \cdots, n$. Then,

$$n = \sum_{l=1}^{n} lC_l \equiv n_0 \bmod 2$$

where n_0 denotes the number of odd cycles, since even number of cycle is deleted by mod 2. So if we denote $c(\Lambda) = n_o + n_e$ where n_e denotes the number of components having even number of vertices, then

$$r(\Lambda) = n - c(\Lambda) = n - n_0 - n_e \equiv n_e \bmod 2.$$

Now fix an elementary spanning subgraph Λ. How many different π can lead to the same Λ? For each cycle in Λ, there are two choices of generating it using π, one is clockwise and the other is counter-clockwise. Note that single edge has only one choice. Thus, the number of π generating same Λ is $2^{s(\Lambda)}$. So done. □

Proposition 6.5. *If G is a simple graph, and $ch(G, \lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$, then*

$$(-1)^i c_i = \sum_{\Lambda} (-1)^{r(\Lambda)} 2^{s(\Lambda)}$$

where Λ ranges over all elementary subgraphs of G with i vertices.

Proof. Note that each principal minor is the determinant of adjacency matrix of subgraph generated by chosen vertices and edges between them. Thus, just apply the theorem on the principal minor to get above result. □

For example, if $i = 3$, then triangle is the only elementary spanning subgraph, so $r = 3 - 1 = 2, s = 1$, thus

$$(-1)^3 c_3 = \sum_{\text{triangle}} (-1)^2 2 = 2 \cdot \#\Delta$$

If $i = 4$, then two types of elementary spanning graphs are here; rectangular or two single edges. A rectangular has $r = 3, s = 1$, and two single edges give $r = 2, s = 0$, so

$$c_4 = \#\text{ of pairs of disjoint edges} - 2 \cdot \#\text{ of 4-cycles.}$$

Now we know Cayley-Hamilton, from the linear algebra, that

Theorem 6.6. *If $p(\lambda) = \det(\lambda I - A)$ then $p(A) = 0$, i.e., $I, A, A^2, \cdots, A^n$ are linearly dependent.*

And a monic polynomial of minimal degree $k(t)$ such that $k(A) = 0$ is called the ***minimal polynomial of*** A.

Proposition 6.7. *If $\lambda_1, \cdots, \lambda_n$ are distinct eigenvalue of A, then $k(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)$*

Proof is from the linear algebra. From this we know below;

Proposition 6.8. *Let G be a connected simple graph. If G has diameter d, with $|E(G)| = m$, then it has at least $d + 1$ distinct eigenvalues.*

Proof. It suffices to show that $I, A, A^1, A^2, \cdots, A^d$ are linearly independent, since otherwise, the minimal polynomial has degree d, which implies that there is only d distinct eigenvalues from the above proposition, contradiction.

Note that $(A^2)_{ij}$ is the number of length 2 walks from v_i to v_j. In general, $(A^d)_{ij}$ is the number of length d walks from v_i to v_j. Now by the definition of diameter, there exists two vertex v_i, v_j in G such that the

minimum length of walk from v_i to v_j is d. Which implies that $(A^k)_{ij} = 0$ for all $0 \leq k < d$. Thus, A^d cannot be made from linear combination of $I, A, A^2, \cdots, A^d$. Now by renumbering let $P : v_1 \to v_2 \to \cdots \to v_d$ are such path. Then $v_1 \to v_{d-1}$ is the minimum length walk from v_1 to v_{d-1}; otherwise it contradicts the minimality of P by replacing it with new minimal edge on P. This implies $(A^k)_{1,d-1} = 0$ for all $0 \leq k < d-1$, but $(A^k)_{1,d-1} = 1$ when $k = d-1$. By the same argument we can show that any set $\{I, A, \cdots, A^k\}$ is linearly independent. Thus, $I, A, A^1, A^2, \cdots, A^d$ are linearly independent, done. □

Definition 6.9. Cospectral graph *of a graph G is a graph that share the same graph spectrum.*

Proposition 6.10. *Let G be k-regular, connected graph. Then,*

1. *k is an eigenvalue of G*
2. *Multiplicity of k is 1,*
3. *For any eigenvalue λ of G, we have $|\lambda| \leq k$.*

Proof. For the first one, take $\vec{x} = (1, 1, \cdots, 1)^t$. Then,

$$A\vec{x} = k\vec{x}$$

from the regularity.

For the second one, assume $\vec{y} = (y_1, \cdots, y_n)^t \neq 0$ is an eigenvector of A. Then, without loss of generality, assume that y_j is positive and y_j has maximum absolute value among all elements in $\vec{y}$. Then,

$$A\vec{y} = k\vec{y}$$

which implies

$$y_{r_1} + \cdots + y_{r_k} = ky_j$$

where $v_{r_1}, \cdots, v_{r_k}$ are from $N(v_j)$. However,

$$|y_{r_1} + \cdots + y_{r_k}| \leq |y_{r_1}| + \cdots + |y_{r_k}| \leq ky_j.$$

and equality holds if and only if $y_{r_i} = y_j$ for all $i \in [k]$. Then, by the connectivity, all y_i's are the same value. Hence eigenspace corresponding to eigenvalue k has dimension 1 generated by $\vec{x} = (1, 1, \cdots, 1)^t$

For the third one, Let λ be an eigenvalue with $\vec{y} = (y_1, \cdots, y_n)^t \neq 0$ as an eigenvector. Also assume that y_j is the greatest value among all elements in $\vec{y}$. Then by the above argument,

$$y_{r_1} + \cdots + y_{r_k} = \lambda y_j$$

Thus,

$$|\lambda||y_j| = |y_{r_1} + \cdots + y_{r_k}| \leq \sum_{i=1}^{k} |y_{r_i}| \leq k|y_j|$$

So $|\lambda| \leq k$. □

Note that if $(\lambda, \vec{x})$ is a pair of eigenvalue and eigenvector, than so is $(-\lambda, -\vec{x})$.

Definition 6.11. *A $n \times n$ matrix S is called* **circulant** *if $S_{ij} = S_{1,j-i+1 \ mod \ n}$, i.e.,*

$$S = \begin{pmatrix} a & b & c & \cdots & z \\ z & a & b & \cdots & y \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b & c & d & \cdots & a \end{pmatrix}$$

Let W be a $n \times n$ circulant matrix starting with $(0, 1, 0, \cdots, 0)$. Then it is known that eigenvalue of W is all n-th roots of unity. Call it $w, w^2, \cdots, w^n$ where $w = e^{\frac{2\pi}{n}i}$. Then, for any circulant matrix S starting with a sequence $(s_1, \cdots, s_n)$,

$$S = \sum_{j=}^{n} s_j W^{j-1}$$

So eigenvalue of S is

$$\lambda_r = \sum_{j=1}^{n} s_j w^{r(j-1)}$$

where $r = 0, 1, \cdots, n-1$.

For example,

- The adjacent matrix of K_n, say A, is a circulant matrix with first row $(0, 1, \cdots, 1)$. Then, for $r \in [n-1] \cup \{0\}$, by the above argument,

  $$\lambda_r = \sum_{j=1}^{n-1} w^{r-j}$$

  Thus its eigenvalue is of form

  $$\lambda_r + 1 = 1 + w^r + w^{2r} + \cdots + w^{r(n-1)} = \begin{cases} n & \text{if } r = 0 \\ 0 & \text{o.w.} \end{cases}$$

  To see this, we use

  $$\lambda^n - 1 = (\lambda - 1)(1 + \lambda + \cdots + \lambda^{n-1})$$

  and the fact that $(w^r)^n = 1$. Thus, the spectrum of K_n, which is

  $$\begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$$

  where the first row is of eigenvalues and the second row is of multiplicities.

- C_n's adjacency matrix is a circulant with first row $(0, 1, 0 \ldots, 0, 1)$. Then,

  $$\lambda_r = w^r + w^{r(n-1)} = 2\cos\left(\frac{2\pi r}{n}\right).$$

  hence

  $$Spec(C_n) = \begin{cases} \begin{pmatrix} 2 & 2\cos\left(\frac{2\pi r}{n}\right) & \cdots & \left(\frac{(n-1)\pi r}{n}\right) \\ 1 & 2 & \cdots & 2 \end{pmatrix} & \text{if } n \text{ is odd} \\ \begin{pmatrix} 2 & 2\cos\left(\frac{2\pi r}{n}\right) & \cdots & \left(\frac{(n-1)\pi r}{n}\right) & -2 \\ 1 & 2 & \cdots & 2 & 1 \end{pmatrix} & \text{if } n \text{ is even.} \end{cases}$$

- The hyper octahedral graph, also known as cocktail party graph, H_s corresponds to a graph $K_{2s} - \{sdiagonaledges\}$. For example, $H_2 = K_4 - \{2 \text{ diagonal edges }\} = C_4$. It has an adjacency matrix A which is circulant matrix with first row $(0, 1, \cdots, 1, 0, 1, \cdots, 1)$ where 0 appears on the first and $(s+1)$-th position. And its spectrum is

  $$Spec(H_s) = \begin{pmatrix} 2s-2 & 0 & -2 \\ 1 & s & s-1 \end{pmatrix}$$

  To see this,

  $$\lambda_r = \sum_{j=1}^{s-1} w^{rj} + \sum_{j=s+1}^{2s-1} w^{rj} = (w^{rs} + 1) \sum_{j=1}^{s-1} w^{rj} = \begin{cases} 2s-2 & \text{if } r = 0 \\ -1 - w^{rs} & \text{if } r \neq 0 \end{cases},$$

where w is $2s$-th roots of unity, i.e., $w = e^{\frac{\pi i}{s}}$. Thus, $w^{rs} = (-1)^r$, hence

$$\lambda_r = \begin{cases} 2s-2 & \text{if } r = 0 \\ -1 + (-1)^{r+1} & \text{if } r \neq 0 \end{cases}$$

Hence when r is nonzero even, the eigenvalue is -2, and when r is nonzero odd, the eigenvalue is 0, and multiplicities are just the number of nonzero even and nonzero odd, respectively.

Definition 6.12 (Line graph of G)**.** *Let $G = (V, E)$ Then $L(G)$ is a* **line graph of** *G defined as follow; $V(L(G)) = E(G)$. And $E(L(G)) = \{e_i \sim e_j : \ e_i$ and e_j share a common vertex$\}$.*

For example, line graph of C_n is still C_n. Now let $n = |V(G)|, m = |E(G)|$.

Definition 6.13 (Incidence matrix)**.** *We call a $n \times m$ matrix X is* **incidence matrix of** *G if*

$$X_{ij} = \begin{cases} 1 & \textit{if } v_i \textit{ incident to } v_j \\ 0 & \textit{o.w.} \end{cases}$$

Then, each column sum is 2, each row sum is the degree of v_i.

Proposition 6.14. *$X^T X = A_{L(G)}$ and if G is k-regular, then $XX^T = A_G - kI_n = L_G$.*

Proof. Note that column of X^T represents an edge, so

$$(X^T X)_{ij} = \begin{cases} 0 & \text{if disjoint} \\ 1 & \text{if connceted} \\ 2 & i = j \end{cases}$$

And by the same consideration,

$$(XX^T)_{ij} = \begin{cases} 0 & \text{if there is no edge between } v_i, v_j \\ 1 & \text{if it has edge} \\ \deg(v_i) & \text{if } i = j \end{cases}$$

□

Quick result

Proposition 6.15. *For all $\lambda \in Spec(L(G))$, $\lambda \geq -2$.*

Proof. Eigenvalue of $A_{L(G)} + 2$ is the eigenvalue of $X^T X$, which is positive semidefinite, which implies all eigenvalues are ≥ 0. □

Proposition 6.16 (Sachs)**.** *G is k-regular, $k \geq 2$, then $ch(L(G), \lambda) = (\lambda + 2)^{m-n} ch(G, \lambda + 2 - k)$, where $m = |E(G)|, n = |V(G)|$.*

Corollary 6.17. *If $Spec(G)$ is $\lambda_1 \geq \cdots \geq \lambda_n$, then $Spec(L(G))$ is -2 of multiplicity of $m - n$ and $2k - 2, \lambda_2 + k - 2, \cdots, \lambda_n + k - 2$.*

Actually corollary is just holds from the roots of polynomial for such change of coefficients.

Proof of the Theorem. Note that $\det(AB) = \det(A)\det(B)$. Let $U = \begin{pmatrix} \lambda I_n & -X \\ 0 & I_m \end{pmatrix}, V = \begin{pmatrix} I_n & X \\ X^t & \lambda I_m \end{pmatrix}$. Then,

$$UV = \begin{pmatrix} \lambda I_n - XX^T & 0 \\ X^t & \lambda I_m \end{pmatrix}, VU = \begin{pmatrix} \lambda I_n & 0 \\ \lambda X^t & \lambda I_m - X^t X \end{pmatrix}.$$

Since $\det(UV) = \det(VU)$, we know

$$\lambda^m \det(\lambda I_n - XX^t) = \det(VU) = \lambda^n \det\left(\lambda I_m - X^t X\right),$$

from the formula of determinant of block matrix. Thus,

$$ch(L(G), \lambda) = \det(\lambda I_n - A_{L(G)}) = \det((\lambda+2)I_m - X^t X) = (\lambda+2)^{m-n} \det\left((\lambda+2)I_n - XX^t\right)$$

Then from k-regularity, $XX^t = kI_n + A_G$, so

$$ch(L(G), \lambda) = (\lambda+2)^{m-n} \det\left((\lambda+2-k)I_n - A_G\right) = (\lambda+2)^{m-n} ch(G; \lambda+2-k).$$

□

For example, to calculate $Spec(L(K_n))$, we know

$$Spec(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix} \implies Spec(L(K_n)) = \begin{pmatrix} 2n-4 & n-4 & -2 \\ 1 & n-1 & \binom{n}{2} - n \end{pmatrix}$$

by the corollary. So if $n = 4$,

$$Spec(L(K_4)) = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 3 & 2 \end{pmatrix}.$$

6.1 Matrix Tree Theorem

Let G be a connected graph. Allow multiple edges. We want to have a formula for the number of spanning trees of G.

- Step 1: Binet-Cauchy Theorem

  Theorem 6.18. *Let $A \in Mat_{m\times n}, B \in Mat_{n\times m}$. If $m > n$, then $det(AB) = 0$ since $rank(AB) = n < m$, which means singular. Otherwise, if $n \geq m$,*

  $$\det(AB) = \sum_S \det(A[S]) \det(B[S])$$

  where S ranges over all m-element subsets of $[n]$, and $A[S]$ is a $m \times m$ submatrix of A consisting of columns whose set of indices is S, and $B[S]$ is a $m \times m$ submatrix of A consisting of rows whose set of indices is S,

  For example, if $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}, B = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{pmatrix}$. Then

  $$\det(AB) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \begin{vmatrix} c_1 & d_1 \\ c_3 & d_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix}$$

  where each term is from $S = \{1,2\}, \{1,3\}, \{2,3\}$ respectively. Proof is omitted.

- Define signed incidence matrix. Fix for each edge an orientation. Then, ***signed incidence matrix*** is a size $n \times m$ matrix defined as following;

  $$(M)_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the head of } e_j \\ -1 & \text{if } v_i \text{ is the tail of } e_j \\ 0 & \text{if } v_i \text{ is not incident to } e_j. \end{cases}$$

  Then i-th row sum of M is $outdeg(v_i) - indeg(v_i)$, and column sum of M is just 0. Now define the ***Laplacian matrix***.$L := MM^t$. Then,

  $$(L)_{ij} = \begin{cases} -m_{ij} & \text{if there are } m_{ij} \text{ edges between } v_i, v_j \\ \deg(v_i) & \text{if } v_i = v_j. \end{cases} = D(G) - A(G)$$

  where $D(G)$ is a degree matrix, which is a diagonal matrix whose diagonal term is just degree. If G is k-regular, then $L = kI_n - A_G$.

- Let S be a set of $n-1$ edges of G. Then, let M_0 be $(n-1)\times m$ submatrix of m by removing the last row. Then, $M_0[S]$ is $(n-1)\times(n-1)$ submatrix of A consisting of columns whose set of indices is S. We claim
$$\det(M_0[S]) = \begin{cases} 0 & \text{if } S \text{ doesn't form a spanning tree} \\ \pm 1 & S \text{ form a spanning tree} \end{cases}.$$
To see this. note that S forms a spanning tree or S contains cycle. Without loss of generality, if $e_1, \cdot, e_k$ in S forms a cycle, then their (signed) column on M sum is 0, by forming a directed cycle (what I care about by this statement is a case of edges such as $1 \to 2 \to 3$ and $1 \to 3$; in this case we should sum first two columns representing $1 \to 2 \to 3$ and subtract third column representing $1 \to 3$ to get notrivial linear combination which gives 0. Thus, columns are not linearly dependent, hence determinant is 0. Since column sum and row sum is always 0, deleting last row doesn't effect it. So done.

  If S has no cycle, then their column is linearly independent; to see this, if not, there exists a nontrivial combination of columns giving 0. Since each column consists of $1, 0, -1$, we can just make a path from column to column by removing each 1's. For example, let's think about the case
$$2(1,0,-1,0)^t + (-1)\cdot(0,1,-1,0)^t + 1(-1,1,0,0)^t + 1(-1,0,0,1)^t + 1(0,0,-1,1)^t = 0$$
We can rewrite this sum, so that 1) each term is just consists of vectors in M_0 with coefficient ± 1. 2) And each partial sum from first term to i-th term is always a vector consists of $1, 0-1$. Proving it is not difficult' 1) is just from the property that each elements sum is 0. 2) is from the fact that rewritten always give cancellation of some element and adding 1 on another position. For example, above case can be rewritten as
$$(1,0,-1,0)^t - (0,1,-1,0)^t + (-1,1,0,0)^t + (1,0,-1,0)^t - (0,0,-1,1)^t + (-1,0,0,1)^t = 0$$
From this we can get cycle, by walking edges in reverse order; for example, $(1,0,-1,0)$ means an edge v_1v_3, so connecting each edges, we get
$$v_1v_3 \to v_3v_2 \to v_2v_1 \to v_1v_3 \to v_3v_4 \to v_4v_1$$
thus it has cycle, done.

From above facts, we can get below theorem.

Theorem 6.19 (Matrix Tree Theorem)**.** *Let L_0 be a matrix obtained from $L(G)$ by removing the last row and the last column. Then,* $\det(L_0) = t(G)$*, the number of spanning tree.*

Proof. Note that $L = MM^t$. And each element in the last row of L denotes the product of last row of M and each corresponding column of M^t. Similarly, each element in the last column of L denotes the product of each corresponding row of M and the last column of M^t.

From the Cauchy-Binet,
$$\det(L_0) = \sum_S \det(M_0[S])\det(M_0[S]^t)$$
where S ranges over all $(n-1)$ subsets of $E(G)$. By the above result, $\det(M_0[S]) = \pm 1$ if and only if S forms a spanning tree. Also, $\det(M_0[S]^t) = \det(M_0[S])$ Hence,
$$\sum_S \det(M_0[S])\det(M_0[S]^t) = \sum_{S\text{: spanning tree of } G} 1 = t(G).$$
□

Remark that this is true for L_0 obtained from L by removing 1 row and the one column, i.e., not necessarily choose the last row and last column. (This is because we can do the same argument of determinant of M_j, where M_j means deleting j-th row on M.) Which menas the (i,j)-cofactor of L is always $t(G)$. Thus, $\det(L) = 0$, and
$$L \cdot adj(L) = \det(L)\cdot I$$
How to express $\det(L_0)$ in terms of L directly? The answer is

Claim 6.20.

$$(-1)^{n-1} n \det(L_0) = \textit{ coefficient of } \lambda \textit{ in } \det(\lambda I - L).$$

Proof. By linear algebra, we know

$$\det(\lambda I - L) = \det \begin{vmatrix} \lambda_{M_{1,1}} & -M_{ij} \\ -M_{ij} & \lambda - M_{n,n} \end{vmatrix}.$$

Add all row to the last one we get $(1, \cdots, 1)$ on the last row, and its submatrix gives $\det(L_0)$, which gives the claim. □

Now note that spectrum of L gives $\mu_1 \geq \cdots \geq \mu_n$ where $\mu_n = 0$, since L is symmetric real. Thus,

$$\det(\lambda I - L) = (\lambda - \mu_1) \cdots (\lambda - 0).$$

So coefficient of λ is

$$(-1)^{n-1} \mu_1 \cdots \mu_{n-1}$$

Corollary 6.21. $t(G) = \frac{1}{n} \mu_1 \cdots \mu_{n-1}$.

Proof. Apply above claim. □

If G is k-regular, then

$$L = kI - A$$

so

$$t(G) = \frac{1}{n}(k - \lambda_1) \cdots (k - \lambda_{n-1})$$

where $\lambda_1, \cdots, \lambda_{n-1}, k$ are spectrum of G.

- K_n is $(n-1)$regular, so
$$t(K_n) = \frac{1}{n}(n - 1 - (-1))^{n-1} = n^{n-2}.$$
- Q_n has $Spec(Q_n) = n - 2i$ with multiplicity $\binom{n}{i}$ from $0 \leq i \leq n$. Hence,
$$t(Q_n) = \frac{1}{n} \prod (2i)^{\binom{n}{i}}$$

Now we can generalize the matrix tree theorem.

Definition 6.22. *A digraph D is* **Eulerian** *iff forall $v \in V(D)$,*

$$indeg(v) = outdeg(v)$$

called **balanced**, *and D is connected.*

Given D is connected and balanced, how many Eulerian trails? To see this, fix and arc e, and look at the number of Eulerian trail starting at e. To see this, think about oriented rooted tree in D rooted at v. Let $\tau(D, V)$ be the number of oriented rooted tree in D with root v. Then, the number of eulerian trail containing e is equal to

$$\tau(D, v) \prod_{u \in D} (outdeg(u) - 1)$$

Theorem 6.23 (BEST theorem)**.** *Let laplacian of D has element in (i, j)position as $-m_{ij}$ if $i \neq j$ or outdegree of v_i if $i = j$. Then, let L_0 be matrix removing the last column and last row. Then,*

$$\tau(D, V_n) = \det(L_0).$$

BEST stands for de Brujin, van Aardensee-Ehrenfest, Smith, and Tutte.

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