Real Variable I - Lecture Note

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Abstract

This note is based on the lecture of Real Variable I given by professor Ken Dykema on Spring 2017 at Texas A&M University. Much part of this note was TEX-ed after class. Every blemish on this note is Byeongsu's own.

Contents

1 Selected review

1.1 Notation and ordering

$$\mathbb{N} := \{1, 2, \cdots\}, \mathbb{Z} := \{\cdots, -1, 0, 1, \cdots\}, \mathbb{Q} = \left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\},$$
$$\mathbb{R} := \text{ set of real number}, \mathbb{C} = \mathbb{R} + i\mathbb{R}$$

Definition 1.1 (Relation). *A* **relation** *on a set X is a subset $R \subset X \times X$. $(x, y) \in R$ can be written as xRy.*

Definition 1.2 (Partial order, total order). *A* **partial order** *on X is a relation R on X such that*

(i) xRx,

(ii) xRy and $yRx \implies x = y$

(iii) xRy and $yRz \implies xRz$

A partial order R is **linear or total ordering** *if $\forall x, y \in X$, either xRy or yRx.*

Example 1.3. *(i) $\leq$ on $\mathbb{R}$ is total*

(ii) $\subseteq$ on $\mathcal{P}(X) = \{S : S \subseteq X\}$. This is partial order but not total order if $|X| \geq 2$.

(iii) Lexical ordering of words is a total ordering.

Definition 1.4 (Poset). *A* **partially ordered set** *or* **poset** *is $(X, \leq)$ where X is a set and $\leq$ is a partial order on X.*

Definition 1.5 (Maximal element). *If X is a poset, there is a maximal element $z \in X$, such that $\forall y \in X$ and $z \leq y$, then $y = z$.*

Now we have four basic propositions related to each other.

Lemma 1.6 (Zorn's Lemma). *Let $(X, \leq)$ be a poset and suppose that every totally ordered subset L of X has an upper bound. (i.e., $\forall L \subseteq X$, endowed with the partial ordering inherited from X is totally ordered; namely, $\forall u \in X$ such that $\forall t \in L$, $t \subseteq u$.)*

Then, X has a maximal element.

Axiom 1.7 (Axiom of choice). *Let X be a set and let $\{X_\alpha\}_{\alpha \in \Lambda}$ be a collection of nonempty subset of X. Then, $\exists f : \Lambda \to X$ such that $\forall \alpha \in \Lambda, f(\alpha) \in X_\alpha$.*

Definition 1.8 (Well-ordering). *A* **well-ordering** *on a set X is a total ordering such that every subset of X has a smallest element.*

Example 1.9. *$\mathbb{N}$ with usual ordering is well-ordering. However, $\mathbb{Z}$ with usual ordering is not an well-ordering. Nor is the usual ordering on $\mathbb{R}$.*

Principle 1.10 (The well-ordering principle). *Every set can be endowed with a well-ordering.*

Principle 1.11 (The Hausdorff Maximal Principle). *Every poset has a maximal totally ordered subset E, i.e., if $\exists E' \subseteq X$ such that $E \subseteq E' \subseteq X$, then E' is linearly ordered.*

Theorem 1.12. *Zorn's lemma, Axiom of choice, the well-ordering principle, and the Hausdorff maximal principle are equivalent and independent with ZF axioms of set theory.*

Proof.

1. $(Haus) \implies (Zorn)$ Take a maximal linearly ordered subset E of X given by the Hausdorff maximal principle. Then it has an upper bound z by the condition of Zorn's lemma. Then, $z \in E$, otherwise E is not a maximal linearly ordered subset. Then, z is a maximal element, since if $\exists z' \in X$ such that $z \leq z'$, then $e \leq z', \forall e \in E$, thus $z' \in E$, contradicting the maximality of E.

2. $(Zorn) \implies (Haus)$ Let $(\mathcal{P}(X), \subseteq)$ be a collection of linearly ordered subsets of X and inclusion. This is a poset. Also, every totally ordered subset L of $\mathcal{P}(X)$ has an upper bound $\cup_{l \in L} l$. Thus, by the Zorn's lemma, it has a maximal element E. Note that E is linearly ordered by construction. Also, it is maximal linearly ordered, since if there exists E' such that $E \subset E' \subseteq X$ and E' is linearly ordered, then $E' \in \mathcal{P}(X)$, thus E is not a maximal element, contradiction.

3. $(Zorn) \implies (Well)$ Let $W = \{(\leq_i .E_i) : \leq_i \text{ is a partial ordering on } E_i\}$. Then, give a partial ordering on W as inclusion; i.e., $(\leq_i .E_i) \leq (\leq_j .E_j)$ if $E_i \subset E_j$, $\leq_i$ and $\leq_j$ agrees on E_i, and $\forall x \in E_j \setminus E_i$ and $\forall y \in E_j$, $y \leq_j x$. Then, if $W' \subseteq W$ be a totally ordered subset of W, then it has an upper bound, by $(\leq_\infty, \cup_{E \in W'} E)$. Hence, by Zorn's lemma, it has a maximal element $(\leq, E)$. And $E = X$, otherwise, $\exists x \in X \setminus E$, thus, we can have well ordering on $E \cup \{x\}$ by extending $\leq$ as $y \leq x$ for all $y \in E$, contradicting to maximality.

4. $(Well) \implies (AC)$ Let $\{X_\alpha\}_{\alpha \in A}$ is a nonempty collection of nonempty sets, given the condition of AC. Then let $X = \cup_{\alpha \in A} X_\alpha$. By the well ordering principle, there exists an well ordering on X. Thus, subset X_α has a minimal element by the well ordering principle. Let $f : A \to \cup_{\alpha \in A} X_\alpha$ by $f(\alpha) =$ the minimal element of well ordering on X_α. Hence, $f(\alpha) \in X_\alpha$ by definition.

5. $(AC) \implies (Haus)$ It uses transfinite induction, but I don't understand.

□

1.2 Cardinality

Cardinal is the "size" of a set.

Definition 1.13. *Let X, Y be sets. Then, $card(X) \leq card(Y)$ means*

$$\exists f : X \xhookrightarrow{one-to-one} Y.$$

$card(X) = card(Y)$ means

$$\exists f : X \xrightarrow{bijection} Y.$$

$card(X) \geq card(Y)$ means

$$\exists f : X \xrightarrow{surjection} Y.$$

Theorem 1.14 (Schroder-Bernstein Theorem, Theorem 0.8 in [1] p. 7)**.**

$$card(X) \geq card(Y) \text{ and } card(X) \leq card(Y) \implies card(X) = card(Y).$$

Also,

$$card(X) \leq card(Y) \iff card(Y) \leq card(X).$$

Proof. Suppose $card(X) \leq card(Y)$. Then, $\exists f : X \to Y$ which is injective. Let $x_0 \in X$ and define $g : Y \to X$ by

$$g(y) = \begin{cases} f^{-1}(y) \text{ if } y \in f(X), \\ x_0 \text{ otherwise.} \end{cases}$$

Then g is surjective.

Conversely, if $card(Y) \geq card(X)$, there exists $g : Y \to X$ which is surjective. Thus, $g^{-1}(\{x\})$ is a nonempty by surjectivity, and disjoint with other pull back of singleton because g is a function. Thus, $\exists f \in \prod_{x \in X} g^{-1}(\{x\})$, and it is an injection from X to Y. □

Proposition 1.15 (Proposition 0.7 in [1] p.7)**.** *For any X, Y sets, either $card(X) \leq card(Y)$ or $card(Y) \leq card(X)$.*

Proof. Let J be a set of all injections from subsets of X to Y. Then, $\forall f \in J, f \subseteq X \times Y$. Thus J have a poset by inclusion. Thus, every totally ordered subset of J has an upper bound derived by unioning all members of the subset. Hence, by Zorn's lemma, it has a maximal element $f : A \subseteq X \to B \subseteq Y$. If both $x \in X \setminus A$ and $y \in Y \setminus B$ exists, we can extend $f : A \cup \{x\} \to B \cup \{y\}$ as $f(x) = y$, contradicting maximality. Thus, either $A = X$ or $B = Y$, this implies either $card(X) \leq card(Y)$ or $card(Y) \leq card(X)$. □

Definition 1.16 (Countable Set)**.** *A set X is called* **countable** *if it is either finite or $card(X) = card(\mathbb{N})$. Otherwise, it is called* **uncountable.**

Proposition 1.17. *If X_α is countable for any $\alpha \in A$ and A is also countable, then $\cup_{\alpha \in A} X_\alpha$ is countable.*

Proof. Since X_α is countable, $\exists f_\alpha : \mathbb{N} \to X_\alpha$ which is surjective. Thus, let $f : \mathbb{N} \times A \to \cup_{\alpha \in A} X_\alpha$ by $f(n, \alpha) = f_\alpha(n)$ is also surjective. And since we also have a surjective map $\mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times A$, done. □

Theorem 1.18. *$\mathbb{R}$ is uncountable, and $card(\mathbb{R}) = card(\mathcal{P}(\mathbb{N}))$*

We need a lemma for showing $\mathbb{R}$ is uncountable.

Lemma 1.19. *For any set X, $card(X) < card(\mathcal{P}(X))$*

proof of the lemma. $f : X \to \mathcal{P}(X)$ by $x \mapsto \{x\}$ is an injection. Thus, $card(X) \leq card(\mathcal{P}(X))$. Also, if $g : X \to \mathcal{P}(X)$, then let $Y = \{x \in X : x \notin g(x)\}$. Then, $Y \notin g(X)$, otherwise $\exists x' \in X$ such that $g(x') = Y$, thus absurdity comes; if $x' \in Y$, then $x' \notin Y$ by definition of Y, otherwise if $x' \notin Y = g(x')$, then $x \in Y$ by defintion of Y. Thus, g is not a surjective. □

proof of the theorem. Let

$$f : \mathcal{P}(\mathbb{N}) \to \mathbb{R} \text{ by } f(A) = \begin{cases} \sum_{n \in A} 2^{-n} \text{ if } |A| = \infty \\ 1 + \sum_{n \in A} 2^{-n} \text{ otherwise.} \end{cases}$$

Thus, f is injective. Conversely, let

$$g : \mathcal{P}(\mathbb{Z}) \to \mathbb{R} \text{ by } g(A) = \begin{cases} \log(\sum_{n \in A} 2^{-n}) \text{ if } A \text{ is bounded,} \\ 0 \text{ otherwise.} \end{cases}$$

Then g is surjective since every positive real number has a base-2 decimal expansion.

$card(\mathbb{R}) = card(\mathcal{P}(\mathbb{N})) > card(\mathbb{N})$, so $\mathbb{R}$ is uncountable. □

Definition 1.20 (Equivalence Relation)**.** *A relation $\sim$ on X is an equivalence relation if*

(i) $\forall x \in X, x \sim x$

(ii) $\forall x, y \in X, x \sim y \implies y \sim x$

(iii) $\forall x, y, z \in X, x \sim y, y \sim z \implies x \sim z$

If $\sim$ is an equivalence relation on X, we write $[x] = \{y \in X : x \sim y\}$ for the equivalence relation of $x \in X$, and $\{[x] : x \in X\}$ forms a partition of X, i.e., disjoint subsets of X.

Proof. If $x \neq\sim y$ but $[x] \cap [y] \neq \emptyset$, $\exists z \in [x] \cap [y], x \sim z \sim y \implies x \sim y$, contradiction. □

2 Measure theory

2.1 Introduction and σ-algebra

Naturally, $\mu([a,b]) = b-a$, which is weight of $[a,b]$. However, this usual concept has defect when we integrate by Riemann sum; not all function is integrable.

Definition 2.1. *A* **measure** *on a set X assigns values $\mu(E) \in [0,+\infty]$ to subset $E \subseteq X$.*

Remark 2.2. *Desired properties for a volume measure on $\mathbb{R}^n$*

(1) If $E_1, E_2, \cdots$ is a sequence of (pairwise) disjoint subsets of X, then

$$\mu(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j).$$

(2) If E and $F \subseteq \mathbb{R}^n$ are congruent by translation, then $\mu(E) = \mu(F)$.

(3) $\mu([0,1]^n) = 1$

However,

Theorem 2.3. *$\nexists$ such a volume measure defined on the set $\mathcal{P}(\mathbb{R}^n)$ of all subsets of $\mathbb{R}^n$.*

Proof. It suffices to prove the case when $n = 1$. Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $x \sim y \in \mathbb{Q}$. It is well defined since $x - x = 0 \implies x \sim x, x - y \in \mathbb{Q} \implies y - x \in \mathbb{Q}$ and $x - y, y - z \in \mathbb{Q} \implies x - z = (x-y) + (y-z) \in \mathbb{Q}$.

Let $D \subseteq [0,1)$ be a subset containing exactly one element from each equivalence class of $\sim$. Clearly, it depends upon the Axiom of Choice.

Given $r \in \mathbb{Q} \cap [0,1)$, let

$$D_r = ((D+r) \cap [0,1)) \cup ((D+r-1) \cap [0,1)) = \{x + r : x \in D \cap [0, 1-r)\} \cup \{x + r - 1 : x \in D \cap [1-r, 1)\}.$$

If $y \in (D+r) \cap (D+r-1)$, then $y = x + r = z + r - 1$ for some $x, z \in D$, hence, $y - r = x \in D, y - r + 1 = z \in D$. However, since $x = z - 1, x \sim z \iff y - r \sim y - r + 1$.

Claim 2.4. *If $r, s \in \mathbb{Q} \cap [0,1]$, and $r \neq s$, then $D_r \cap D_s = \emptyset$.*

Proof. If $y \in D_r \cap D_s$, then either $y - r, y - s \in D$ or $y - r + 1, y - s \in D$. For the first case, $y - r - (y - s) = s - r \in \mathbb{Q}$, thus $y - r \sim y - s$. Also note that $y - r \neq y - s$ since $r \neq s$. However, D has only one element from each equivalence set, contradiction. For the second case, $y - r + 1 - (y - s) = s - r + 1 \in \mathbb{Q}$, thus $y - r + 1 \sim y - s$. The only possible situation is $y - r + 1 = y - s$, then $r - s = 1$, contradiction since $r, s \in [0,1)$. $\square$

Claim 2.5. $\sum_{r \in \mathbb{Q} \cap [0,1)} D_r = [0,1)$.

Proof. Let $a \in [0,1)$ then a belongs to some equivalence class of $\sim$. So, $\exists q \in \mathbb{Q}$ such that $a - q \in D$. Thus $0 \leq a - q < 1$, since $D \subset [0,1)$. Since $0 \leq a < 1$,

$$-1 < -a \leq -q < 1 - a < 1 \implies -1 < q < 1.$$

If $0 \leq q < 1$, then $a \in D_q$, otherwise, $a \in D_{q+1}$. In any case, $a \in \sum_{r \in \mathbb{Q} \cap [0,1)} D_r = [0,1)$. Since other direction of inclusion is obvious, done. $\square$

Now we can go back to the main argument. Suppose for contradiction, a countable additive volume measure μ on $\mathcal{P}(\mathbb{R})$ exists. Then,

$$\mu(D_r) = \mu(r + D \cap [0, 1-r)) + \mu((r-1) + D \cap [1-r, 1)) = \mu(D \cap [0, 1-r)) + \mu(D \cap [1-r, 1)) = \mu(D).$$

So,

$$1 = \mu([0,1)) = \mu(\cup_{r \in \mathbb{Q} \cap [0,1)} D_r = \sum_{r \in \mathbb{Q} \cap [0,1)} \mu(D_r) = \sum_{\mathbb{N}} \mu(D),$$

since $\mathbb{Q} \cap [0,1)$ is countable. Hence, if $\mu(D) = 0$, $1 = 0$, contradiction. If $\mu(D) \neq 0$, $1 = \infty$, contradiction. □

Theorem 2.6 (Banach Tarski Paradox)**.** *Let $n \geq 3$ and $U, V \subseteq \mathbb{R}^n$ be bounded set. Then, $\exists k \in \mathbb{N}, E_1, \cdots, E_k, F_1, \cdots, F_k$, and $\{E_j\}$ and $\{F_j\}$ are pairwise disjoint, such that $U = \sum_{j=1}^k E_j, V = \sum_{j=1}^k F_j$ but E_j and F_j are congruent to each other for all j. A and B are congruent to each other means that we can get one from the other by a composition of translation, rotation, and reflection.*

Proof. See https://people.math.umass.edu/∼weston/oldpapers/banach.pdf □

Corollary 2.7. *For $n \geq 3$, $\nexists \mu : \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$ satisfying*

(1) μ is finitely additive, i.e., if $k \in \mathbb{N}, E_1, \cdots E_k \subseteq \mathbb{R}^n$ are disjoint then $\mu(\cup_{j=1}^k E_j) = \sum_{j=1}^k \mu(E_j)$.

(2) E, F are congruent, then $\mu(E) = \mu(F)$.

(3) $\mu([0,1)^n) = 1$.

Definition 2.8. *Let X be sets, $\mathfrak{a} \subseteq \mathcal{P}(X)$, and $\mathfrak{a} \neq \emptyset$. We say $\mathfrak{a}$ is an* **algebra** *of subsets of X if*

(i) $n \in \mathbb{N}, E_1, \cdots, E_n \in \mathfrak{a} \implies \cup_{j=1}^n E_j \in \mathfrak{a}$.

(ii) $E \in \mathfrak{a} \implies E^c = X \setminus E \in \mathfrak{a}$.

We say $\mathfrak{a}$ is a **σ-algebra** *if also belows are hold;*

(i') $E_1, E_2, \cdots \in \mathfrak{a} \implies \cup_{j=1}^\infty E_j \in \mathfrak{a}$

Observation 2.9. *If $\mathfrak{a}$ is an algebra of subsets of X, then*

(1) $X = E \cup E^c \in \mathfrak{a}$ and $\emptyset = X^c \in \mathfrak{a}$

(2) If $E_1, E_2, \cdots E_n \in \mathfrak{a}$, then $\cap_{j=1}^n E_j = \left(\cup_{j=1}^n E_j^c\right)^c \in \mathfrak{a}$ (De Morgan)

(3) If $\mathfrak{a}$ is a σ-algebra, then $\cap_{j=1}^\infty E_j = \left(\cup_{j=1}^\infty E_j^c\right)^c \in \mathfrak{a}$ (De Morgan)

Example 2.10. *(1) $\mathfrak{a} = \mathcal{P}(X)$.*

(2) $\mathfrak{a} = \{\emptyset, X\}$.

(3) $\mathfrak{a} = \{E \subseteq X :$ either E or E^c is countable $\}$

(4) $\xi \subseteq \mathcal{P}(X)$ then we denote **the σ-algebra generated by** *ξ as*

$$\sigma\text{-}alg(\xi) := \bigcap_{\substack{\mathfrak{a} \subset \mathcal{P}(X) \\ \mathfrak{a} \text{ is a } \sigma\text{-algebra} \\ \xi \subseteq \mathfrak{a}}} \mathfrak{a},$$

e.g., if $\xi = \{B\}$ for some $B \subseteq X$ then σ-alg$(\xi) = \{\emptyset, X, B, B^c\}$.

Lemma 2.11. *If $\xi \subseteq \mathcal{F} \leq \mathcal{P}(X)$, then σ-alg$(\xi) \leq \sigma$-alg$(\mathcal{F})$.*

Proof. σ-alg$(\mathcal{F})$ contains ξ, thus it contains σ-alg(ξ) □

(5) Let X be a topological space, e.g., X is a metric space such as $\mathbb{R}^n$. Let $\mathcal{T}_X$ be the set of all open subsets of X. Then, **Borel σ-algebra** *of X is $\mathcal{B}_X := \sigma\text{-alg}(\mathcal{T}_X)$ And its member is called* **Borel sets**.

Definition 2.12 (G_δ-set and F_σ-set)**.** *Denote* **G_δ-set** *be a countable intersection of open sets. Also, denote* **F_σ-set** *be a countable union of closed sets. Similarly, let* **$G_{\delta\sigma}$-set** *be a countable union of G_δ-set, and* **$F_{\sigma\delta}$-set** *be countable intersection of F_σ-set.*

Proposition 2.13 (Proposition 1.2 in [1])**.** *$\mathcal{B}_{\mathbb{R}}$ is generated by any of the following:*

(a) the open intervals; $\xi_1 = \{(a,b) : a,b \in \mathbb{R}, a < b\}$

(b) the closed intervals; $\xi_2 = \{[a,b] : a,b \in \mathbb{R}, a < b\}$

(c) the half-open intervals; $\xi_3 = \{(a,b] : a,b \in \mathbb{R}, a < b\}$ *or* $\xi_4 = \{[a,b) : a,b \in \mathbb{R}, a < b\}$

(d) the open rays; $\xi_5 = \{(a,\infty) : a \in \mathbb{R}\}$ *or* $\xi_6 = \{(-\infty,a) : a \in \mathbb{R}\}$

(e) the closed rays; $\xi_7 = \{[a,\infty) : a \in \mathbb{R}\}$ *or* $\xi_8 = \{(-\infty,a] : a \in \mathbb{R}\}$

Proof. First of all, $\xi_2 \subseteq \mathcal{B}_{\mathbb{R}}$ since the complement of closed set is open, and $\xi_5,\xi_6,\xi_7,\xi_8 \subseteq \mathcal{B}_{\mathbb{R}}$ since they are either open or closed sets. Also, ξ_3,ξ_4 are G_δ-set, which is in $\mathcal{B}_{\mathbb{R}}$ since it is closed under countable intersections. Thus, σ-alg$(\xi_i) \subseteq \mathcal{B}_{\mathbb{R}}$ for all $i = 1,2,\cdots,8$

For (a), since $\xi_1 \subset \mathcal{T}_{\mathbb{R}}$, σ-alg$(\xi_1) \subseteq \mathcal{B}_{\mathbb{R}}$. To show the other direction of inclusion, it suffices to show that $\mathcal{T}_{\mathbb{R}} \subseteq \sigma$-alg$(\xi_1)$. Note that

$$(-\infty,b) = \bigcup_{m=1}^{\infty}(b-n,b) \in \sigma\text{-alg}(\xi_1), (a,\infty) = \bigcup_{n=1}^{\infty}(a,a+n) \in \sigma\text{-alg}(\xi_1).$$

Note that every open subsets of $\mathbb{R}$ is a countable union of disjoint open intervals, so $\mathcal{T}_{\mathbb{R}} \subseteq \sigma$-alg$(\xi_1) \implies \mathcal{B}_{\mathbb{R}} \subseteq \sigma$-alg$(\xi_1)$.

For (b), note that $(a,b) = \bigcup_{n=1}^{\infty}[a+n^{-1}, b-n^{-1}] \in \sigma$-alg$(\xi_2)$, thus $\xi_1 \subseteq \sigma$-alg(ξ_2). Therefore, $\mathcal{B}_{\mathbb{R}} = \sigma$-alg$(\xi_1) \subseteq \sigma$-alg$(\xi_2) \subseteq \mathcal{B}_{\mathbb{R}}$, done.

For (c), note that $(a,b) = \bigcup_{n=1}^{\infty}(a, b-n^{-1}] \in \sigma$-alg$(\xi_3)$ or $(a,b) = \bigcup_{n=1}^{\infty}[a+n^{-1}, b) \in \sigma$-alg$(\xi_4)$, thus $\xi_1 \subseteq \sigma$-alg$(\xi_3) \cap \sigma$-alg(ξ_4). Therefore, $\mathcal{B}_{\mathbb{R}} = \sigma$-alg$(\xi_1) \subseteq \sigma$-alg$(\xi_3) \subseteq \mathcal{B}_{\mathbb{R}}$ and $\mathcal{B}_{\mathbb{R}} = \sigma$-alg$(\xi_1) \subseteq \sigma$-alg$(\xi_4) \subseteq \mathcal{B}_{\mathbb{R}}$, done.

For (d), note that $(a,b] = (a,\infty) \cap (b,\infty)^c \in \sigma$-alg$(\xi_5)$, thus $\xi_3 \subseteq \sigma$-alg(ξ_5), therefore $\mathcal{B}_{\mathbb{R}} = \sigma$-alg$(\xi_3) \subseteq \sigma$-alg$(\xi_5) \subseteq \mathcal{B}_{\mathbb{R}}$. Also, $[a,b) = (-\infty,a)^c \cap (b,\infty) \in \sigma$-alg$(\xi_6)$, thus $\xi_4 \subseteq \sigma$-alg(ξ_6), therefore $\mathcal{B}_{\mathbb{R}} = \sigma$-alg$(\xi_4) \subseteq \sigma$-alg$(\xi_6) \subseteq \mathcal{B}_{\mathbb{R}}$. done.

For (e), note that $[a,b) = [a,\infty) \cap [b,\infty)^c \in \sigma$-alg$(\xi_7)$, thus $\xi_6 \subseteq \sigma$-alg(ξ_7), therefore $\mathcal{B}_{\mathbb{R}} = \sigma$-alg$(\xi_6) \subseteq \sigma$-alg$(\xi_7) \subseteq \mathcal{B}_{\mathbb{R}}$. Also, $(a,b] = (-\infty,a]^c \cap (-\infty,b] \in \sigma$-alg$(\xi_8)$, thus $\xi_3 \subseteq \sigma$-alg(ξ_8), therefore $\mathcal{B}_{\mathbb{R}} = \sigma$-alg$(\xi_3) \subseteq \sigma$-alg$(\xi_8) \subseteq \mathcal{B}_{\mathbb{R}}$. done. □

Let A be a set and $\forall\alpha \in A$, let X_α be a nonempty set, and let m_α be a σ-algebra of subsets of X_α. Let

$$X = \prod_{\alpha\in A} X_\alpha := \{(x_\alpha)_{\alpha\in A} : x_\alpha \in X_\alpha\} = \{f : A \to \bigcup_{\alpha\in A} X_\alpha \text{ s.t. } \forall\alpha\in A, f(\alpha) \in X_\alpha\}.$$

(Note that it uses the Axiom of Choice when A is infinite.) For example, if $A = \{1,2,\cdots,n\}$,

$$\prod_{\alpha\in A} X_\alpha = \prod_{j=1}^{n} X_j = X_1 \times X_2 \times \cdots \times X_n.$$

For $\beta \in A$, we let $\pi_\beta : \prod_{\alpha\in A} X_\alpha \to X_\beta$ is the coordinate projection given by $\pi_\beta((X_\alpha)_{\alpha\in A}) = X_\beta$.

Definition 2.14. *Let m_α be a σ-algebra on X. The* **product σ-algebra** $m = \bigotimes_{\alpha \in A} m_\alpha$ *is the σ-algebra of subsets of X generated by $\{\pi_\alpha^{-1}(E) : \alpha \in A, E \in m_\alpha\}$.*

Proposition 2.15 (Proposition 1.3 in [1])**.** *If A is countable, then $m = \otimes_{\alpha \in A} m_\alpha$ is the σ-algebra generated by $\xi = \{\prod_{\alpha \in A} E_\alpha : E_\alpha \in m_\alpha\}$.*

Proof. Clearly, $m \subseteq \sigma\text{-alg}(\xi)$ since for any $F \in m_\beta$,

$$\pi_\beta^{-1}(F) = \prod_{\alpha \in A} E_\alpha$$

where $E_\alpha = \begin{cases} F \text{ if } \alpha = \beta \\ X_\alpha \text{ otherwise.} \end{cases}$

But in general, $\prod_{\alpha \in A} E_\alpha = \cap_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha) \in m$ if A is countable. Thus, $\sigma\text{-alg}(\xi) \subseteq m$, done. $\square$

Proposition 2.16 (Proposition 1.4 in [1])**.** *Suppose $m_\alpha = \sigma\text{-alg}(\xi_\alpha)$ for some $\xi_\alpha \subseteq \mathcal{P}(X_\alpha)$. Let $m = \otimes_{\alpha \in A} m_\alpha$. Then,*

1) $m = \sigma\text{-alg}(\mathcal{T}_1)$ where $\mathcal{T}_1 = \{\pi_\alpha^{-1}(E) : \alpha \in A, E \in \xi_\alpha\}$.

2) If A is countable, then $m = \sigma\text{-alg}(\mathcal{T}_2)$ where $\mathcal{T}_2 = \{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \xi_\alpha\}$.

Proof. For (1), note that $m = \sigma\text{-alg}\left(\{\pi_\alpha^{-1}(E) : \alpha \in A, E \in m_\alpha\}\right)$ by definition. Since $\mathcal{T}_1 \subseteq \xi_\alpha$, $\sigma\text{-alg}(\mathcal{T}_1) \subseteq \sigma\text{-alg}(\xi_\alpha) = m_\alpha$, thus, $\sigma\text{-alg}(\mathcal{T}_1) \subseteq m$.

To show $m \subseteq \sigma\text{-alg}(\mathcal{T}_1)$, we must show that $\forall F \in m_\alpha, \pi_\alpha^{-1}(F) \in \sigma\text{-alg}(\mathcal{T}_1)$. Fix $\alpha \in A$. Let

$$n = \{G \subseteq X_\alpha : \pi_\alpha^{-1}(G) \in \sigma\text{-alg}(\mathcal{T}_1)\}.$$

Claim 2.17. *n is a σ-algebra on X_α.*

proof of the claim. If $G_i \in n$ for $i \in \mathbb{N}$,

$$\pi_\alpha^{-1}\left(\bigcup_{n=1}^{\infty} G_n\right) = \bigcup_{n=1}^{\infty} \pi_\alpha^{-1}(G_n) \in \sigma\text{-alg}(\mathcal{T}_1) \implies \bigcup_{n=1}^{\infty} G_n \in n.$$

Also, $\forall G \in n$, $\pi_\alpha^{-1}(X_\alpha \setminus G) = X \setminus \pi_\alpha^{-1}(G) \in \sigma\text{-alg}(\mathcal{P}_1)$. So $X \setminus G \in n$. Thus, n is closed under countable union and complementary. $\square$

Now it suffices to show that $m_\alpha \subseteq n$, since if it is true, then for any $E \in m_\alpha$, $\pi_\alpha^{-1}(E) \in \sigma\text{-alg}(\mathcal{T}_1)$, thus $m \subseteq \sigma\text{-alg}(\mathcal{T}_1)$. Note that

$$\forall E \in \xi_\alpha, \pi_\alpha^{-1} \in \mathcal{T}_1 \subseteq \sigma\text{-alg}(\mathcal{T}_1) \implies E \in n \implies \xi_\alpha \subseteq n,$$

by definition of $\mathcal{T}_1$, n. Thus,

$$m_\alpha = \sigma\text{-alg}(\xi_\alpha) \subseteq \sigma\text{-alg}(n) = n.$$

For (2), $m \subseteq \sigma\text{-alg}(\mathcal{T}_2)$ since for any $F \in m_\beta$,

$$\pi_\beta^{-1}(F) = \prod_{\alpha \in A} E_\alpha$$

where $E_\alpha = \begin{cases} F \text{ if } \alpha = \beta \\ X_\alpha \text{ otherwise.} \end{cases}$

But in general, $\prod_{\alpha \in A} E_\alpha = \cap_{\alpha \in A} \pi_\alpha^{-1}(E_\alpha) \in m$ if A is countable. Thus, $\sigma\text{-alg}(\xi) \subseteq m$, done. $\square$

2.2 Measures

Definition 2.18 (Measurable space and measure). *A* **measurable space** (X, m) *is a pair of nonempty set* X *and* $\sigma-algebra$ m *of subsets of* X. *A* **measure** *on* (X, m) *is* $\mu : m \to [0, \infty]$ *satisfying*

(i) $\mu(\emptyset) = 0$

(ii) μ *is countably additive, namely, if* $E_1, E_2, \cdots$ *are disjoint, then* $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Then, we say (X, m, μ) *is a* **measure space.**

Note that countably additive implies finite additive.

Definition 2.19 (Properties of a measure space). *A measure space* (X, m, μ) *is*

- *A* **probability space** *if* $\mu(X) = 1$.
- **finite** *if* $\mu(X) < +\infty$
- σ**-finite** *if* $\exists E_1, E_2, \cdots$ *such that* $X = \bigcup_{j=1}^{\infty} E_j$, $\mu(E_j) < \infty, \forall j \in \mathbb{N}$.
- **semi-finite** *if* $\forall E \in m$ *s.t.* $\mu(E) = \infty$, $\exists F \in m$ *s.t.* $F \subset E$ *and* $0 < \mu(F) < \infty$.

Note that σ-finite implies semifinite.
Our principal goal is to construct a nice measure space.

Example 2.20 (Examples). .

1. **Counting measure**. $(X, \mathcal{P}(X), \mu)$, $\mu(E) = \begin{cases} |E| \text{ if } E \text{ is finite} \\ \infty \text{ otherwise.} \end{cases}$

2. **Dirac mass** $(X, \mathcal{P}(X), \delta_x)$ *for some* $x \in X$, *s.t.* $\delta_x(E) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ otherwise.} \end{cases}$

3. (X, m, μ), $\mu(E) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ otherwise.} \end{cases}$

4. *Let* X *be any uncountable set.* $m = \{E \subseteq X : \text{either } E \text{ or } E^c \text{ is countable}\}$. $\mu(E) = \begin{cases} 0 \text{ if } E \text{ is countable} \\ \infty \text{ otherwise.} \end{cases}$

Remark 2.21 (Counter-example). *Let* X *be any uncountable set.* $m = \{E \subseteq X : \text{either } E \text{ or } E^c \text{ is countable}\}$. $\mu(E) = \begin{cases} 0 \text{ if } E \text{ is finite} \\ \infty \text{ otherwise.} \end{cases}$ *Then, it is not countably additive. However,* μ *is finitely additive, i.e., if* $E_1, \cdots, E_n \in m$ *are disjoint, then* $\mu(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i)$. *So it is called* μ *is* **finite additive measure**.

Theorem 2.22 (Theorem 1.8 in [1]). *Let* (X, m, μ) *be a measure space*

(a) **Monotonicity.** *If* $E, F \in m$ *and* $E \subseteq F$, *then* $\mu(E) \leq \mu(F)$.

(b) **Subaddiditivity.** *If* $E_1, E_2, \cdots \in m$, $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_j)$

(c) **Continuity from below.** *If* $E_1 \subseteq E_2 \subseteq \cdots$, *then* $\mu(\bigcup_{k=1}^{\infty} E_k) = \lim_{k \to \infty} \mu(E_k)$.

(d) **Continuity from above (limited).** *If* $E_1 \supseteq E_2 \supseteq \cdots$, *and* $\mu(E_j) < \infty$ *for some* $j \in \mathbb{N}$, *then* $\mu(\cap_{n=1}^{\infty} E_n) = \lim_{n \to \infty} \mu(E_n)$.

Proof. For (a), note that $F = E \cup (F \setminus E)$, thus

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E).$$

For (b), let $F_1 = E_1, F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i\right)$. Then, F_i are disjoint, and $\cup_{k=1}^n E_k = \cup_{k=1}^n F_k$. Thus,

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \mu(\bigcup_{k=1}^{\infty} F_k) = \sum_{k=1}^{\infty} \mu(F_k) \leq \sum_{k=1}^{\infty} \mu(E_k),$$

by monotonicity.

For (c), let $F_1 = E_1, F_n = E_n \setminus E_{n-1}$. Then, $\cup_{k=1}^{\infty} E_k = \cup_{k=1}^{\infty} F_k$, and F_k are disjoint. Thus,

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(F_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} F_k\right) = \lim_{n\to\infty} \mu(E_n).$$

For (d), we need a lemma;

Lemma 2.23. *If $E \subseteq F$, $E, F \in m$, and $\mu(F) < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$.*

proof of the lemma. By (a), $\mu(E) \leq \mu(F)$, and $\mu(F) = \mu(E) + \mu(F \setminus E)$. Thus, $\mu(F) - \mu(E) = \mu(F \setminus E)$. ☐

Without loss of generality, assume $\mu(E_1) < \infty$. Then,

$$\mu(E_1) - \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(E_1 \setminus E_n) = \mu\left(\bigcup_{n=1}^{\infty} (E_1 \setminus E_n)\right) = \mu\left(E_1 \setminus \left(\bigcap_{n=1}^{\infty} E_n\right)\right) = \mu(E_1) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$$

where first and last equality comes from the lemma, and second equality comes from (c), and the other's are comes from set theoretic operation. Thus, $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$ ☐

Example 2.24. *Suppose $(\mathbb{R}, \mathcal{P}(\mathbb{R})$, counting measure). Then, $\mu((0,1)) = \infty$. $\mu(\cap_{k=1}^n (0, \frac{1}{k})) = \mu(\emptyset) = 0$. This shows that $\mu(E_j) < \infty$ for some j is needed for making (d) true.*

Definition 2.25 (Null set)**.** *Let (X, m, μ) be a measure space. A* **null set** *is $E \in m$ such that $\mu(E) = 0$. A statement is said to hold* **almost everywhere**, **μ-almost everywhere**, **a.e.**, *or* **$\mu-$a.e.** *if it holds for all x in the complement of a null set.*

Example 2.26. *For example, $f : X \to \mathbb{R}$ vanishes almost everywhere means $\exists$ a null set E s.t. $f(x) = 0$ for all $x \in E^c$.*

Definition 2.27. *A measure space (X, m, μ) is said to be* **complete** *if $E \in m, \mu(E) = 0$ and $F \subseteq E \implies F \in m, \mu(F) = 0$.*

Theorem 2.28 (Theorem 1.9 in [1])**.** *Let (X, m, μ) be a measure space and let $\mathcal{N} = \{N \in m : \mu(N) = 0\}$. Let $\bar{m} = \{E \cup F : E \in m, \exists N \in \mathcal{N} \text{ s.t. } F \subseteq N\}$ Then, $\bar{m}$ is a σ-algebra and $\exists!$ extension $\bar{\mu} : \bar{m} \to [0, \infty]$ of μ s.t. $\bar{\mu}$ is a measure on $(X, \bar{m})$.*

Definition 2.29. *$(X, \bar{m}, \bar{\mu})$ is the completion of $(X, m\mu)$.*

Proof. Show $\bar{m}$ is a σ-algebra. Let $G_1, G_2, \cdots \in \bar{m}$. We have $G_j = E_j \cup F_j$ for some $E_j \in m$ and $F_j \subseteq N_j \in \mathcal{N}$. Then,

$$G := \bigcup_{j=1}^{\infty} G_j = \left(\bigcup_{j=1}^{\infty} E_j\right) \cup \left(\bigcup_{j=1}^{\infty} F_j\right).$$

Note that $\bigcup_{j=1}^{\infty} E_j \in m$ by countable additivity of m and

$$\left(\bigcup_{j=1}^{\infty} F_j\right) \subseteq \bigcup_{j=1}^{\infty} N_j \in \mathcal{N},$$

thus $\left(\bigcup_{j=1}^{\infty} F_j\right) \in \mathcal{N}$. This implies $G \in \bar{m}$.

Let $G = E \cup F \in \bar{m}$, where $E \in m$, $F \subseteq N \in \mathcal{N}$. Since $F \subseteq N$, $F^c = N^c \cup (N \setminus F)$. Thus,

$$G^c = E^c \cap F^c = E^c \cap (N^c \cup (N \setminus F)) = (E^c \cap N^c) \cup (E^c \cap (N \setminus F)).$$

Since $E^c \cap N^c \in m$ since it is σ-algebra and $E^c \cap (N \setminus F) = E^c \cap N \cap F^c \subseteq N$. Hence $G^c \in \bar{m}$. Therefore, $\bar{m}$ is a σ-algebra.

Let's define $\bar{\mu} : \bar{m} \to [0, \infty]$ extending μ by $\bar{\mu}(E \cup F) = \mu(E)$.

Claim 2.30. *μ is well-defined.*

Proof. If $E_1, E_2 \in m$, $F_1 \subseteq N_1$, $F_2 \subseteq N_2$, with $N_1, N_2 \in \mathcal{N}$ and $E_1 \cup F_1 = E_2 \cup F_2$. It suffices to show that $\mu(E_1) = \mu(E_2)$. Note that

$$E_1 \subseteq E_2 \cup F_2 \subseteq E_2 \cup N_2 \implies \mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2).$$

Similarly,

$$E_2 \subseteq E_1 \cup F_1 \subseteq E_1 \cup N_1 \implies \mu(E_2) \leq \mu(E_1) + \mu(N_1) = \mu(E_1).$$

This implies $\mu(E_1) = \mu(E_2)$. $\square$

Thus, $\bar{\mu}$ extends μ. Thus, $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$. And,

Claim 2.31. *$\bar{\mu}$ is countably additive. Let $G_1, G_2, \cdots \in \bar{m}$ be disjoint. Write each $G_i = E_i \cup F_j$ for some $E_j \in m$, $F_j \subseteq N_j \in \mathcal{N}$. Note that E_i's are disjoint. Let $G = \cup_{j=1}^{\infty} G_j$. Then*

$$G = \left(\bigcup_{j=1}^{\infty} E_j\right) \cup \left(\bigcup_{j=1}^{\infty} F_j\right).$$

Since $\left(\bigcup_{j=1}^{\infty} E_j\right) \subseteq \left(\bigcup_{j=1}^{\infty} N_j\right) \in \mathcal{N}$, $G \in \bar{m}$. Thus,

$$\bar{\mu}(G) = \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \bar{\mu}(G_j).$$

For uniqueness, suppose $\mu' : \bar{m} \to [0, \infty]$ is a measure that extends μ. It suffices to show $\mu' = \bar{\mu}$. Let $G \in \bar{m}$. Then, $G = E \cup F$ for $E \in m$, $F \subseteq N \in \mathcal{N}$. Then,

$$\mu(E) = \mu'(E) \leq \mu'(G) \leq \mu'(E \cup N) = \mu(E \cup N) = \mu(E) + \mu(N) = \mu(E).$$

Thus, $\mu'(G) = \mu(E) = \bar{\mu}(G) \implies \mu' = \mu$. $\square$

2.3 Outer measure

Definition 2.32 (Outer measure)**.** *An* **outer measure** *on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ s.t.*

(i) $\mu^*(\emptyset) = 0$

(ii) **Monotonicity.** $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$.

(iii) **Subadditivity.** $\mu^*(\cup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$.

Proposition 2.33 (Proposition 1.10 in [1])**.** *Let $\xi \subseteq \mathcal{P}(X)$ s.t. $\emptyset \in \xi, X \in \xi$. Let $\rho : \xi \to [0, \infty]$ be a function s.t. and $\rho(\emptyset) = 0$. Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be given by*

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_1, E_2, \cdots \in \xi, A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

Then μ^* is an outer measure.

Proof. (i) $\mu^*(\emptyset) = 0$, because $\emptyset \subseteq \bigcup_{j=1}^{\infty} \emptyset \implies 0 \leq \mu^*(\emptyset) \leq \sum_{j=1}^{\infty} \rho(\emptyset) = 0$.

(ii) ***Monotonicity*** If $A, B \in \mathcal{P}(X)$, $A \subseteq B$, then whenever $E_j \in \xi$ and $B \subseteq \bigcup_{j=1}^{\infty} E_j$, then

$$A \subseteq B \subseteq \bigcup_{j=1}^{\infty} E_j \implies \mu^*(A) \leq \sum_{j=1}^{\infty} \rho(E_j) \implies \mu^*(A) \leq \mu^*(B)$$

by taking infimum. So, $\mu^*(A) \leq \mu^*(B)$, by taking the infimum.

(iii) ***Subadditivity*** Suppose $A_1, A_2, \cdots \in \mathcal{P}(X)$. We want to show $\mu^*(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu^*(A_k)$. Let $\epsilon > 0$. Choose $E_{k,1}, E_{k,2}, \cdots \in \xi$ such that

$$A_k \subseteq \bigcup_{j=1}^{\infty} E_{k,j} \text{ and } \mu^*(A_k) \leq \sum_{j=1}^{\infty} \rho(E_{k,j}) \leq \mu^*(A_k) + \frac{\epsilon}{2^k}.$$

Existence of such sets is assured by the definition of infimum. Then,

$$A := \bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} E_{k,j} \implies \mu^*(A) \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \rho(E_{k,j}) = \sum_{k=1}^{\infty} \mu^*(A_k) + \frac{\epsilon}{2^k} = \sum_{k=1}^{\infty} \mu^*(A_k) + \epsilon.$$

Since ϵ was arbitrary, $\mu^*(A) \leq \sum_{k=1}^{\infty} \mu^*(A_k)$.

□

Definition 2.34 (μ^*-measurable)**.** *Let μ^* be an outer measure on X, and let $A \subseteq X$. We say A is μ^*-measurable if*

$$\forall E \subseteq X, \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Note that $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ is true by subadditivity.

Theorem 2.35 (Caratheodory extension theorem)**.** *Let μ^* be an outer measure on X. Let $\mathcal{M}$ be the collection of all μ^*-measurable subsets of X. Then, $\mathcal{M}$ is a σ-algebra and the restriction of μ^* to $\mathcal{M}$ is a complete measure.*

Proof. Step by step approach.

Step I: ***Show*** $\emptyset \in \mathcal{M}$**.** Let $E \subseteq X$. Then, $\mu^*(E \cap \emptyset) + \mu^*(E \cup \emptyset) = 0 + \mu^*(E) = \mu^*(E)$.

Step II: ***Show*** $A \in \mathcal{M} \implies A^c \in \mathcal{M}$***.*** If $A \in \mathcal{M}, E \subseteq X$, then

$$\mu^*(E \cap A^c) + \mu^*(E \cap (A^c)^c = \mu^*(E \cap A^c) + \mu^*(E \cap A) = \mu^*(E)$$

from the μ^* measurability of A.

Step III: ***Show*** $\mathcal{M}$ ***is an algebra of subsets of*** X ***and*** $\mu^*|\mathcal{M}$ ***is finitely additive.*** Let $A, B \in \mathcal{M}$.

(i) ***Show*** $A \cup B \in \mathcal{M}$***.*** Let $E \subseteq X$. Then,

$\mu^*(E) = \mu^*(E\cap A)+\mu^*(E\cap A^c) = \mu^*(E\cap A\cap B)+\mu^*(E\cap A\cap B^c)+\mu^*(E\cap A^c\cap B^c)+\mu^*(E\cap A^c\cap B)$.

since $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, by subadditivity,

$\mu^*(E\cap(A\cup B)) \leq \mu^*(E\cap(A\cap B))+\mu^*(E\cap(A\cap B^c))+\mu^*(E\cap(A^c\cap B)) = \mu^*(E)-\mu^*(E\cap A^c\cap B^c)$.

Thus,

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \geq \mu^*(E),$$

where the last inequality also comes from the subadditivity. Thus, $A \cup B \in \mathcal{M}$.

(ii) ***Show if*** $A \cap B = \emptyset$ ***then*** $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$***.*** Assume $A \cap B = \emptyset$. Let $E = A \cup B$. From the μ^*-measurability of A,

$$\mu^*(A \cup B) = \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(A) + \mu^*(B).$$

Step IV: ***Show*** $\mathcal{M}$ ***is a*** σ***-algebra, and*** $\mu^*|_{\mathcal{M}}$ ***is countable additive.***

(i) ***Show*** $\mathcal{M}$ ***is a*** σ***-algebra.*** Let $D_j \in \mathcal{M}, D := \bigcup_{j=1}^{\infty} D_j$. We want to show $D \in \mathcal{M}$. Let $A_k = D_k \setminus \bigcup_{j=1}^{k-1} D_j$. Then, $A_1, A_2, \cdots$ are disjoint. Let $B_k := \bigcup_{j=1}^{k} A_j = \bigcup_{j=1}^{k} D_j$. Then, $B_1 \subseteq B_2 \subseteq \cdots$ and $D := \bigcup_{k=1}^{\infty} B_k$. Then, $\forall k \in \mathbb{N}$, $A_k, B_k \in \mathcal{M}$.
We will show by induction on k, that $\forall E \subseteq X$,

$$\mu^*(E \cap B_k) = \sum_{j=1}^{k} \mu^*(E \cap A_j). \tag{1}$$

If $k = 1$, then $B_1 = A_1$, thus the equation (1) hold. If $K \geq 2$, since $A_k \in \mathcal{M}$,

$$\mu^*(E \cap B_k) = \mu^*(E \cap B_k \cap A_k) + \mu^*(E \cap B_k \cap A_k^c)$$

Since $B_k \cap A_k = A_k, B_k \setminus A_k = \bigcup_{j=1}^{k-1} A_j = B_{k-1}$, since A_i's are disjoint. Thus,

$$\mu^*(E \cap B_k) = \mu^*(E \cap A_k) + \mu^*(E \cap B_{k-1}) = \sum_{j=1}^{k} \mu^*(E \cap A_j),$$

where the last equality comes from the inductive hypothesis.
Thus, for any $k \in \mathbb{N}$.

$$\mu^*(E) = \mu^*(E\cap B_k^c)+\mu^*(E\cap B_k) = \mu^*(E\cap B_k^c)+\sum_{j=1}^{k} \mu^*(E\cap A_j) \geq \mu^*(E\cap D^c)+\sum_{j=1}^{k} \mu^*(E\cap A_j),$$

since $B_k \subseteq D \implies B_k^c \supseteq D^c$. Therefore, by taking limit,

$$\mu^*(E) \geq \mu^*(E \cap D^c) + \sum_{j=1}^{\infty} \mu^*(E \cap A_j).$$

Also, since $D = \bigcup_{j=1}^{\infty} A_j$, thus by countable subadditivity,

$$\mu^*(E \cap D) \leq \sum_{j=1}^{\infty} \mu^*(E \cap A_j).$$

So,

$$\mu^*(E) \geq \mu^*(E \cap D^c) + \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \geq \mu^*(E \cap D^c) + \mu^*(E \cap D) \geq \mu^*(E).$$

So we have equality and $D \in \mathcal{M}$.

(ii) ***Show*** $\mu^*|_{\mathcal{M}}$ ***is countable additive.*** We will use the setting in (i). Suppose $D_1, D_2, \cdots$ are disjoint set in (i). Then, $\forall j \in \mathbb{N}, A_j = D_j$. Take $E = D$ above. Then,

$$\mu^*(D) = \mu^*(E \cap D^c) + \sum_{j=1}^{\infty} \mu^*(E \cap A_j) = \mu^*(\emptyset) + \sum_{j=1}^{\infty} \mu^*(A_j) = \sum_{j=1}^{\infty} \mu^*(A_j).$$

Step V: ***Show*** $\mu^*|_{\mathcal{M}}$ ***is complete,*** i.e., suppose $N \in \mathcal{M}, \mu^*(N) = 0$, and $F \subseteq N \implies F \in \mathcal{M}$. Let $E \subseteq X$. We want to show that $\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \cap F^c)$. Since $E \cap F \subseteq N$,

$$\mu^*(E \cap F) \leq \mu^*(N) = 0$$

by monotonicity. So, $\mu^*(E \cap F) = 0$. Then,

$$\mu^*(E) \leq \mu^*(E \cap F) + \mu^*(E \cap F^c) = \mu^*(E \cap F^c) \leq \mu^*(E).$$

Thus $\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \cap F^c) \implies F \in \mathcal{M}$.

□

Caratheodory extension can be used for extending algebra with premeasure to σ-algebra and measure.

Definition 2.36 (Premeasure)**.** *A* **premeasure** *on a set X is $\mu_0 : \mathfrak{a} \to [0, +\infty]$, where $\mathfrak{a}$ is an algebra of subsets of X, and*

1. $\mu_0(\emptyset) = 0$
2. *if $A_1, A_2, \cdots \in \mathfrak{a}$, disjoint and if $A := \bigcup_{j=1}^{\infty} A_j \in \mathfrak{a}$,*

$$\mu_0(A) = \sum_{j=1}^{\infty} \mu_0(A_j).$$

The motivating example is following; Let

$$\mathfrak{a} := \{\emptyset\} \cup \left\{ \bigcup_{j=1}^{n} (a_j, b_j] : n \in \mathbb{N}, -\infty \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \leq \infty \right\}$$

with

$$\mu_0 \left(\bigcup_{j=1}^{\infty} (a_j, b_j] \right) = \sum_{j=1}^{n} (b_j - a_j).$$

Since it is countably additive and monotone, μ_0 is a premeasure.

Theorem 2.37 (Theorem 1.13 and 1.14 in [1] p. 31)**.** *Let $\mathfrak{a}$ be an algebra of subsets of X and let $\mu_0 : \mathfrak{a} \to [0, +\infty]$ be a premeasure. Let $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ be the outer measure constructed from μ_0 as a proposition 1.10., i.e.,*

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathfrak{a}, E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}.$$

Then,

(i) $\mu^*|_{\mathfrak{a}} = \mu_0$

(ii) If $A \in \mathfrak{a}$, A is μ^*-measurable.

Thus, by the Caratheodory extension theorem, $(\sigma\text{-}alg(\mathfrak{a}), \mu^*|_{\sigma\text{-}alg(\mathfrak{a})})$ is a measure on $\sigma\text{-}alg(\mathfrak{a})$. Also,

(iii) If $\nu : \sigma\text{-}alg(\mathfrak{a}) \to [0, +\infty]$ is any measure such that $\nu|_{\mathfrak{a}} = \mu_0$, then $\forall E \in \sigma\text{-}alg(\mathfrak{a}), \nu(E) \leq \mu^*(E)$ with equality if $\mu^*(E) < \infty$.

(iv) If μ_0 is σ-finite, then $\mu^*|_{\sigma\text{-}alg(\mathfrak{a})}$ is the unique measure on $\sigma\text{-}alg(\mathfrak{a})$ extending μ_0.

Proof. 1. ***Show*** $\mu^*|_{\mathfrak{a}} = \mu_0$**.** Let $E\mathfrak{a}$. For all $i \in \mathbb{N}$, $\begin{cases} A_i = E_1 \text{ if } i = 1 \\ A_i = \emptyset \text{ otherwise.} \end{cases}$ Then, $\mu^*(E) \leq \mu_0(E)$ by subadditivity. To show $\mu^*(E) \geq \mu_0(E)$, let $A_j \in \mathfrak{a}$ such that $E \leq \bigcup_{j=1}^{\infty} A_j$. Let

$$B_n := E \cap \left(A_n \setminus \bigcup_{j=1}^{n-1} A_j \right).$$

Then, $B_n \in \mathfrak{a}$, B_i's are disjoint. Also,

$$\bigcup_{n=1}^{\infty} B_n = E \cap \left(\bigcup_{j=1}^{\infty} A_j \right) = E.$$

Since μ_0 is a premeasure,

$$\mu_0(E) = \sum_{n=1}^{\infty} \mu_0(B_n) \leq \sum_{n=1}^{\infty} \mu_0(A_n).$$

By taking the infimum on $\{A_n\}_{n=1}^{\infty}$,

$$\mu_0(E) \leq \mu^*(E).$$

2. ***If*** $A \in \mathfrak{a}$***,*** A ***is*** μ^****-measurable.*** Let $E \subseteq X$. We must show that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Let $\epsilon > 0$, choose $B_1, B_2, \dots \in \mathfrak{a}$ such that $E \subseteq \bigcup_{j=1}^{\infty} B_j$, and

$$\mu^*(E) + \epsilon \geq \sum_{j=1}^{\infty} \mu_0(B_j).$$

By definition of μ^*, such B_js exist. Then,

$$\begin{aligned}
\mu^*(E)+\epsilon &\geq \sum_{j=1}^{\infty}\mu_0(B_j) \\
&= \sum_{j=1}^{\infty}\left(\mu_0(B_j\cap A)+\mu_0(B_j\cap A^c)\right),\text{ since } B_j\cap A \text{ and } B_j\cap A\in\mathfrak{a}. \\
&= \sum_{j=1}^{\infty}\mu_0(B_j\cap A)+\sum_{j=1}^{\infty}\mu_0(B_j\cap A^c) \\
&\geq \mu^*(E\cap A)+\mu^*(E\cap A^c),\text{ since } E\cap A\subseteq\bigcup_{j=1}^{\infty}B_j\cap A.
\end{aligned}$$

By taking $\lim_{\epsilon\to 0}$ on both sides, we get desired inequality.

3. ***If*** $\nu:\sigma$***-alg***$(\mathfrak{a})\to[0,+\infty]$ ***is any measure such that*** $\nu|_{\mathfrak{a}}=\mu_0$, ***then*** $\forall E\in\sigma$***-alg***$(\mathfrak{a}),\nu(E)\leq\mu^*(E)$ ***with equality if*** $\mu^*(E)<\infty$. Note that μ is a measure derived by caratheodory extension theorem and μ^*. Let $A_n\in\mathfrak{a}$ such that $E\subseteq\bigcup_{n=1}^{\infty}A_n$. We want to show that $\nu(E)\leq\sum_{n=1}^{\infty}\mu_0(A_n)$.

   By subadditivity of ν,

$$\nu(E)\leq\nu(\bigcup_{n=1}^{\infty}A_n)\leq\sum_{n=1}^{\infty}\nu(A_n)=\sum_{n=1}^{\infty}\mu_0(A_n)\implies\nu(E)\leq\mu(E). \tag{2}$$

   Let $\epsilon>0$. Choose $A_n\in\mathfrak{a}$ such that $E\subseteq\bigcup_{n=1}^{\infty}A_n$ and $\mu(E)+\epsilon\geq\sum_{n=1}^{\infty}\mu_0(A_n)$. Let $A:=\bigcup_{n=1}^{\infty}A_n$. Then, if $\mu(E)<\infty$,

$$\mu(A)=\mu(A\setminus E)+\mu(E)\implies\mu(A\setminus E)=\mu(A)-\mu(E)\leq\left(\sum_{n=1}^{\infty}\mu_0(A_n)\right)-\mu(E)\leq\epsilon.$$

   Also, for any $n\in\mathbb{N}$, $\nu(\bigcup_{j=1}^{n}A_j)=\mu_0(\bigcup_{j=1}^{n}A_j)$ since $\bigcup_{j=1}^{n}A_j\in\mathfrak{a}$, thus by letting $B_n:=\bigcup_{j=1}^{n}A_j$,

$$\nu(A)=\nu\left(\bigcup_{n=1}^{\infty}B_n\right)=\lim_{n\to\infty}\nu(B_n)=\lim_{n\to\infty}\mu(B_n)=\mu(A).$$

   Therefore,

$$\mu(E)\leq\mu(A)=\nu(A)=\nu(E)+\nu(A\setminus E)\leq\nu(E)+\mu(A\setminus E)\leq\nu(E)+\epsilon.$$

   where second equality comes from continuity from below w.r.t. ν, and third equality comes from the inequality (2), the last equality comes from the continuity from below w.r.t. μ. Therefore, by lettign $\epsilon\to 0$, we have $\nu(E)=\mu(E)$ if $\mu(E)<\infty$.

4. ***If*** μ_0 ***is*** σ***-finite, then*** $\mu^*|_{\sigma\text{-}alg(\mathfrak{a})}$ ***is the unique measure on*** σ***-alg***$(\mathfrak{a})$ ***extending*** μ_0***.*** By hypothesis, $\exists A_1,A_2,\dots\in\mathfrak{a}$ such that $X=\bigcup_{i=1}^{\infty}A_i$ with $\mu(A_i)<\infty,\forall i\in\mathbb{N}$. Replcae A_n by $A_n\setminus\bigcup_{j=1}^{n-1}A_j$ if necessary, we may assume that A_ns are disjoint, without loss of generality. Then, $\forall E\in\sigma\text{-alg}(\mathfrak{a})$,

$$\nu(E)=\nu\left(\bigcup_{n=1}^{\infty}E\cap A_n\right)=\sum_{n=1}^{\infty}\nu(E\cap A_n)=\sum_{n=1}^{\infty}\mu(E\cap A_n)=\nu(E).$$

   where second equality comes from the countable additivity of ν, third equality comes from $\nu=\mu$ for any measurable set having finite measure, and the last equality comes from the countable additivity of μ.

$\square$

2.4 Borel measure on the real line

Proposition 2.38 (Special case of proposition 1.7)**.** *Let $\xi \subseteq \mathcal{P}(\mathbb{R})$ be the collection of all half open intervals, closed at the right, i.e.,*

$$\xi = \{(a, b] : a, b \in \mathbb{R}, a < b\} \cup \{(-\infty, b] : b \in \mathbb{R}\} \cup \{(a, +\infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}.$$

Let $\mathfrak{a}$ be the set of all finite union of elements of ξ. Then $\mathfrak{a}$ is an algebra of subsets of $\mathbb{R}$.

If $I_1, I_2 \in \xi$ and $I_1 \cap I_2 \neq \emptyset$ or if $dist(I_1, I_2) := \inf\{|x - y| : x \in I_1, y \in I_2\} = 0$, then $I_1 \cup I_2 \in \xi$. Thus, each elements of $\mathfrak{a}$ can be written uniquely as a finite union of separated elements of ξ, where I_1 and I_2 are **separated** *if $dist(I_1, I_2) > 0$.*

Proof. We must show followings;

(i) $\emptyset \in \mathfrak{a}$; it holds by definition of $\mathfrak{a}$.

(ii) ***If*** $A_1, \cdots, A_n \in \mathfrak{a} \implies \cup_{j=1}^n A_j \in \mathfrak{a}$. ; it holds by definition of $\mathfrak{a}$.

(iii) ***If*** $A \in \mathfrak{a} \implies A^c \in \mathfrak{a}$**.** Suppose $A = \bigcup_{j=1}^n I_j$ with $I_1, I_2, \cdots I_n \in \xi$, where they are separated. Suppose each I_j is bounded. Thus, $I_j = (a_j, b_j]$ for some $a_j, b_j \in \mathbb{R}$ for $1 \leq j \leq n$. By renumbering if necessary, we may assume that $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$. Then,

$$A^c = (-\infty, a_1] \cup (b_1, a_2] \cup \cdots \cup (b_{n-1}, a_n] \cup (b_n, \infty) \in \mathfrak{a}.$$

Thus, $\mathfrak{a}$ is an algebra. □

Notation 2.39 (Construction of Lebesgue-Stieltjes measures)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing (i.e. non-decreasing) functions that is right continuous, (i.e., $\forall a \in \mathbb{R}, \lim_{x \to a+} F(x) = F(a)$.) Set*

$$F(-\infty) := \lim_{x \to -\infty} F(x) \in \mathbb{R} \cup \{-\infty\}, F(\infty) := \lim_{x \to \infty} F(x) \in \mathbb{R} \cup \{+\infty\}.$$

Let $\mu_0 : \xi \to [0, \infty]$ be

$$\begin{aligned} \mu_0((a, b]) &= F(b) - F(a) \\ \mu_0((a, +\infty)) &= F(+\infty) - F(a) \\ \mu_0((-\infty, b]) &= F(b) - F(-\infty) \\ \mu_0(\mathbb{R}) &= F(+\infty) - F(-\infty) \\ \mu_0(\emptyset) &= 0. \end{aligned}$$

Then, we extend μ_0 to $\mathfrak{a}$ as follow; If $A \in \mathfrak{a}$, then A has unique representation as $A = \bigcup_{j=1}^n I_j$ for some separated $I_1, \cdots, I_n \in \xi$, by the previous proposition. Let $\mu_0(A) := \sum_{j=1}^n \mu_0(I_j)$

Proposition 2.40 (Proposition 1.15 in [1] p.33)**.** *μ_0 is a premeasure on $\mathfrak{a}$.*

Proof. Note that $\mu_0(\emptyset) = 0$. We must show that if $A_1, A_2, \cdots \in \mathfrak{a}$ are disjoint and if $\bigcup_{j=1}^\infty A_j \in \mathfrak{a}$, then

$$\mu_0(\bigcup_{j=1}^\infty A_j) = \sum_{j=1}^\infty \mu_0(A_j).$$

Without loss of generality, assume $\forall j \in \mathbb{N}, A_j \in \xi$. Since $\bigcup_{j=1}^\infty A_j \in \mathfrak{a}$, we may write

$$\bigcup_{j=1}^\infty A_j = \bigcup_{k=1}^n I_k$$

for some separated $I_1, I_2, \cdots, I_n \in \xi$, by the previous proposition. Let $F_k = \{j \in \mathbb{N} : A_j \subseteq I_k\}$. It will suffice to show that

$$\mu_0(I_k) = \sum_{j \in F_k} \mu_0(A_j);$$

since $\mu_0(\bigcup_{j=1}^{\infty} A_j) = \sum_{k=1}^{n} \mu_0(I_k)$. Thus we may assume that

$$\bigcup_{j=1}^{\infty} A_j = I = (a, b] \in \xi,$$

and we want to show

$$\mu_0(I) = \sum_{j=1}^{\infty} \mu_0(A_j).$$

Suppose $I = (a, b]$. We may write $A_j = (a_j, b_j]$. To show $\mu_0(I) \geq \sum_{j=1}^{\infty} \mu_0(A_j)$, it suffices to show

$$\sum_{j=1}^{n} \mu_0(A_j) \leq \mu_0(I)$$

for all $n \in \mathbb{N}$. After renumbering, we may assume that

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b.$$

Thus,

$$\begin{aligned}
\mu_0(I) &= F(b) - F(a) \geq F(b_n) - F(a_1) \\
&\geq F(b_n) - (F(b_{n-1}) - F(a_n)) - (F(b_{n-2}) - F(a_{n-1})) - \cdots - (F(b_1) - F(a_2)) - F(a_1) \\
&= \sum_{j=1}^{n} \mu_0(A_j).
\end{aligned}$$

By letting $n \to \infty$, we have $\mu_0(I) \geq \sum_{j=1}^{\infty} \mu_0(A_j)$.

To show reverse direction, i.e., $\mu_0(I) \leq \sum_{j=1}^{\infty} \mu_0(A_j)$., we need compactness argument. Let $\epsilon > 0$. From the right continuity of F, we can get $\delta > 0$ such that

$$F(a + \delta) - F(a) < \epsilon < \epsilon$$

and $\delta_j > 0$ such that

$$F(b_j + \delta_j) - F(b_j) \leq \frac{\epsilon}{2^j},$$

for each a and b_j which we setup in the previous paragraph. Then,

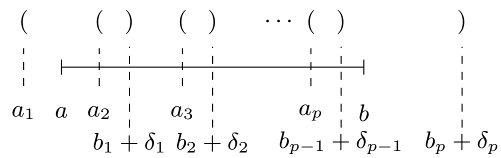
$$[a + \delta, b] \subseteq I \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j + \delta_j),$$

where second subset equation is derived from the assumption $\bigcup_{j=1}^{\infty} A_j = I$. Since $[a + \delta, b]$ is closed bounded set, thus it is compact. Therefore, $((a_j, b_j + \delta_j))_{j=1}^{\infty}$ has a finite subcover. So by renumbering if necessary, we can assume that $[a + \delta, b] \subseteq \bigcup_{j=1}^{p} (a_j, b_j + \delta_j)$. Also, we can assume that $\{(a_j, b_j + \delta_j\}_{j=1}^{p}$ is a minimal subcover.

After renumbering again, we can assume

$$\begin{aligned}
a_1 &< a + \delta < b_1 + \delta \\
a + \delta \le a_2 &< b_1 + \delta < b_2 + \delta_2 \\
a_3 &< b_2 + \delta_2 < b_3 + \delta_3 \\
&\cdots \\
a_{l+1} &< b_l + \delta_l < b_{l+1} + \delta_{l+1} \\
&\cdots \\
a_p &< b_{p-1} + \delta_{p-1} \le b < b_p + \delta_p.
\end{aligned}$$

We can draw above inequalities in a line segment.



Thus,

$$\begin{aligned}
\mu(I) &= F(b) - F(a) \\
&< F(b) - F(a + \delta) + \epsilon \\
&\le F(b_p + \delta_p - F(a + \delta) + \epsilon \\
&\le F(b_p + \delta_p) - F(a_1) + \epsilon \\
&\le F(b_p + \delta_p) + \sum_{l=1}^{p-1} (F(b_l + \delta_l) - F(a_{l+1}) - F(a_1) + \epsilon \\
&= \sum_{l=1}^{p} (F(b_j + \delta_j) - F(a_j)) + \epsilon
\end{aligned}$$

Since $F(b_j + \delta_j) < \frac{\epsilon}{2^j} + F(b_j)$,

$$\begin{aligned}
\mu(I) &\le \sum_{l=1}^{p} (F(b_j + \delta_j) - F(a_j)) + \epsilon \\
&\le \sum_{l=1}^{p} (F(b_j) - F(a_j) + \frac{\epsilon}{2^{\sigma(j)}}) + \epsilon \\
&\le \sum_{l=1}^{p} (F(b_j) - F(a_j)) + 2\epsilon \\
&\le \sum_{l=1}^{p} \mu_0(A_j) + 2\epsilon \\
&< \sum_{j=1}^{\infty} \mu_0(A_j) + 2\epsilon
\end{aligned}$$

where σ is a permutation of $\mathbb{N}$ accounting for the reordering. Thus,

$$\mu_0(I) < \sum_{j=1}^{\infty} \mu_0(A_j) + 2\epsilon,$$

and by letting $\epsilon \to 0$,

$$\mu_0(I) \leq \sum_{j=1}^{\infty} \mu_0(A_j) + 2\epsilon.$$

Thus we are done if a, b are finite.

If $a = -\infty$, then, using the cover $[-M, b]$, we get an inequality

$$F(b) - F(-M) \leq \sum_{j=1}^{\infty} \mu_0(I_j) + 2\epsilon.$$

Also, if $b = \infty$, then we get inequality $F(M) - F(a) \leq\leq \sum_{j=1}^{\infty} \mu_0(I_j) + 2\epsilon$. In any case, by letting $M \to \infty$ and $\epsilon \to 0$, we get desired inequality. □

Note that compactness is used to use such interlacing argument. Without compactness, it is too complicated.

Theorem 2.41 (Theorem 1.16 in [1] p.36)**.** *(a) If $F : \mathbb{R} \to \mathbb{R}$ is nondecreasing and right continuous function, there exists unique measure $\mu_F : \beta_{\mathbb{R}} \to [0, \infty]$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}, a < b$.*

(b) If $G : \mathbb{R} \to \mathbb{R}$ is also right increasing right continuous function, then $\mu_G = \mu_F$ if and only if $G - F = c$ for some constant c.

(c) If $\mu : \beta_R \to [0, +\infty]$ is a measure such that $\mu((a, b]) < +\infty$ for every $a, b \in \mathbb{R}$ with $a < b$, then letting

$$F(x) = \begin{cases} \mu((0, x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x, 0]) & x < 0. \end{cases}$$

Then $\mu = \mu_F$.

(d) **(c*)** *If $\mu : \mathcal{B}_{\mathbb{R}} \to [0, k]$ for some $k > 0$ is a finite measure, then $\mu = \mu_G$ where $G(x) = \mu((-\infty, x])$.*

For example, $\mu_F((a, b]) = F(b) - F(a) = \mu((0, b]) + \mu((a, 0])$ if $a < 0, b > 0$.

Proof. Take the premeasure μ_0 constructed in the proposition 1.15 on $\mathcal{A}$ which is an algebra generated by the half open interval (h-interval). Use the general construction of an outer measure from the proposition 1.10. Use the Caratheodory's extension theorem (1.11) and proposition 1.13 and theorem 1.14 to conclude that $\mu^*|_{\sigma\text{-alg}(\mathcal{A})}$ is a measure extending μ_0.

For (b), if $G - F = c$, then trivially holds since $\mu_G((a, b]) = G(b) - G(a) = F(b) - F(a) = \mu_F((a, b])$. Conversely, if $\mu_G = \mu_F$, then $G(b) - F(b) = G(a) - F(a)$ for any a, b, thus, $G - F = c$ for some constant.

For (c), and (c*), it is derived from the uniqueness of μ as a extension of μ_0, since both has the same premeasure and both are finite measure. □

We call μ_F in (a) the ***Lebesgue Stieltjes measure associated to*** F. Recall that the Caratheodory's extension theorem actually gives us a complete measure on the σ-algebra of all μ^*-measurable sets. We let $\mathcal{M}_F$ denote this σ-algebra (possibly larger than $\mathcal{B}_{\mathbb{R}}$) and we denote also by $\mu_F : \mathcal{M}_F \to [0, \infty]$ the measure constructed on $\mathcal{M}_F$. (And call also this the ***Lebesgue Stieltjes measure associated to*** F.

Example 2.42 (Special Cases)**.**

1. *$F(x) = x$. Then $\mu_F((a, b]) = b - a$. Then μ_F is called the* **Lebesgue measure.** *We write $\mathcal{L} = \mathcal{M}_F$ called the* **set of Lebesgue measurable sets.** *We may write $m = \mu_F$.*

2. Fix $x \in \mathbb{R}$. Let $F(t) = \begin{cases} 0 & \text{if } t < x \\ 1 & \text{if } t \geq x. \end{cases}$ *Then* $\mu((a,b]) = \begin{cases} 1 & \text{if } x \in (a,b] \\ 0 & o.w. \end{cases}$ *and* $\mathcal{M}_F = \mathcal{P}(\mathbb{R})$ *and* $\mu_F(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & o.w. \end{cases}$ *This* $\mu_F = \delta_x$ *is called* **Dirac Mass**.

Remark 2.43. *For any Lebesgue Stieltjes measure,*

$$\mu_F(\{x\}) = \lim_{\delta \to 0} \mu_F((x-\delta, x]) = F(x) - \lim_{\delta \to 0} F(x-\delta) \underbrace{=}_{right\ cts} F(x) - F(x-).$$

2.5 Regularity Properties

Now fix increasing right continuous function F, let $\mathcal{M} = \mathcal{M}_F$, $\mu = \mu_F$.

Lemma 2.44 (Lemma 1.17 in [1], p.35)**.** *For any* $E \in \mathcal{M}$,

$$\mu_E = \inf\{\sum_{j=1}^{\infty} \mu((a_j, b_j)) : a_j, b_j \in \mathbb{R}, a_j < b_j, E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j)\}.$$

Proof. We know that $LHS \leq RHS$ since

$$\mu(E) \underbrace{\leq}_{\text{monotonicity}} \mu\left(\bigcup_{j=1}^{\infty} (a_j, b_j)\right) \underbrace{\leq}_{\text{subadditivity}} \sum_{j=1}^{\infty} \mu(a_j, b_j).$$

For $LHS \geq RHS$, if $\mu(E) = +\infty$, then there is nothing to show. Thus, assume that $\mu(E) < +\infty$. Let $\epsilon > 0$. By construction, $\exists A_j \in \mathcal{A}$ such that $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and $\mu(E) + \epsilon \geq \sum_{j=1}^{\infty} \mu(A_j)$, from subadditivity on RHS. Without loss of generality, we can assume that $A_j = (c_j, d_j]$ for $c_j, d_j \in \mathbb{R}, c_j < d_j$. Choose $b_j > d_j$ such that

$$F(b_j) < F(d_j) + \frac{\epsilon}{2^j}.$$

Then,

$$\mu((c_j, d_j]) \leq \mu((c_j, b_j]) = F(b_j) - F(c_j) \leq F(d_j) - F(c_j) + \frac{\epsilon}{2^j} = \mu((c_j, d_j]) + \frac{\epsilon}{2^j} = \mu(A_j) + \frac{\epsilon}{2^j}.$$

Thus,

$$\mu(E) + \epsilon \geq \sum_{j=1}^{\infty} \mu(A_j) \geq \sum_{j=1}^{\infty} \left(\mu((c_j, b_j) - \frac{\epsilon}{2^j}\right).$$

Hence,

$$\mu(E) + 2\epsilon > \sum_{j=1}^{\infty} \left(\mu((c_j, b_j)\right).$$

Since $E \subseteq \bigcup_{j=1}^{\infty} (c_j, b_j)$, we can take infimum on both sides to conclude that

$$\mu(E) + 2\epsilon > \inf\{\sum_{j=1}^{\infty} \mu((a_j, b_j)) : a_j, b_j \in \mathbb{R}, a_j < b_j, E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j)\}.$$

By letting $\epsilon \to 0$, we get desired inequality. $\square$

Theorem 2.45 (Theorem 1.18 in [1] p.36)**.** *If* $E \in \mathcal{M}$*, then*

$$\mu(E) = \inf\{\mu(U) : U \text{ is open in } \mathbb{R}, E \subseteq U\} \tag{3}$$
$$\mu(E) = \sup\{\mu(K) : K \text{ is compact in } \mathbb{R}, K \subseteq E\}. \tag{4}$$

Proof. For (3), $\leq$ is clear by the monotonicity of μ. For $\geq$, if $\mu(E) = +\infty$, there is nothing to show. Assume $\mu(E) < +\infty$. By the Lemma 1.17, given $\epsilon > 0$, $\exists a_j < b_j$ such that $E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j)$ and

$$\mu(E) + \epsilon > \sum_{j=1}^{\infty} \mu((a_j, b_j)) \underbrace{\geq}_{\text{subadditivity}} \mu\left(\bigcup_{j=1}^{\infty}(a_j, b_j)\right).$$

Since $\bigcup_{j=1}^{\infty}(a_j, b_j)$ is open set containing E, we can take infimum on both sides, to get

$$\mu(E) + \epsilon > \inf\{\mu(U) : U \text{ is open in } \mathbb{R}, E \subseteq U\}.$$

By letting $\epsilon \to 0$, we have the desired inequality.

For (4), also $\geq$ is clear by the monotonicity. To show $\leq$, assume first that E is bounded, say $E \subseteq [-r, r].r > 0$. So, $\mu(E) \leq \mu([-r, r]) < +\infty$. Then $\overline{E}$ is closed and bounded thus $\overline{E}$ is compact and $\mu(\overline{E}) < +\infty$. Thus,

$$\mu(\overline{E} \setminus E) = \mu(\overline{E}) - \mu(E) < +\infty.$$

By (3), $\exists U$ which is open in $\mathbb{R}$ such that $\overline{E} \setminus E \subseteq U$ and $\mu(U) < \mu(\overline{E} \setminus E) + \epsilon$. Let

$$K = \overline{E} \setminus U \subseteq \overline{E} \subseteq [-r, r].$$

Then K is also closed and bounded, thus K is compact. Also, $K \subseteq \overline{E} \setminus \left(\overline{E} \setminus E\right) = E$ and

$$\mu(K) = \mu(\overline{E} \setminus U) = \mu(\overline{E} \setminus (\overline{E} \cap U)) = \mu(\overline{E}) - \mu(\overline{E} \cap U) \geq \mu(\overline{E}) - \mu(U) > \mu(\overline{E}) - (\mu(\overline{E} \setminus E) + \epsilon) = \mu(E) - \epsilon.$$

Hence,

$$\mu(K) \geq \mu(E) - \epsilon$$

By taking supremum on both sides,

$$\mu(E) - \epsilon < \sup\{\mu(K) : K \text{ is compact and } K \subseteq E\}.$$

Letting $\epsilon \to 0$, we get $\leq$ inequality of (4). □

Corollary 2.46. *Take general $E \in \mathcal{M}$. Then,*

$$\begin{aligned}\mu(E) &= \sup\{\mu(E \cap [-n, n]) : n \in \mathbb{N}\} \\ &= \sup_{n \in \mathbb{N}}\{\sup\{\mu_K : K \textit{ is compact and } K \subseteq E \cap [-n, n]\}\} \\ &= \sup\{\mu(K) : K \textit{ is compact and } K \subseteq E\}.\end{aligned}$$

Proof. When E is bounded, everything is clear. If E is not bounded, then first two equality is clear. And the last equality is also clear since K in the left side is contained in the rightside, and vice versa, from the Heine-Borel theorem that any compact set in $\mathbb{R}$ is bounded. □

Lemma 2.47. *If $E \in \mathcal{M}$, then $\forall \epsilon > 0, \exists V$ an open subset of $\mathbb{R}$ such that $E \subset V, \mu(V \setminus E) < \epsilon$.*

Proof. If $\mu(E) < +\infty$, then $\mu(V \setminus E) = \mu(V) - \mu(E)$ and we find V using theorem 1.18. If $\mu(E) = +\infty$, then we can write $E = \bigcup_{n \in \mathbb{Z}}(E \cap (n, n+1])$. For each $n \in \mathbb{Z}$, we find V_n is open in $\mathbb{R}$ such that $E \cap [n, n+1] \subset V_n$, and $\mu(V_n \setminus (E \cap (n, n+1])) < \frac{\epsilon}{2^{|n|}}$.

Let $V = \bigcup_{n \in \mathbb{Z}} V_n$ then V is open in $\mathbb{R}$, thus

$$\mu(V \setminus E) = \mu\left(\left(\bigcup_{n \in \mathbb{Z}}\right) \setminus E\right) = \mu\left(\bigcup_{n \in \mathbb{Z}}(V_n \setminus E)\right) \leq \sum_{n \in \mathbb{Z}} \mu(V_n \setminus E) \leq \sum_{n \in \mathbb{Z}} \mu(V_n \setminus (E \cap (n, n+1]) \leq \sum_{n \in \mathbb{Z}} \frac{\epsilon}{2^{|n|}} = 3\epsilon.$$

□

Theorem 2.48 (Theorem 1.19 in [1]p. 36)**.** *Let $E \subseteq \mathbb{R}$. Then, the following are equivalent.*

1. $E \in \mathcal{M}$.
2. $E = V \setminus N$ *for some* G_δ*set* V *and* $N \in \mathcal{M}$ *such that* $\mu(N) = 0$.
3. $E = H \setminus N$ *for some* F_σ*set* H *and* $N \in \mathcal{M}$ *such that* $\mu(N) = 0$.

Note that G_δ *set is a set of countable intersection of open sets and* F_σ *set is a set of countable union of closed sets.*

Proof. $(ii) \implies (i)$ and $(iii) \implies (i)$ are clear. To show $(i) \implies (ii)$, by the previous lemma, $\forall n \in \mathbb{N}, \exists V_n$, which is open in $\mathbb{R}$ such that $E \subseteq V_n$ and $\mu(V_n \setminus E) < \frac{1}{n}$. Replacing V_n by $\cap_{k=1}^{n} V_n$ if necessary, we may without loss of generality assume $V_1 \supseteq V_2 \supseteq \cdots \supseteq E$. Let $V = \bigcap_{n=1}^{\infty} V_n$. Then, $V \in G_\delta$ and $V \supseteq E$. Let $N = V \setminus E$. Then, $E = V \setminus N$ and

$$\mu(N) \leq \mu(V_n \setminus E) \leq \frac{1}{n}, \forall n \in \mathbb{N}.$$

Hence $\mu(N) = 0$.

To see $(i) \implies (iii)$, let $E \in \mathcal{M}$. By (ii), $E^c = V \setminus N$, where $V \in G_\delta, N \in \mathcal{M}$ such that $\mu(N) = 0$. Thus,

$$E = (V \cap N^c)^c = V^c \cup N$$

and $V^c \in F_\sigma$, since the complement of G_δ set is F_σ set. □

Corollary 2.49. *If* $L \in \mathcal{M}$ *such that* $\mu(L) = 0$. *Then,* $\exists$ *a* G_δ*-set* $V \in \mathcal{B}_\mathbb{R}$ *such that* $L \subseteq E$ *and* $\mu(V) = 0$. *Thus, the Lebesgue Stieltjes measure is the completion of* $\mathcal{B}_\mathbb{R}$. *(Every union of borel set with a null set is Lebesgue Stieltjes measurable and every null set in Lebesgue Stieltjes measurable set is contained in a set of null sets in* $\mathcal{B}_\mathbb{R}$.*) Thus* μ *is the completion of* $\mu|_{\mathcal{B}_\mathbb{R}}$.

Proof. It is just direct application of the theorem. □

From this fact we can use the notation

$$\mathcal{L} := \mathcal{M}_F \text{ where } F(x) = x, m := \mu_F.$$

Theorem 2.50 (Theorem 1.21 in [1] p.37)**.** *If* $E \in \mathcal{L}$, *and* $s, r \in \mathbb{R}$, *then* $s+E, rE \in \mathcal{L}$ *and* $m(s+E) = m(E)$, $m(rE) = |r|m(E)$.

Proof. Since $\mathcal{A}$ is invariant under translation and dilations, so does $\mathcal{B}_\mathbb{R}$. Since $m(E) = m(E+s)$ and $m(rE) = |r|m(E)$ for any finite unions of intervals E, which implies $m|_\mathcal{A}$ has a translation invariant and dilation invariant as a premeasure. Thus, by the theorem 1.14, these property holds for $\mathcal{B}_\mathbb{R}$. Now for any $E \in \mathcal{L}$ with $m(E) = 0$, $\exists F \in \mathcal{B}_\mathbb{R}$ such that $m(F) = 0$ with $E \subseteq F$, then

$$m(E+s) \leq m(F+s) = m(F) = 0 \text{ and } m(rE) = |r|m(E) \leq |r|m(F) = 0 \implies m(E+s) = 0 = m(rE).$$

Since every measurable set in $\mathcal{L}$ can be represented as union of $F_\sigma$set and a null set, and measures of these two sets are invariant under dilation and translation, we can conclude that every measure of measurable set is invariant under dilation and translation. □

2.6 Revisit Cardinal Numbers.

Definition 2.51 (Cardinal Numbers)**.** *Two sets* A *and* B *have the* **same size** *if* $\exists f : A \to B$ *that is one-to-one and onto* B. *There is a class of sets, i.e., a collection that is too big to be a set, called* **cardinal numbers**. *The* **Cardinality of a set** A *is the unique cardinal number that has the same size as* A.

Remark 2.52. *For example,* $0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}, \cdots, \mathbb{N}_0 = \aleph_0$, *called aleph zero.*

Definition 2.53. *If κ and λ are cardinal numbers, we write $\kappa \leq \lambda$ if for sets A and B having cardinalities κ and λ respectively, $\exists$ a one-to-one function $f : A \to B$.*

Theorem 2.54 (Schroder-Bernstein Theorem)**.** *If $\kappa \leq \lambda$ and $\lambda \leq \kappa$, then $\kappa = \lambda$*

Proof. Already proved above. $\square$

Proposition 2.55 (Proposition 0.7 in [1] p.7)**.** *For any X, Y sets, either $card(X) \leq card(Y)$ or $card(Y) \leq card(X)$.*

Proof. Already proved above. $\square$

Definition 2.56 (Cardinal Arithmetic)**.** *Suppose $card(A) = \kappa, card(B) = \lambda$. Then,*

$$\kappa + \lambda := card(A \uplus B), \kappa \cdot \lambda = card(A \times B), \kappa^\lambda = card(A^B)$$

where $\uplus$ implies the disjoint union of two sets, and $A^B := \{f : B \to A\}$.

Thus, $card(2^A) = card(\mathcal{P}(A)) = 2^\kappa$.

Theorem 2.57. *If κ, λ are infinite cardinals, then*

$$\kappa + \lambda = \max(\kappa, \lambda) = \kappa \cdot \lambda$$

Also, for any cardinals κ, λ, μ,

$$(\kappa^\lambda)^\mu = (\kappa^{\lambda \cdot \mu}).$$

Proof. Incomplete. $\square$

Theorem 2.58 (Cantor)**.** *If κ is any cardinal then $\kappa \leq 2^\kappa$ but $\kappa \neq 2^\kappa$.*

Proof. Let A be a set with $card(A) = \kappa$. Then there exists a one-to-one function

$$f : A \to 2^A \text{ by } a \mapsto f_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\kappa \leq 2^\kappa$. Suppose for $\kappa = 2^\kappa$. Then, there exists a bijection $g : A \to 2^A = \{0, 1\}^A$. Let

$$F : A \times A \to \{0, 1\} \text{ with } F(a_1, a_2) = (g(a_1))(a_2).$$

Let $\phi : A \to \{0, 1\}$ be defined by

$$\phi(a) \neq F(a, a) \text{ for all } a \in A.$$

Then, $\phi \in \{0, 1\}^A = g(A)$, thus $\exists a \in A$ such that $g(a) = \phi$. Then, $\phi(a) = g(a)(a) = F(a, a)$, contradiction. $\square$

Theorem 2.59. *$card(\mathbb{R}) = 2^{\aleph_0} = card(\mathcal{P}(\mathbb{N}))$, and this give $[0, 1]$ to its decimal expansions.*

Proof. It is already proved in theorem 1.18. $\square$

Theorem 2.60 (Theorem 1.21 in [1] p. 37, revisited)**.** *If $E \in \mathcal{L} = \mathcal{M}_F$, $F(x) = x$, $m = \mu_F$ then $\forall r, s \in \mathbb{R}$, $s + E \in \mathcal{L}$ and $rE \in \mathcal{L}$, and $m(s + E) = m(E)$ and $m(rE) = |r|m(E)$.*

Proof. Let $\mathcal{B}' = \{E \in \mathcal{B}_{\mathbb{R}} : s + E \in \mathcal{B}_{\mathbb{R}}\}$. Then $\mathcal{B}'$ is a sub σ-algebra of $\mathcal{B}_{\mathbb{R}}$ with $(a,b) \in \mathcal{B}'$ for all $a, b \in \mathbb{R}$, with $a < b$. Thus, $\mathcal{B}' = \mathcal{B}_{\mathbb{R}}$. Let $\mathcal{B}'' = \{E \in \mathcal{B}_{\mathbb{R}} : m(s+E) = m(E)\}$. Then $\mathcal{B}''$ is a sub σ-algebra of $\mathcal{B}_{\mathbb{R}}$ since

$$\mathcal{B}'' \supset \{(a,b) : a, b \in \mathbb{R}, a < b\},$$

hence for all $E \in \mathcal{B}_{\mathbb{R}}$, $m(E) = \inf\{m(U) : U \text{ open in } \mathbb{R}, E \subseteq U\}$ implies

$$m(s+E) = \inf\{m(s+U) : E \subseteq U, U \text{ is open in } \mathbb{R}\} = \inf\{m(U) : E \subseteq U, U \text{ is open in } \mathbb{R}\} = m(E).$$

Now let $E \in \mathcal{L}$. Then, $\exists F \in \mathcal{B}_{\mathbb{R}}$, $N' \in \mathcal{B}_{\mathbb{R}}$ such that $m(B') = 0$ and $E = F \cup N$ for some $N \subseteq N'$. Therefore,

$$m(s+E) = m(s+F \cup s+N) = m(s+F) + m(s+N) \le m(F) + m(N') = m(F) = m(E).$$

So $s + E \in \mathcal{L}$ and $m(s+E) = m(E)$. By the similar argument, $m(rE) = |r|m(E)$. □

Remark 2.61. *A totally ordered set $(\Omega, \le)$ is* **well-ordered** *if every nonempty subset of Ω has a minimal element. Let $\tilde{\Omega}$ be an well-ordered set. For $x \in \tilde{\Omega}$, let*

$$P_x := \{y \in \tilde{\Omega} : y \le x \text{ and } y \ne x\},$$

Let

$$U = \{x \in \tilde{\Omega} : P_x \text{ is uncountable}\}.$$

If $U = \emptyset$, let $\Omega = \tilde{\Omega}$. Otherwise, let $z = \min(U)$, which exists by the well-ordering property of $\tilde{\Omega}$. Then, let

$$\Omega := \{x \in \tilde{\Omega} : x < z\} = P_z.$$

Then, Ω is uncountable since $z \in U$ and $\Omega = P_z$. Moreover, $\forall x \in \Omega$, P_x is countable since z is the least element in U. Also, by restricting $\le$ to Ω gives an well-ordering on Ω.

Proposition 2.62. *Every countable subset of Ω has an upper bound.*

Proof. Suppose $E \subseteq \Omega$ is countable. Suppose for contradiction, E has no upper bound. Then, $\forall y \in \Omega, \exists e \in E$ such that $e \not\le y$, i.e., $y < e$, from the total ordering. Thus, $y \in P_e$, $\Omega = \bigcup_{e \in E} P_e$. However, P_e is countable, and countable union of countable sets is also countable. Thus, Ω is countable, contradiction. □

Proposition 2.63. *If Ω' is any uncountable set endowed with a well-ordering $\le'$ having the property that $\forall x \in \Omega'$, $\{y \in \Omega' : y \le' x, x \ne y\}$ is countable, then $(\Omega', \le')$ is order isomorphic to $(\Omega, \le)$, i.e., there exsits a bijection function preserving order.*

Proof. It is standard Zorn's lemma argument. Let $A = \{P_y : \exists \text{ sub-order isomophism } P_y \to \Omega\}$. Then, every chain of A has maximal element by the total-ordering. Hence, A has the maximal element, by the Zorn's lemma. Since for all $y \in \Omega'$, P_y has sub-order isomorphism with Ω using bijection of $\mathbb{N}$. Thus, its maximal element must contain every P_y, which is Ω' itself. □

Definition 2.64 (Hereditary)**.** *Given $H \subseteq \Omega$, we say H is hereditary if $x \in H, y \in \Omega$, $y \le x \implies y \in H$. Let*

$$\text{Ⓗ} := \{\phi : H \to H' : H \text{ is a hereditary subset of } \Omega, H' \text{ is a hereditary subset of } \Omega', \phi \text{ is an order-isomorphism}\}.$$

We order Ⓗ by defining $\phi_1 \le \phi_2$ if ϕ_2 extends ϕ_1. This is a partial ordering, so Zorn's lemma says that if Ⓗ is not empty and every totally ordered subset of Ⓗ has an upper bound, then Ⓗ has a maximal element.

Proof. Note that Ⓗ is not empty since the map $\{\min(\Omega)\} \mapsto \{\min(\Omega')\} \in$ Ⓗ. Also, if $\{\phi_\lambda : \lambda \in \Lambda\}$ is a totally ordered subset of Ⓗ, we can construct an upper bound of the subset as

$$\phi : \bigcup_{\lambda \in \Lambda} dom(\phi_\lambda) \to \bigcup_{\lambda \in \Lambda} range(\phi_\lambda) \text{ by } \phi(x) = \phi_\lambda(x) \text{ since for some } \lambda, x \in dom(\phi_\lambda).$$

Then, ϕ is order-isomorphism and $\bigcup_{\lambda \in \Lambda} dom(\phi_\lambda), \bigcup_{\lambda \in \Lambda} range(\phi_\lambda)$ are hereditary in Ω, Ω' respectively. Thus, ϕ is in Ⓗ and it is an upper bound of $\{\phi_\lambda : \lambda \in \Lambda\}$.

Hence, by the Zorn's Lemma, there exists a maximal element of Ⓗ, say ψ. □

Proposition 2.65. *ψ is a surjective order-isomorphism, i.e., $dom(\psi) = \Omega, range(\psi) = \Omega'$.*

Proof. Suppose for contradiction, let $dom(\psi) \neq \Omega$. Then, $dom(\psi) \leq P_x$ for some x since $dom(\psi)$ is hereditary. Without loss of generality, such x is minimal; i.e., there is no y such that $y \leq x$, $y \neq x$ but $dom(\psi) \subseteq P_y$. This is possible since Ω is totally ordered set. Then, P_x is countable by the construction of Ω. So $range(\psi)$ is countable, which implies $range(\psi) \subsetneq \Omega'$.

Let $y = \min(\Omega' \setminus range(\psi))$. Then, $dom(\psi) \cup \{x\}$ is also hereditary by choice of x, as is $range(\psi) \cup \{y\}$. Let $\tilde{\psi} : dom(\psi) \cup \{x\} \to range(\psi) \cup \{y\}$ be

$$\tilde{\psi}(a) = \begin{cases} \psi(a) & \text{if } a \neq x \\ y & \text{if } a = x. \end{cases}$$

Then, $\tilde{\psi} \in \textcircled{H}$, $\psi \leq \tilde{\psi}$, $\psi \neq \tilde{\psi}$, thus ψ is not a maximal element, contradiction. □

Definition 2.66 (Ordinal)**.** *Let $(\Omega, \leq)$ constructed in the above is the first uncountable ordinal. Then,* $0 := \min(\Omega), 1 := \min(\Omega \setminus \{0\}), 2 = \min(\Omega \setminus \{0, 1\}), \cdots, w = \min(\Omega \setminus \mathbb{N}_0), w + 1 = \min(\Omega \setminus (\mathbb{N}_0 \cup \{w\})), \cdots$. *Thus, countable ordinals are*

$$0, 1, 2, \cdots, w, w + 1, w + 2, \cdots, 2w, 2w + 1, \cdots, 3w, 3w + 1, \cdots, w^2, w^2 + 1, \cdots,$$

The first uncountable cardinal is $card(\Omega)$, denoted $card(\Omega) = \aleph_1$.

Remark 2.67 (Continuum Hypothesis)**.** $\alpha_1 = c = 2^{\aleph_0}$.

Example 2.68 (Transfinite recursion)**.** *Let's construct σ-alg(ξ) for some $\xi \subseteq \mathcal{P}(X)$. We will construct $\mathcal{R}_a$ for every $a \in \Omega$ to show that $\bigcup_{a \in \Omega} \mathcal{R}_a = \sigma\text{-}alg\,(\xi)$. Let $R_0 = \xi \cup \{\emptyset\}$. If $a \in \Omega$ has an immediate predecessor b, i.e., if a is a successor ordinal, i.e., if $b = \max(\{x \in \Omega : x < a\}$ exists, then we let $\mathcal{R}_a$ be a set of all countable union and complements of such union of elements from $\mathcal{R}_b$. If a is not a successor ordinal, then we let $\mathcal{R}_a = \bigcup_{x \in \Omega : x < a} \mathcal{R}_x$.*

Claim 2.69. $\bigcup_{a \in \Omega} R_a = \sigma\text{-}alg\,(\xi)$.

Proof. $\leq$ is clear from the construction. More correctly, it follows by the transfinite induction on W that $\mathcal{R}_a \subseteq \sigma\text{-alg}\,(\xi), \forall a \in \Omega$. Indeed, $\mathcal{R}_0 \subseteq \sigma\text{-alg}\,(\xi)$. If $a \in \Omega$ and if $\mathcal{R}_x \in \sigma\text{-alg}\,(\xi)$ for all $x < a$, then we must shown that $\mathcal{R}_a \subseteq \sigma\text{-alg}\,(\xi)$. If $a = b + 1$, then $\mathcal{R}_a \subseteq \sigma\text{-alg}\,(\mathcal{R}_b) \subseteq \sigma\text{-alg}\,(\xi)$ by the inductive hypothesis. If a is not a successor ordinal, then $\mathcal{R}_a = \bigcup_{x < a} \mathcal{R}_x \subseteq \sigma\text{-alg}\,(\xi)$, by the inductive hypothesis that for each $x < a$, $\mathcal{R}_x \subseteq \sigma\text{-alg}\,(\xi)$.

For $\geq$, we must show that $\mathcal{R} := \bigcup_{a \in \Omega} \mathcal{R}_a$ is σ-algebra. If $E \in \mathcal{R}_a$ for some $a \in \Omega$, then $E^c \in \mathcal{R}_a$ by the construction of $\mathcal{R}_a$. Suppose $E_1, E_2, \cdots \in \mathcal{R}$. Let $a(j) \in \Omega$ such that $E_j \in \mathcal{R}_{a(j)}$ for all $j \in \mathbb{N}$. By the proposition, $\{a(j)\}_{j \in \mathbb{N}}$ has an upper bound $b \in \Omega$. Thus, $\forall k \in \mathbb{N}, E_k \subseteq \bigcup_{j=1}^{\infty} \mathcal{R}_{a(j)} \subseteq \mathcal{R}_b$. And by construction, $\bigcup_{k=1}^{\infty} E_k \in \mathcal{R}_{b+1} \implies \bigcup_{k=1}^{\infty} E_k \in \mathcal{R}$.

Thus, $\mathcal{R}$ is σ-algebra. □

Proposition 2.70. *Suppose $\xi \subseteq \mathcal{P}(X)$ is countably infinite. Then, $card(\sigma\text{-}alg\,(\xi)) = c$.*

Proof. We have $\sigma\text{-alg}\,(()\,\xi) = \bigcup_{\alpha \in \Omega} \mathcal{R}_\alpha$, where Ω is the first uncountable ordinal. Then, by our construction, $card(R_0) = card(\xi \cup \{\emptyset\}) = \aleph_0$.

Claim 2.71. $\forall \alpha > 0$, *$card(\mathcal{R}_\alpha) = c$.*

Proof. By the transfinite induction, we assume that the assertion holds for all $\alpha \in \Omega, \alpha < \beta \in \Omega$. And show it holds for β. Note that

$$card(\mathcal{R}_1) \leq card(\mathcal{P}(\xi)) = 2^{\aleph_0} = c.$$

1. ***Case 1:*** $\beta \in \Omega$ ***is*** $\beta = \beta_0 + 1$***.*** Then, $\mathcal{R}_\beta$ consists of all countable union of sets from $\mathcal{R}_{\beta_0}$, and their complement B, so

$$c = 2^{\alpha_0} \leq card(R_\beta) \leq card(\mathcal{R}_{\beta_0}^{\mathbb{N}}) = c^{\alpha_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c.$$

2. ***Case 2:*** $\beta \in \Omega$ ***is not a successor ordinal.*** Then,

$$\mathcal{R}_\beta = \bigcup_{\substack{\alpha < \beta \\ \alpha \in \Omega}} \mathcal{R}_\alpha \text{ and } card(\mathcal{R}_\beta) \leq \aleph_0 \cdot c = c$$

   Note that $\{\alpha \in \Omega : \alpha < \beta\}$ is countable, by construction of Ω.

□

By the claim,

$$card(\sigma\text{-alg}\,(\xi)) = card\left(\bigcup_{a \in \Omega} R_a\right) \leq card\Omega \cdot c = \aleph_1 \cdot c \leq c\dot{c} = c.$$

□

Corollary 2.72. $card(\mathcal{B}_{\mathbb{R}}) = c$

Proof. $\{(a,b) : a, b \in \mathbb{Q}\}$ generates $\mathcal{B}_{\mathbb{R}}$ as a σ-algebra. □

Definition 2.73 (The Cantor Set)**.** *Let* $C_0 = [0,1]$*.* $C_1 = C_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$*, and* $C_2 = C_1 \setminus \left((\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})\right)$*, and so on. Now let* $C = \bigcap_{n=1}^\infty C_n$*. Then* C *is totally disconnected, i.e., no two points are connected by the line segment. However, it is still uncountable.*

Elements of $[0,1]$ have base 3 expansions $x = \sum_{j=1}^\infty \frac{a_j}{3^j}$ for $a_j \in \{0,1,2\}$, and these expansions are unique except when $x = \frac{n}{3^p}, n \in \mathbb{N}_0$ when x will has two such expansions

$$a_j = 0 \forall j > p \text{ or } a_j = 2 \forall j < p.$$

And we choose the expansion tha avoids having any $a_j = 1$ if possible. Actually,

$$C = \{x \in [0,1] : x = \sum_{j=1}^\infty \frac{a_j}{3^j} \text{ for some } a_j \in \{0,2\}\}.$$

Then we have a bijection $\theta : C \to \{0,2\}^{\mathbb{N}}$. Thus, $card(C) = c$

Lemma 2.74. $m(C) = 0$.

Proof.

$$m(C) = \lim_{n\to\infty} m(C_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0,$$

by the continuity from above. □

Proposition 2.75. $card(\mathcal{L}) = 2^c$.

Proof. Note that $\mathcal{P}(C) \subseteq \mathcal{L} \subseteq \mathcal{P}(\mathbb{R})$ and $card(\mathcal{P}(C)) = 2^c = card(\mathcal{P}(\mathbb{R}))$ gives the desired result. □

3 Integration

3.1 Measurable function

Definition 3.1 (Measurable function). *Let (X, m) and (Y, n) are measurable spaces and let $f : X \to Y$. f is* **measurable** *or* (m, n)**-measurable** *if $\forall E \in m$, $f^{-1}(E) \in m$.*

Observation 3.2. *If also $(Z, \mathfrak{a})$ is a measurable space and if $g : Y \to Z$ is $(n, \mathfrak{a})$-measurable, then $g \circ f : X \to Z$ is (m, α)-measurable.*

This is because every inverse image of α-measurable set is n-measurable, and its inverse image of f is also m-measurable.

Proposition 3.3 (Proposition 2.1 in [1] p.43). *If $n = \sigma\text{-}alg\,(\xi)$, then $f : X \to Y$ is (m, n)-measurable iff $\forall E \in \xi$, $f^{-1}(E) \in m$.*

Proof. If f is (m, n)-measurable, then trivially $f^{-1}(E) \in m$ for all $E \in \xi \subseteq \sigma\text{-alg}\,(\xi)$.

If $\forall E \in \xi$, $f^{-1}(E) \in m$, let $A = \{E \subseteq Y : f^{-1}(E) \in m\}$. Then, $\xi \subseteq A$. It suffices to show that $\sigma\text{-alg}\,(\xi) \subseteq A$. Note that A is closed under complement since $f^{-1}(E^c) = X - f^{-1}(E) = f^{-1}(E)^c$ for any $E \in A$. Also, $\emptyset \in A$ since $f^{-1}(\emptyset) = \emptyset \in m$. Finally, A is closed under countable union since

$$f^{-1}(\bigcup_{j=1}^{\infty} E_j) = \bigcup_{j=1}^{\infty} f^{-1}(E_j) \in m.$$

Hence, A is a σ-algebra, containing ξ. Thus, $\sigma\text{-alg}\,(\xi) \subseteq A$, hence f is (m, n)-measurable. ☐

Corollary 3.4 (Corollary 2.2 in [1] p.44). *If X and Y are topological spaces and if $f : X \to Y$ is continuous function, then f is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.*

Proof. Note that

$$\mathcal{B}_Y = \sigma\text{-alg}\,(\{V \subset Y : V \text{ is open}\})\,, \mathcal{B}_X = \sigma\text{-alg}\,(\{U \subset X : U \text{ is open}\})\,,$$

and V is open in Y implies $f^{-1}(V)$ is open. Thus, by the above proposition, f is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable. ☐

Remark 3.5 (Convention). *If (X, m) is a measurable space and if $f : X \to \mathbb{R}$ (or $f : X \to \mathbb{C}$), then we will say f is* **measurable** *or* m**-measurable** *if it is $(m, \mathcal{B}_{\mathbb{R}})$-measurable (or $(m, \mathcal{B}_Y)$-measurable).*

Observation 3.6. *If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both $\mathcal{L}$-measurable, i.e., Lebesgue measurable, then by the convention, f is $(\mathcal{L}, \mathcal{B}_{\mathbb{R}})$-measurable, thus $g \circ f$ need not be $\mathcal{L}$-measurable.*

Proposition 3.7 (Proposition 2.3 in [1] p. 44). *Given (X, m) and $f : X \to \mathbb{R}$, the followings are equivalent.*

(i) f *is* m*-measurable*

(ii) $\forall a \in \mathbb{R}$, $f^{-1}((-\infty, a)) \in m$.

(iii) $\forall a \in \mathbb{R}$, $f^{-1}((-\infty, a]) \in m$.

(iv) $\forall b \in \mathbb{R}$, $f^{-1}((b, \infty)) \in m$.

(v) $\forall b \in \mathbb{R}$, $f^{-1}([b, \infty)) \in m$.

Proof. Note that $(i) \implies (ii), (iii), (iv), (v)$, since the given sets are all Borel set. Conversely, from the proposition 1.2, we know that the given sets can generates the Borel σ-algebra, therefore from the proposition 2.1, we can concluded that f is m-measurable. ☐

Definition 3.8 (σ-algebra generated by $\{f_\alpha\}_{\alpha\in A}$)**.** *Let X be a set. Fix a set A and $\forall\alpha \in A$, fix (Y_α, m_α) a measurable space. Suppose $f_\alpha : X \to Y_\alpha$ is a function. Then,* **σ-algebra generated by** *$\{f_\alpha\}_{\alpha\in A}$ is σ-alg$\left(\{f_\alpha^{-1}(E) : \alpha \in A, E \in m_\alpha\}\right)$. This is the smallest σ-algebra of subsets of X making all the functions f_α measurable.*

For example, let $Y = \prod_{\alpha\in A} Y_\alpha$ and let $\pi_\alpha : Y \to Y_\alpha$. Denote the α-th coordinate projection, then the product σ-algebra is $\otimes_{\alpha\in A} n_\alpha$, which is the σ-algebra generated by $(\pi_\alpha)_{\alpha\in A}$. (See definition 2.14 of this note.)

Lemma 3.9. *Let A be a set and $\forall\alpha \in A$, let (Y_α, n_α) be a measurable space. Let Z be a set, and let $f_\alpha : Z \to Y_\alpha$ be a function for each $\alpha \in A$. Let $n \subseteq \mathcal{P}(Z)$ be the σ-algebra generated by $(f_\alpha)_{\alpha\in A}$. Let (X, m) be a measurable space. Let $g : X \to Z$ be a function.*

Then, g is (m, n)-measurable iff $\forall\alpha \in A$, $f_\alpha \circ g$ is (m, n_α)-measurable.

Proof. To show if part, note that n is generated by $\{f_\alpha^{-1}(E) : \alpha \in A, E \in n_\alpha\}$. Thus, for g to be (m, n)-measurable, it suffices to show that $\forall\alpha \in A, \forall E \in n_\alpha$, $g^{-1}(f_\alpha^{-1}(E)) \in m$. And it is true, since for any $E \in n_\alpha$, $f_\alpha^{-1}(E) \in n$ by definition, and $g^{-1}(F) \in m$ for any $F \in n$. Let $F = f_\alpha^{-1}(E)$ and we're done.

To see only if part, suppose g is (m, n)-measurable. Then, by definition of n, f_α is (n, n_α) measurable for each $\alpha \in A$. Thus, $f_\alpha \circ g$ is (m, n_α)-measurable; done. □

Proposition 3.10 (Proposition 2.4 in [1] p.44)**.** *Let (X, m) and $\forall\alpha \in A$, (Y_α, n_α) be measurable spaces. Let $Y = \prod_{\alpha\in A} Y_\alpha$ and $n = \otimes_{\alpha\in A} n_\alpha$. Let $g : X \to Y$. Then, g is (m, n) measurable if and only if $\forall\alpha \in A, \pi_\alpha \circ g : X \to Y_\alpha$ is (m, n_α)-measurable.*

Proof. Just special case of the above lemma. □

Remark 3.11 (Recall a product of metric space)**.** *A* **metric space** *is (X, d) where d is a metric on X. And $U \subseteq X$ is called* **open** *if $\forall x \in U$, $\exists\epsilon > 0$ such that $B_\epsilon(X) := \{y \in X : d(x, y) < \epsilon\} \subseteq U$.*

If $(X_1, d_1), \cdots, (X_n, d_n)$ are metric spaces, then we equip $X = \prod_{j=1}^n X_j$ with the product metric e.g.,

$$d^{(\infty)}((x_j)_{j=1}^n, (y_j)_{j=1}^n) = \max_{1\le j\le n} d_j(x_j, y_j),$$

which is equivalent to any of

$$d^{(p)}((x_j)_{j=1}^n, (y_j)_{j=1}^n) = \left(\sum_{j=1}^n d_j(x_j, y_j)^p\right)^{\frac{1}{p}}.$$

Definition 3.12 (Separability)**.** *A metric space (X, d) is separable if there exists a countable subset $D \subseteq X$ that is dense in X.*

Proposition 3.13 (Proposition 1.5 in [1] p.23)**.** *Let (X_j, d_j), $1 \le j \le n$ be metric space and let $X = \prod_{j=1}^n X_j$ be equipped with $d^{(\infty)}$. Then $\otimes_{j=1}^n \mathcal{B}_{X_j} \subseteq \mathcal{B}_X$. If $(X_1, d_1), \cdots (X_n, d_n)$ are all separable, then we get $\otimes_{j=1}^n \mathcal{B}_{X_j} = \mathcal{B}_X$.*

Proof. By definition of product σ-algebra, $\otimes_{j=1}^n \mathcal{B}_{X_j} = \sigma$-alg$\left(\{\pi_j^{-1}(U) : 1 \le j \le n, U \text{ is open in } X_j\}\right)$. since each π_j is continuous from X to X_j, each $\pi_j^{-1}(U)$ is open, so $\pi^{-1}(U) \in \mathcal{B}_X$. Suppose $\forall j \in [n], X_j$ has a countable dense set D_j. Then, $D = \prod_{j=1}^n D_j$ is dense in X, and D is countable. Hence X has countable dense subset D. And we note the standard result;

Remark 3.14 (Standard result)**.** *For D, which is countable dense in X, every open set in X is the union of a subfamily of $\{B_{\frac{1}{l}}(x) : x \in D, l \in \mathbb{N}\}$, where $B_{\frac{1}{l}} := \{y \in X : d(x, y) < \frac{1}{l}\}$.*

Proof. See the topology stuff and prove it. □

Now we want to prove $\otimes_{j=1}^n \mathcal{B}_{X_j} \supseteq \mathcal{B}_X$. Let $U \subseteq\in X$ be open. It suffices to show that $U \in \otimes_{j=1}^n \mathcal{B}_{X_j}$. Since U is countable union of sets of $B_{\frac{1}{l}}(x), x \in D$, it suffices to show that each $B_{\frac{1}{l}}(x) \in \otimes_{j=1}^n \mathcal{B}_{X_j}$. We choose $d = d^{(\infty)}$, defined above. Then, $B_{\frac{1}{l}}(x) = \prod_{j=1}^n B_{\frac{1}{l}}(x_j) \in \otimes_{j=1}^n \mathcal{B}_{X_j}$. ☐

Corollary 3.15 (Corollary 1.6 in [1] p.23 and more.)**.** $\mathcal{B}_{\mathbb{C}} \cong \mathcal{B}_{\mathbb{R}^2} \cong \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$. $\mathcal{B}_{\mathbb{R}^n} = \otimes_{j=1}^n \mathcal{B}_{\mathbb{R}}$.

Corollary 3.16 (Corollary of proposition 2.4 in [1])**.** *If (X, m) is a measurable space and $f : X \to \mathbb{C}$, then f is $(m, \mathcal{B}_{\mathbb{C}})$-measurable iff Ref, Imf are $(m, \mathcal{B}_{\mathbb{R}})$-measurable.*

Proof. From the above corollary, $\mathcal{B}_{\mathbb{C}} \cong \mathcal{B}_{\mathbb{R}^2} \cong \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$. ☐

Proposition 3.17 (Proposition 2.6 in [1] p.45)**.** *If $f, g : X \to \mathbb{C}$ are m-measurable, then $f + g, fg$ are also m-measurable.*

Proof. Let $H : X \to \mathbb{C} \times \mathbb{C}$ be $H(x) = (f(x), g(x))$. Then, H is (by proposition 2.4 of the [1]) $(m, \mathcal{B}_{\mathbb{C}} \otimes \mathcal{B}_{\mathbb{C}})$-measurable. Let $\phi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be $\phi(z, w) = z + w$ and $\varphi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be $\varphi(z, w) = zw$. Then, ϕ, φ are continuous, thus, they are $(\mathcal{B}_{\mathbb{C}^2}, \mathcal{B}_{\mathbb{C}})$-measurable. Thus, $\phi \circ H = f + g$ and $\varphi \circ H = fg$ are $(m, \mathcal{B}_{\mathbb{C}})$-measurable. ☐

Remark 3.18. *Consider $\bar{\mathbb{R}} = [-\infty, \infty]$. We topologize it by making an order preserving bijection $[-\infty, \infty] \to [-1, 1]$ into a homeomorphism. If (X, m) is a measurable space and $f : X \to [-\infty, +\infty]$ we say f is m-mesaurable if it is $(m, \mathcal{B}_{\bar{\mathbb{R}}})$-measurable.*

Lemma 3.19. *Let $f : X \to [-\infty, \infty]$ is m-measurable iff $\forall a \in \mathbb{R}, f^{-1}((a, +\infty]) \in m$.*

Proof. We proved it in the proposition 2.3 in [1]. ☐

Proposition 3.20 (Proposition 2.7 in [1] p. 45)**.** *Suppose $f_j : X \to [-\infty, \infty], j \in \mathbb{N}$ is m-measurable. Then,*

$$g_1 = \sup_{j \in \mathbb{N}} f_j, g_2 = \inf_{j \in \mathbb{N}} f_j, g_3 = \lim \sup_{j \to \infty} f_j g_4 = \lim \sup_{j \to \infty} f_j$$

are measurable.

Proof.

$$g_1^{-1}((a, +\infty]) = \bigcup_{j=1}^{\infty} f_j^{-1}((a, +\infty]) \in m$$

since

$$\sup_{j \in \mathbb{N}} f_j(x) > a \iff \exists j \in \mathbb{N} \text{ such that } f_j(x) > a.$$

So g_1 is m-measurable. Since $g_2 = -\sup_{j \in \mathbb{N}}(-f_j(x))$, so g_2 is also m-measurable. And note that

$$g_3 = \inf_{n \in \mathbb{N}} \sup_{j \geq n} f_j(x) = \inf_{n \in \mathbb{N}} h_n(x),$$

where $h_n(x) = \sup_{j \geq n} f_j$. Note that $h_n(x)$ is also m-measurable, since it is also supremum of countably many functions up to renumbering. Hence, g_3 is m-measurable. Similarly,

$$g_4 = \sup_{n \in \mathbb{N}} \inf_{k \geq n} f_j(x)$$

is also m-measurable. ☐

Corollary 3.21 (Corollary 2.8 in [1] p.46)**.** *If $f, g : X \to [-\infty, \infty]$ are m-measurable, then* $\max(f, g)(x) := \max(f(x), g(x))$ *and* $\min(f, g)(x) := \min(f(x), g(x))$ *are m-measurable.*

Proof. Let $f_1 = f, f_n = g$ for any $n \geq 2$, then $\max(f, g)(x) = \sup_{n \in \mathbb{N}} f_n, \min(f, g)(x) = \inf_{n \in \mathbb{N}} f_n$, thus they are m-measurable. ☐

Corollary 3.22 (Corollary 2.9 in [1] p.46). *If $f_j : X \to [-\infty, \infty]$ is m-measurable, and if $f(x) := \lim_{j\to\infty} f_j(x)$ exists for all x, then $f(x)$ is m-measurable. If we replace $[-\infty, \infty]$ to $\mathbb{C}$, the statement still holds.*

Proof. If $f(x) := \lim_{j\to\infty} f_j(x)$ exists for all x, then $f(x) = \limsup_{j\to\infty} f_j(x) = \liminf_{j\to\infty} f_j(x)$, thus it is m-measurable by the proposition 2.7. Also, in case of $\mathbb{C}$, we know that $f(x) := \lim_{j\to\infty} f_j(x)$ exists iff $Ref(x) := \lim_{j\to\infty} Ref_j(x)$ exists and $Imf(x) := \lim_{j\to\infty} Imf_j(x)$ exists, from the corollary 2.5 in [1]. Thus, we know Ref and Imf are m-measurable by the statement for the case $[-\infty, \infty]$, which we just proved. Thus, $f = Ref + iImf$ is also m-measurable, from proposition 2.6. ☐

Definition 3.23 (Characteristic function and Simple function). *Let X be a set and $E \subseteq X$, then the* **characteristic function** *of E is* $1_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$ *A simple function on X is a function of the form $f = \sum_{j=1}^{n} a_j 1_{E_j}$ for some $n \in \mathbb{N}$ and some $a_j \in \mathbb{C}$ and $E_j \subseteq X$.*

Observation 3.24. *Let $f : X \to \mathbb{C}$ is a simple function iff $im(f)$ is a finite set. Suppose $\{z_1, \cdots, z_n\} = im(f)$. Then,*

$$f = \sum_{j=1}^{n} z_j 1_{E_j}$$

where $E_j = f^{-1}(\{z_j\})$. We call the above representation as the **standard form** *of the simple function f.*

Definition 3.25 (Positive and negative part, polar decomposition). *If $f : X \to \bar{\mathbb{R}}$, we define the* **positive** *and* **negative parts** *of f to be*

$$f^+(x) = \max(f(x), 0), \qquad f^-(x) = \max(-f(x), 0).$$

Then $f = f^+ - f^-$, and $|f| = f^+ + f^-$. Thus, if f is measurable, then f^+, f^- are measurable by the corollary 2.8, thus $|f|$ is measurable by the proposition 2.6. Conversely, if f^+ and f^- are measurable, then by proposition 2.6, f is measurable. Also, if $f : X \to \mathbb{C}$, we have a **polar decomposition***:*

$$f = (sgnf)|f|, \text{ where } sgn(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Again, if f is measurable, then so are $|f|$ by the proposition 2.6. And note that if $U \subseteq \mathbb{C}$ is open, then $sgn^{-1}(U)$ is open or a form $V \cup \{0\}$, where V is also open, thus $V \cup \{0\}$ is a Borel measurable set. Thus, $sgn(z)$ is Borel measurable by proposition 2.1. Thus, $sgn(f)$ is measurable.

Remark 3.26. *If f, g are simple, then so are $f + g$ and fg, since their image is also finitely many, therefore we can represent it as the standard form.*

Observation 3.27. *If (X, m) is measurable space and if $f = \sum_{j=1}^{n} z_j 1_{E_j}$ is a simple function in standard form, then f is m-measurable iff $\forall j, E_j \in m$.*

Proof. Note every inverse image is the union of some E_j's, therefore, f is measurable iff E_j's are in m. ☐

Theorem 3.28 (Theorem 2.10 in [1] p.47). *Let (X, m) be a measurable space.*

(a) Let $f : X \to [0, \infty]$ be m-measurable. Then, $\exists$ a sequence $(\phi_n)_{n=1}^{\infty}$ of m-measurable simple functions such that $\phi_1 \leq \phi_2 \leq \cdots \leq f$ and $\phi_n \to f$ pointwise with uniform convergence on subsets of X where f is bounded.

(b) If $f : X \to \bar{\mathbb{R}}$ or $f : X \to \mathbb{C}$ is m-measurable, then $\exists$ a sequence $(\phi_n)_{n=1}^{\infty}$ of m-measurable simple functions such that $|\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$, $\phi_n \to f$ pointwise with uniform convergence on subsets of X where f is bounded.

Proof of (a). Let $E_k^{(n)} = f^{-1}([\frac{k-1}{2^n}, \frac{k}{2^n}])$ for all $k \in [2^n]$. Let $F^{(n)} = f^{-1}([2^n, +\infty])$. Let

$$\phi_n = \left(\sum_{k=1}^{2^{2n}} \frac{k-1}{2^n} 1_{E_k^{(n)}} \right) + 2^n 1_{F^{(n)}}.$$

See below picture for understanding the ϕ_n.

Thus, $\phi_n \leq \phi_{n+1}$ since the latter is cut more finer than the former one, and the value of function is always greater then or equal to that of the former. Also, $\phi_n \leq f$ since it is defined to have a minimum value of each inverse image of intervals, which f has a value.

If $Y \subseteq X$ and $f(Y) \subseteq [0, 2^p]$, then $\forall x \in Y, \forall n \geq p$,

$$|f(x) - \phi_n(x)| \leq \frac{1}{2^n}.$$

Thus, f_n uniformly converge to f on any subset Y of X, which f is bounded. □

Proof of (b). Suppose $f : X \to \bar{R}$. Let $(\psi_n^{(+)})_{n=1}^{\infty}$ and $(\psi_n^{(-)})_{n=1}^{\infty}$ be approximating sequences for f_+ and f_- obtained from part (a). Then, $\forall n \in \mathbb{N}, \psi_n^{(+)} \psi_n^{(-)} = 0$. Thus, we know $|f| = f_+ + f_-$ and $f_+ f_- = 0$. Hence,

$$|\psi_n| := \psi_n^{(+)} + \psi_n^{(-)}$$

has the desired properties.

If $f : X \to \mathbb{C}$, then letting $(\psi_n)_{n=1}^{\infty}$ and $(\rho_n)_{n=1}^{\infty}$ be the approximating sequence of Ref and Imf. Then, $\phi_n := \psi_n + i\rho_n$ have the desired properties. □

Definition 3.29 (Almost everywhere)**.** *If $P(x)$ is a property that depends on $x \in X$, and if (X, M, μ) is a measure space, we say P holds μ-**almost everywhere** or μ-**a.e.** if it holds for all $x \in E$ for some $E \in m$ such that $\mu(E^c) = 0$.*

Lemma 3.30. *Suppose (X, m, μ) is a complete measure space, $E \in m$, $B \subseteq X$ such that $B \Delta E \in m$, where Δ is symmetric difference. Suppose $\mu(B \Delta E) = 0$. Then, $B \in m$.*

Proof. By the completeness of μ, $E \setminus B$ and $B \setminus E \in m$ since they are subset of $B \Delta E$. Thus,

$$B = (B \cap E) \cup (B \setminus E) = (E \setminus (E \setminus B)) \cup (B \setminus E) \in m.$$

□

Proposition 3.31 (Proposition 2.11 in [1] p.47)**.** *Let (X, m, μ) be a complete measure space and suppose (Y, n) is a measurable space.*

(a) If $f, g : X \to Y$, f is measurable, and $f = g$ μ-a.e., then g is measurable.

(b) Suppose $f_j : X \to \mathbb{C}$ (or $\bar{\mathbb{R}}$) and $f : X \to \mathbb{C}$ (or $\bar{\mathbb{R}}$) such that $f_j \to f$ μ-a.e. and $\forall j \in \mathbb{N}$, f_j is measurable, then f is measurable.

Note that *a.e.* implies there exists $E \in m$ such that $\mu(E^c) = 0$ and $\forall x \in E$, (a) $f(x) = g(x)$ or (b) $\lim_{j \to \infty} f_j(x) = f(x)$.

Proof. For (a), Let E be as described. Let $A \in n$. It suffices to show that $g^{-1}(A) \in m$. But $g^{-1}(A) \Delta f^{-1}(A) \subseteq E^c$, and we know $\mu(E^c) = 0$. Thus, $g^{-1}(A) \Delta f^{-1}(A) \in m$ from the completeness of μ, and $\mu(g^{-1}(A) \Delta f^{-1}(A)) = 0$. However, $f^{-1}(A) \in m$. By the above lemma, $g^{-1}(A) \in m$.

For (b), let $\tilde{f}_j = f_j 1_E$ and $\tilde{f} = f 1_E$ where E is described above. Then, $\tilde{f}_j = f_j$ a.e. and $\tilde{f} = f$ a.e. and $\lim_{j \to \infty} \tilde{f}_j = \tilde{f}$. By part (a), $\forall j \in \mathbb{N}, \tilde{f}_j$ is measurable. By proposition 2.7, $\tilde{f}$ is measurable. By part (a) f is measurable. □

Proposition 3.32 (Proposition 2.12 in [1] p.48)**.** *Let (X, m, μ) and let $(X, \bar{m}, \bar{\mu})$ be its completion. If $f : X \to \mathbb{C}$ (or $\bar{\mathbb{R}}$) is $\bar{m}$-measurable, then there exists $g : X \to \mathbb{C}$ (or $\bar{\mathbb{R}}$) such that g is m-measurable and $f = g$ $\bar{\mu}$-a.e.*

Proof. Using the theorem 2.10, there exists a sequence $(\phi_n)_{n=1}^{\infty}$ of simple $\bar{m}$-measurable functions, $\phi_n : X \to \mathbb{C}$ (or $\bar{\mathbb{R}}$) such that $\phi_n \to f$ pointwisely. Writing

$$\phi_n = \sum_{j=1}^{k(n)} z_j^{(n)} 1_{E_j^{(n)}}$$

in standard form, where $z_j^{(n)} \in \mathbb{C}, E_j^{(n)} \in \bar{m}$. Since $\bar{m}$ is the completion of m, we have $E_j^{(n)} = F_j^{(n)} \cup A_j^{(n)}$ for some $F_j^{(n)} \in m$ and $A_j^{(n)} \subseteq N_j^{(n)}$ where $\mu(N_j^{(n)}) = 0, N_j^{(n)} \in m$. Let $N = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{k(n)} N_j^{(n)} \in m$. Then $\mu(N) = 0$. Let

$$\psi_n := \phi_n 1_{N^c} = \left(\sum_{j=1}^{k(n)} z_j^{(n)} 1_{E_j^{(n)}} \right) 1_{N^c}.$$

ψ_n is an m-measurable simple functions and $\psi_n \to f \cdot 1_{N^c}$ pointwise. By proposition 2.7, $g := f \cdot 1_{N^c}$ is m-measurable and $g = f$ m-a.e. Since N is still null set with respect to $\bar{\mu}$, $g = f$ $\bar{\mu}$-a.e. $\square$

3.2 Integration of Nonnegative Functions

Definition 3.33 (Integration)**.** *Fix a measure space (X, m, μ). Let $\phi : X \to [0, +\infty)$ be a measurable simple function. In standard form*

$$\phi = \sum_{j=1}^{n} c_j 1_{E_j},$$

where $E_1, \cdots, E_n \in m$ are disjoint, and $c_j \geq 0$.

We define the **integral of ϕ with respect to** *μ to be*

$$\int \phi d\mu := \sum_{j=1}^{n} c_j \mu(E_j) \in [0, +\infty].$$

If $A \in m$, we define

$$\int_A \phi d\mu (= \int \phi \cdot 1_A d\mu) := \sum_{j=1}^{n} c_j \mu_j(E_j \cup A).$$

Other notation for integral is as below;

$$\int \phi d\mu = \int \phi \int \phi(x) d\mu(x) = \int \phi(x) \mu(dx),$$

such notations depends on the context.

Proposition 3.34 (Proposition 2.13 in [1] p.49)**.** *Let $\phi, \psi : X \to [0, +\infty]$ be measurable simple functions.*

(a) If $c \geq 0$, then $\int c\phi = c \int \phi$.

(b) $\int \phi + \psi = \int \phi + \int \psi$

(c) If $\phi \leq \psi$, then $\int \phi \leq \int \psi$.

(d) The map $m \ni A \mapsto \int_A \phi d\mu$ is a measure.

Proof. For (a), if $\phi = \sum_{j=1}^{n} c_j 1_{E_j}$ as a standard form, then

$$\int c\phi = \sum_{j=1}^{n} cc_j \mu(E_j) = c \sum_{j=1}^{n} c_j \mu(E_j) = c \int \phi.$$

For $(b), (c)$, write $\phi = \sum_{j=1}^{n} a_j 1_{E_j}, \psi = \sum_{j=1}^{m} b_j 1_{F_j}$ as a standard form. Then, use

$$\phi = \sum_{i,j}^{n,m} a_j 1_{E_j \cap F_j}, \psi = \sum_{i,j}^{n,m} b_j 1_{E_i \cap F_j}.$$

Then,

$$\int \phi + \psi = \sum_{i,j}^{n,m} (a_i + b_j)\mu(E_i \cap F_j) = \sum_{i,j}^{n,m} a_i \mu(E_i \cap F_j) + \sum_{i,j}^{n,m} b_j \mu(E_i \cap F_j) = \int \phi + \int \psi.$$

And, if $\phi \le \psi$, then for each $(i, j) \in [n] \times [m]$, $a_i \le b_j$, thus

$$\int \phi = \sum_{i,j}^{n,m} a_i \mu(E_i \cap F_j) \le \sum_{i,j}^{n,m} b_j \mu(E_i \cap F_j) = \int \psi.$$

For (d), let

$$\nu(A) := \int_A \phi d\mu = \int \phi 1_A d\mu = \sum_{i=1}^{n} a_i \mu(E_i \cap A).$$

Clearly, $\nu() = 0$. Let $A_1, A_2, \cdots \in m$ be disjoint sets, and let $A = \bigcup_{i=1}^{\infty} A_i$. It suffices to show that $\nu(A) = \sum_{j=1}^{\infty} \nu(A_j)$. We have

$$\begin{aligned} \nu(A) &= \sum_{i=1}^{n} a_i \mu(E_i \cap A) \\ &= \sum_{i=1}^{n} a_i (\sum_{j=1}^{\infty} \mu(E_i \cap A_j)) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{n} a_i \mu(E_i \cap A_j) \\ &= \sum_{j=1}^{\infty} \int_{A_j} \phi d\mu \\ &= \sum_{j=1}^{\infty} \nu(A_j), \end{aligned}$$

as desired. □

Remark 3.35 (Remark in class). *If $\psi = \sum_{j=1}^{m} c_j 1_{R_j}$ for $c_j \ge 0$, $R_j \in m$ is not necessarily disjoint, then by the linearlity properties,*

$$\int \phi d\mu = \sum_{j=1}^{m} c_j \mu(R_j).$$

Remark 3.36 (Remark by Byeongsu Yu). *Acutally, in this case, since we deal with $[0, +\infty]$, we don't have to worry about the condition of interchanging two sums. However, for future reference, I leave some proposition which may be useful.*

Proposition 3.37 (Condition of finite and infinite sum)**.** *Let I be non-empty and finite, and suppose that for each $x \in I$ the series $\sum_{y=0}^{\infty} f_{x,y}$ converges. Then the series $\sum_{y=0}^{\infty} \sum_{x\in I} f_{x,y}$ also converges, and we have*

$$\sum_{x\in I} \sum_{y=0}^{\infty} f_{x,y} = \sum_{y=0}^{\infty} \sum_{x\in I} f_{x,y}.$$

Proof. We have

$$\begin{aligned}
\sum_{x\in I} \sum_{y=0}^{\infty} f_{x,y} &= \sum_{x\in I} \left(\lim_{n\to\infty} \sum_{y=0}^{n} f_{x,y} \right) && \text{(by definition)} \\
&= \lim_{n\to\infty} \left(\sum_{x\in I} \sum_{y=0}^{n} f_{x,y} \right) && \text{(since addition is continuous)} \\
&= \lim_{n\to\infty} \left(\sum_{y=0}^{n} \sum_{x\in I} f_{x,y} \right) && \text{(interchanging finite sums)} \\
&= \sum_{n=0}^{\infty} \sum_{x\in I} f_{x,y}. && \square
\end{aligned}$$

□

The proof shows that the result holds in any abelian [topological group][1] (or even semigroup), the minimum structure needed to talk about infinite series. (For instance, this means that you can use it in topological vector spaces without having to resort to vector-valued integration.)

We cannot omit the assumption that for each $x \in I$ the series $\sum_{y=0}^{\infty} f_{x,y}$ converges, as illustrated by the following example.

Example 3.38. *Let $I := \{-1, 1\}$. For all $x \in I$ and $y \in \mathbb{N}$ we define $f_{x,y} := x \cdot y$. Now we have*

$$\sum_{y=0}^{\infty} \sum_{x\in I} f_{x,y} = \sum_{y=0}^{\infty} (-y + y) = \sum_{y=0}^{\infty} 0 = 0,$$

whereas

$$\sum_{x\in I} \sum_{y=0}^{\infty} f_{x,y} = \left(\sum_{y=0}^{\infty} -y \right) + \left(\sum_{y=0}^{\infty} y \right) = -\infty + \infty,$$

which is undefined.

Definition 3.39 (L^+ and integration on measurable function)**.** *Let*

$$L^+ := \{f : X \to [0, +\infty] : f \text{ is } m\text{-measurable}\}.$$

For $f \in L^+$, we define

$$\int f d\mu := \sup\{\int \phi d\mu : \phi : X \to [0, +\infty], \phi \leq f\}.$$

Observation 3.40. *(1) If $\phi : X \to [0, +\infty)$ is a simple measurable function, then $\phi \in L^+$, and*

$$\underbrace{\int \phi d\mu}_{\text{new definition}} = \underbrace{\int \phi d\mu}_{\text{old definition}}.$$

(2) If $f, g \in L^+, f \leq g \implies \int f \leq \int g$, since we are taking supremum of a subset.

(3) If $f \in L^+, c \geq 0$, then $cf \in L^+$ and $\int cf = c \int f$.

Remark 3.41 (Remark for MCT)**.** *Note that if $f_1 \leq f_2 \leq \cdots$, then $\liminf f_n = \limsup f_n$ pointwisely, since monotonic sequence converges in the extended real line. (If it is bounded, then the Monotone Convergence Theorem of the Real number assures such convergence. If it is not bounded, then $\liminf f_n = \infty$, otherwise it is bounded, contradiction. Since f_i can be a constant function, this can be applied to the increasing sequence of the real number. Therefore, limits in the condition of Theorem 2.14 are well-defined.*

Theorem 3.42 (Theorem 2.14 in [1] p.50, Monotone Convergence Theorem (MCT))**.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence in L^+, with $f_1 \leq f_2 \leq \cdots$. Let*

$$f(x) := \lim_{n\to\infty} f_n(x) = \sup_{n\in\mathbb{N}} f_n(x) \in [0, +\infty].$$

Then,

$$f \in L^+$$

and

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu = \sup_{n\in\mathbb{N}} \int f_n d\mu.$$

Proof. By the Corollary 2.9, f is measurable, thus $f \in L^+$. Since $f_n \leq f, \forall n \in \mathbb{N}$,

$$\int f_n \leq \int f, \forall n \in \mathbb{N} \implies \int f \geq \lim_{n\to\infty} \int f_n d\mu.$$

Let $\phi : X \to [0, +\infty]$ be a simple measurable function such that $\phi \leq f$. Let $0 < \alpha < 1$, and define

$$E_n := \{x \in X : f_n(x) \geq \alpha\phi(x)\}.$$

Since $f_n \nearrow f, E_n \nearrow$, i.e.,

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$$

since $f_n \leq f_{n+1}$ for any $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in X$, we have

$$\bigcup_{n=1}^{\infty} E_n = \lim_{n\to\infty} E_n = X.$$

Since $A \mapsto \int_A \phi d\mu$ is a measure by the theorem 2.13 (d), we have

$$\int \phi d\mu = \lim_{n\to\infty} \int_{E_n} \phi d\mu$$

by continuity from below. However,

$$\alpha \int_{E_n} \phi d\mu = \int_{E_n} \alpha\phi d\mu \leq \int_{E_n} f_n d\mu \leq \int f_n d\mu.$$

Thus,

$$\alpha \int \phi d\mu = \lim_{n\to\infty} \alpha \int_{E_n} \phi d\mu \leq \lim_{n\to\infty} \int f_n d\mu.$$

Let $\alpha \to 1$, thus

$$\int \phi d\mu \leq \lim_{n\to\infty} \int f_n d\mu.$$

Take the supremum of all ϕ, we get

$$\int f d\mu \leq \lim_{n\to\infty} \int f_n d\mu.$$

Thus,

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu = \int \lim_{n\to\infty} f_n d\mu.$$

$\square$

Remark 3.43 (Consequence of MCT). *Given $f \in L^+$, from the theorem 2.10, $\exists (\phi_n)_{n=1}^{\infty}$, a sequence of measurable simple functions such that $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$ such that $\lim_{n\to\infty} \phi_n = f$ pointwise. By the MCT, $\int f d\mu = \lim_{n\to\infty} \int \phi_n d\mu$.*

Example 3.44 (MCT needs 'M' for monotonic). *Let $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$. Define $\phi_n := n \cdot 1_{(0,\frac{1}{n})}$. Then, $\phi_n \to 0$ pointwise. However,*

$$\int \phi_n d\mu = 1 \not\to \int 0 d\mu = 0,$$

since $\phi_n \to 0$ is not monotonic convergent.

Theorem 3.45 (Theorem 2.15 in [1] p.51). *Let $f_n \in L^+$, for all $n \in \mathbb{N}$. Let $f = \sum_{n=1}^{\infty} f_n$. Then, $f \in L^+$ since each partial sum is measurable, and limit of measurable function is also measurable, by the Corollary 2.9 in [1].*

Also, $\int f d\mu = \sum_{i=1}^{\infty} \int f_n d\mu$.

Proof. Step 1: ***Show*** $\int f_1 + f_2 d\mu = \int f_1 d\mu + \int f_2 d\mu$. Let

$$\begin{aligned} \phi_1 \leq \phi_2 \leq \cdots \leq f_1 \\ \psi_1 \leq \psi_2 \leq \cdots \leq f_2 \end{aligned}$$

be simple measurable functions with $\phi_n \to f_1, \psi_n \to f_2$. Then, $(\phi_n + \psi_n)_{n=1}^{\infty}$ is an increasing sequence of measurable simple functions converging to $f_1 + f_2$. So,

$$\int (f_1 + f_2) d\mu \underbrace{=}_{\text{(MCT)}} \lim_{n\to\infty} \int (\phi_n + \psi_n) d\mu \underbrace{=}_{\substack{\text{Theorem} \\ \text{2.13(b)}}} \lim_{n\to\infty} (\int \phi_n + \int \psi_n) \underbrace{=}_{\text{(MCT)}} \int f_1 d\mu + \int f_2 d\mu.$$

Step 2: ***Use induction on*** $\mathbb{N}$. If it holds for $N-1$ and 2, such that $N-1 \geq 2$, then

$$\int \sum_{n=1}^{N} f_n = \int \left(\left(\sum_{n=1}^{N-1} f_n \right) + f_N \right) \underbrace{=}_{\substack{\text{inductive} \\ \text{hypothesis} \\ \text{for 2}}} \int \left(\sum_{n=1}^{N-1} f_n \right) + \int f_N \underbrace{=}_{\substack{\text{inductive} \\ \text{hypothesis} \\ \text{for } N-1}} \sum_{n=1}^{N} \int f_n d\mu$$

Step 3: Note that

$$\int \sum_{i=1}^{\infty} f_i = \int \lim_{n\to\infty} \sum_{i=1}^{n} f_i \underbrace{=}_{\text{(MCT)}} \lim_{n\to\infty} \sum_{i=1}^{n} \int f_n d\mu.$$

$\square$

Proposition 3.46 (Proposition 2.16 in [1] p.51). *Let $f \in L^+$, then*

$$\int f = 0 \iff f = 0 \text{ a.e.}$$

Proof. Suppose $f = 0$ a.e.. If $\phi : X \to [0, +\infty)$ is simple and $\phi = 0$ a.e., then in standard form,

$$\phi = \sum_{j=1}^{n} c_j 1_{E_j}$$

and $c_j \neq 0$, thus $\mu(E_j) = 0$ for each $j \in [n]$. Thus,

$$\int \phi = \sum_{j=1}^{n} c_j \mu(E_j) = 0.$$

Then now take the simple function $\psi : X \to [0, +\infty)$ such that $\psi \leq f$. Then, $\psi = 0$ a.e., since $f = 0$ is a.e. and $\psi \leq f$. So, $\int \psi = 0$ as shown above. Thus

$$\int f := \sup \left\{ \int \psi : \psi : X \to [0, +\infty), \psi \text{ is simple}, \psi \leq f \right\} = 0$$

To show the converse, let $E_n = f^{-1}([\frac{1}{n}, +\infty])$. Then,

$$0 \leq \frac{1}{n} 1_{E_n} \leq f \implies 0 \int \frac{1}{n} 1_{E_n} \leq \int f = 0,$$

for all $n \in \mathbb{N}$. Thus,

$$\frac{1}{n} \mu(E_n) = 0 \implies \mu(E_n) = 0,$$

for all n. However, let $E := \bigcup_{n=1}^{\infty} E_n = f^{-1}((0, +\infty])$. Then,

$$\mu(E) \underbrace{\leq}_{\text{subadditivity of } \mu} \sum_{n=1}^{\infty} \mu(E_n) = 0.$$

Therefore, $f = 0$ a.e. □

Corollary 3.47 (Corollary 2.17 in [1] p.51, The almost everywhere MCT)**.** *Suppose $f_n \in L^+$ for all $n \in \mathbb{N}$, and $f \in L^+$. Suppose that for almost every $x \in X$,*

$$f_1(x) \leq f_2(x) \leq \cdots \tag{5}$$

and

$$\lim_{x \to \infty} f_n(x) = f(x) \tag{6}$$

Then,

$$\lim_{n \to \infty} \int f_n = \int f.$$

Proof. By hypothesis, $\exists E \in m$ s.t. $\mu(E) = 0$ and $\forall x \in E^c$, the equation (3) and (4) holds. Then, $f_n 1_{E^c} \nearrow f 1_{E^c}$ pointwisely. Thus, by the Monotone Convergence Theorem,

$$\lim_{n \to \infty} \int f_n 1_{E^c} = \int f 1_{E^c}.$$

However,

$$\int f_n = \int (f_n 1_{E^c} + f_n 1_E) \underbrace{=}_{\text{Theorem 2.15}} \int f_n 1_{E^c} + \underbrace{\int f_n 1_E}_{=0 \text{ by Proposition 2.16}} = \int f_n 1_{E^c} \tag{7}$$

And similarly,

$$\int f = \int (f1_{E^c} + f1_E) \underbrace{=}_{\text{Theorem 2.15}} \int f1_{E^c} + \underbrace{\int f1_E}_{=0 \text{ by Proposition 2.16}} = \int f1_{E^c} \tag{8}$$

Thus,

$$\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n 1_{E^c} = \int f1_{E^c} = \int f,$$

as desired. ∎

Proposition 3.48 (Proposition 2.18 in [1] p.52, Fatou's Lemma). *Let $(f_n)_{n=1}^{\infty}$ be any sequence in L^+. Then,*

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n.$$

Before proving this, note the natural example;

Example 3.49 (Natural Example). *Let $f_n = n \cdot 1_{(0,\frac{1}{n})}, \forall n \in \mathbb{N}$, and let $\mu = m$, the Lebesgue measure. Then, $\liminf_{n\to\infty} f_n = 0$. However, $\int f_n = 1$ for any $n \in \mathbb{N}$. Thus,*

$$\int \liminf_{n\to\infty} f_n = 0 \le 1 = \liminf_{n\to\infty} \int f_n.$$

Proof of the Fatou's Lemma. Let $g_k = \inf_{n\ge k} f_n$. Then,

$$g_1 \le g_2 \le \cdots$$

and define

$$f := \lim_{k\to\infty} g_k = \lim_{k\to\infty} \inf_{n\ge k} f_n = \sup_{k\to\infty} \inf_{n\ge k} f_n = \liminf_{n\to\infty} f_n.$$

By the Monotone convergence theorem,

$$\int f = \lim_{k\to\infty} \int g_k.$$

However, $\forall n \ge k$,

$$g_k \le f_n \implies \int g_k \le \inf_{n\ge k} \int f_n.$$

Thus

$$\int f \underbrace{=}_{\text{(MCT)}} \sup_{k\ge 1} \int g_k \le \sup_{k\ge 1} \inf_{n\ge k} \int f_n \underbrace{=}_{\text{definition}} \liminf_{n\to\infty} \int f_n.$$

∎

Corollary 3.50 (Corollary 2.19 in [1] p.52). *If $f_n \in L^+$ for $(n \ge 1)$ and $f \in L^+$ and if $f_n \to f$ a.e., then*

$$\int f \le \liminf_{n\to\infty} \int f_n.$$

Proof. If $f_n \to f$ everywhere, then apply the Fatou's lemma. In general, use the same method as a proof of the almost everywhere MCT. Let $E \in m$ such that $\mu(E) = 0$, and $\forall x \in E^c$, $f_n(x) \to f(x)$ pointwisely. Then, $f_n 1_{E^c} \to f1_{E^c}$ pointwisely. Thus, by the Fatou's lemma,

$$\int f1_{E^c} \le \liminf_{n\to\infty} \int f_n 1_{E^c}.$$

Also, we already know that $\int f_n = \int f_n 1_{E^c}$ for any $n \in \mathbb{N}$ by the equation (5) and $\int f = \int f1_{E^c}$ by the equation (6). Thus,

$$\int f = \int f1_{E^c} \le \liminf_{n\to\infty} \int f_n 1_{E^c} = \liminf_{n\to\infty} \int f_n,$$

as desired. ∎

Observation 3.51 (Proposition 2.20 in [1] p.52)**.** *If $f \in L^+, \int f < +\infty$, then $\mu(\{x : f(x) = +\infty\}) = 0$, and $\{x : f(x) > 0\}$ is σ-finite.*

Proof. Suppose $\mu(\{x : f(x) = +\infty\}) > 0$. Then,

$$+\infty > \int f \geq \int_{\{x:f(x)=+\infty\}} f = \infty \times \mu(\{x : f(x) = +\infty\}) = \infty,$$

contradiction. Also, note that

$$\int_{\{x:n>f(x)>\frac{1}{n}\}} f \leq \int f < \infty$$

. Thus, $\mu(\{x : n > f(x) > \frac{1}{n}\}) < \infty$. Let $E_n = \{x : n > f(x) > \frac{1}{n}\}$ and $F_n = E_n \setminus \left(\cup_{i=1}^{n-1} E_i\right)$. for any $n \in \mathbb{N}$, and let $F_0 = \{x : f(x) = +\infty\}$. Then, F_n's are disjoint and $\bigcup_{n=1}^{N} F_n = E_N$ for any $N \in \mathbb{N}$, thus $\bigcup_{n=0}^{\infty} = \{x : f(x) > 0\}$. Thus,

$$\{x : f(x) > 0\} \cap F_n \leq F_n \implies \mu(\{x : f(x) > 0\} \cap F_n) \leq \mu(F_n) < \infty.$$

Thus, $\{x : f(x) > 0\}$ is σ-finite. $\square$

3.3 Integration of complex functions

Definition 3.52 (Integrability of extended real-valued function)**.** *Let $f : X \to \bar{\mathbb{R}}$ be in L^+. We have $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Thus, by the Corollary 2.8 in [1], f_+, f_- are in L^+. And note that $f = f_+ - f_-$, and $|f| = f_+ + f_-$. We say f is* **integrable** *if $\int |f| < +\infty$, and then we can define*

$$\int f := \int f_+ - \int f_-.$$

Proposition 3.53 (Proposition 2.21 in [1] p.53)**.** *The set of integrable (thus measurable) real-valued function on X is a real vector space, on which integration ($f \mapsto \int f$) is a linear functional.*

Proof. For the first assertion, let $f, g : X \to \mathbb{R}$ be integrable. Let $a \in \mathbb{R}$. Note that $+$ defined for function already satisfies associativity and commutativity, and the scalar multiplication satisfies compatibility with field multiplication and distibutivity. Also, $1 \in \mathbb{R}$ satisfies $1 \cdot f = f$. Therefore, it suffices to show that $f + g$ and af are integrable, and $\int af = a \int f$, $\int f + g = \int f + \int g$.

Note that

$$|af| = |a||f| \implies \int |af| = \int |a||f| = |a| \int |f| < \infty,$$

thus af is integrable. Also,

$$\int |f + g| \underbrace{\leq}_{\text{triangle ineq.}} \int |f| + |g| = \int |f| + \int |g| < +\infty,$$

thus $f + g$ is also integrable.

If $a > 0$, then $(af)_+ = af_+$ and $(af)_- = af_-$, thus

$$\int af = \int af_+ - \int af_- \underbrace{=}_{\text{Theorem 2.15}} a \left(\int f_+ - \int f_- \right) = a \int f.$$

If $a < 0$, then $(af)_+ = -af_-, (af)_- = -af_+$, so

$$\int af = \int (-af_-) - \int (-af_+) = a \int f_+ - a \int f_- = a(\int f_+ - \int f_-) = a \int f.$$

Let $h = f + g$. Then, $h_+ - h_- = f_+ - f_- + g_+ g_-$, thus

$$h_+ + f_- + g_- = h_- + f_+ + g_+ \implies \int h_+ + f_- + g_- = \int h_- + f_+ + g_+$$

Thus, by the theorem 2.15,

$$\int h_+ + \int f_- + \int g_- = \int h_- + \int f_+ + \int g_+ \implies \int h_+ - \int h_- = \int f_+ - \int f_- + \int g_+ - \int g_- = \int f + \int g.$$

$\square$

Definition 3.54 (Integrable in complex valued function)**.** *Let $f : X \to \mathbb{C}$ be a measurable function. We say f is* **integrable** *if $\int |f| < \infty$. Note that*

$$|f| \leq |Ref| + |Imf| \leq 2|f|.$$

So f is integrable iff Ref and Imf are integrable. We define

$$\int f := \int Ref + i \int Imf.$$

Proposition 3.55. *The set of complex valued integrable function is a complex vector space and integration (i.e. $f \mapsto \int f$) is a linear functional.*

Proof. For the first assertion, let $f, g : X \to \mathbb{R}$ be integrable. Let $a \in \mathbb{C}$. Note that $+$ defined for function already satisfies associativity and commutativity, and the scalar multiplication satisfies compatibility with field multiplication and distibutivity. Also, $1 \in \mathbb{C}$ satisfies $1 \cdot f = f$. Therefore, it suffices to show that $f + g$ and af are integrable, and $\int af = a \int f$, $\int f + g = \int f + \int g$.

Note that

$$|af| = |a||f| \implies \int |af| = \int |a||f| = |a| \int |f| < \infty,$$

thus af is integrable. Also,

$$\int |f + g| \underbrace{\leq}_{\text{triangle ineq.}} \int |f| + |g| = \int |f| + \int |g| < +\infty,$$

thus $f + g$ is also integrable.

If $a = b + ic$, then $af = bRef - cImf + i(bImf + cRef)$ Then,

$$\begin{aligned}
\int af \quad & \underbrace{=}_{Def.} \quad \int (bRef - cImf) + i \int (bImf + cRef) \\
& \underbrace{=}_{\text{Thm 2.21 in [1]}} \quad b \int Ref - c \int Imf + i \left(b \int Imf + c \int Ref \right) \\
& = \quad b \left(\int Ref + i \int Imf \right) + ci \left(\int Ref + i \int Imf \right) = a \int f.
\end{aligned}$$

Also,

$$\int f + g \underbrace{=}_{Def.} \int (Ref + iImf + Reg + iImg) \underbrace{=}_{\text{Thm 2.21 in [1]}} \int Ref + i \int Imf + \int Reg + i \int Img = \int f + \int g$$

Thus, integration is also a linear functional. $\square$

Notation 3.56 (L^1 space)**.** *L^1 or $L^1(\mu)$ denotes*

$$L^1 = L^1(\mu) := \{f : X \to \mathbb{C} : f \text{ is measurable and integrable}\}.$$

Also,

$$L^1_{\mathbb{R}}(\mu) := \{f : X \to \mathbb{R} : f \text{ is measurable and integrable}\}.$$

Proposition 3.57 (Proposition 2.22 in [1] p.53)**.** *If $f \in L^1$, then $|\int f| \le \int |f|$.*

Proof. Case 1: ***when*** $Imf = 0$ ***everywhere.*** Then,

$$\left|\int f\right| = \left|\int f_+ - \int f_-\right| \le \int |f_+| + \left|\int f_-\right| = \int f_+ + \int f_- = \int f_+ + f_- = \int |f|$$

Case 2: ***when*** $\int Imf = 0$**.** Then,

$$\left|\int f\right| = \left|\int Ref\right| \underbrace{\le}_{\text{by case 1}} \int |Ref| \le \int |f|\,.$$

Case 3: ***General Cases.*** Note that $\int f = e^{i\theta}\left|\int f\right|$ for some θ, since it is just complex number. Thus, let $g = e^{-i\theta} f$. Then,

$$\int g = e^{-i\theta}\int f = |\int f| \ge 0.$$

However, we know that

$$\int g = \int Reg + i\int Img.$$

By case 2,

$$\left|\int f\right| = \left|\int g\right| \le \int |g| = \int |f|.$$

$\square$

Definition 3.58. *If we have $f \in L^1$ and if $E \in m$, then define $\int_E f := \int f 1_E$.*

Recall the proposition 2.16 in [1] that if $h \in L^+$, then $\int h = 0 \iff h = 0$ a.e.

Proposition 3.59 (Proposition 2.23 in [1] p.54)**.** *Let $f, g \in L^1$. Then, TFAE.*

1. $\int_E f = \int_E g$ *for all* $E \in m$.
2. $\int |f - g| = 0$.
3. $f = g$ *a.e.*

Proof. $(ii) \iff (iii)$: Since $|f-g| \in L^+$, by the proposition 2.16 in [1], $\int |f-g| = 0$ if and only if $|f-g| = 0$ a.e. Also, $|f - g| = 0$ a.e. if and only if $f = g$ a.e.

$(iii) \implies (i)$: Note that

$$\left|\int_E f - \int_E g\right| = \left|\int (f-g)1_E\right| \underbrace{\le}_{\text{Thm 2.22}} \int |f-g| = 0.$$

$(i) \implies (iii)$: Assume it was not true that $f = g$ a.e. Then it suffices to find $E \in m$ such that $\int_E f \ne \int_E g$. Let $h = f - g$. Then,

$$\int_E f - \int_E g = \int_E h.$$

By hypothesis, it is not true that $h = 0$a.e., and we seek E such that $\int_E h \neq 0$. Without loss of generality, h is real valued. Also, without loss of generality, it is not true that $h_+ = 0$ a.e. Let $E = h^{-1}((0, +\infty))$. Then, $\mu(E) > 0$. So,

$$\int_E h = \int_E h^+ = \int h^+ > 0$$

by the converse statement of proposition 2.16. If h is general complex valued function, then $h = Reh_+ - Reh_- + iImh_+ - iImh_-$, and at least one of terms of h is nonzero a.e. by the assumption. Say Reh_+ is nonzero a.e. Then, let $E = Reh_+((0, \infty))$. So,

$$\int_E h = \int_E h^+ = \int h^+ > 0,$$

by the converse statement of proposition 2.16. □

Definition 3.60 (Pseudometric on L^1)**.** *Let $\rho : L^1 \times L^1(\mu) \to [0, +\infty)$ by $\rho(f, g) = \int |f - g|$. Then, by the proposition 2.23 in [1], $\rho(f, g) = 0 \iff f = g$ a.e. Also, $\rho(g, f) = \rho(f, g)$ and $\rho(f, h) \leq \rho(f, g) + \rho(g, h)$. Thus, ρ is pseudo-metric on L^1. We introduce the equivalence relation*

$$f \sim g \iff f = g \text{ a.e.}$$

Using this, we can redefine

$$L^1 := L^1(\mu) := \{f : X \to \mathbb{C} : f \text{ is measurable and } \int |f| < +\infty\}/\sim .$$

Then ρ becomes a metric on $L^1(\mu)$, the L^1**-metric.**

Remark 3.61 (Abuse of notation)**.** *By (standard) abuse of notation, we continue to write $f \in L^1$ rather than $[f] \in L^1$.*

Notation 3.62. *If $\bar{\mu}$ is the completion of μ, then we have the natural identification of $L^1(\mu) \cong L^1(\bar{\mu})$, since by the proposition 2.12, if $f : X \to \mathbb{C}$ is $\bar{\mu}$-measurable, then there exists $g : X \to \mathbb{C}$ a μ-measurable function such that $f = g$ a.e.*

Theorem 3.63 (Theorem 2.24 in [1] p.54, Dominated Convergence Theorem)**.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence in L^1. Suppose 1) $f_n \to f$ a.e., 2) $\exists g \in L^+$ such that $\int g < +\infty$ and 3) $\forall n \in \mathbb{N}, |f_n| \leq g$ a.e.*

Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.

Proof. By redefining on a null set using the proposition 2.12, we may, without loss of generality, assure that $f_n \to f$ pointwise. And, $\forall n \in \mathbb{N}$, $|f_n| \leq g$ holds everywhere. Thus, f is measurable by the theorem 2.11 (b). Moreover, $|f| \leq g$, and since g is integrable,

$$\int |f| < +\infty \implies f \in L^1.$$

By considering the real and imaginary parts, we may without loss of generality assume that each f_n is real-valued. We have

$$-g \leq f_n \leq g.$$

Thus,

$$g + f_n \in L^+.$$

So,

$$\int g + \int f = \int (g + f) = \int \lim_{n\to\infty} (g + f_n) \underbrace{\leq}_{\text{Fatou's Lemma}} \liminf_{n\to\infty} \int (g + f_n) = \int g + \liminf_{n\to\infty} \int f_n.$$

Thus, $\int f \leq \liminf_{n\to\infty} \int f_n$. Replacing f_n by $-f_n$, we get

$$-\inf f = \int -f \underbrace{\leq}_{\text{Fatou's Lemma}} \liminf \left(-\int f_n \right) = -\lim \sup_{n\to\infty} \int f_n.$$

Thus, $\limsup_{n\to\infty} \int f_n \leq \int f$. Thus, limit exists and

$$\lim_{n\to\infty} \int f_n = \int f.$$

$\square$

Theorem 3.64 (Theorem 2.27 in [1] p.56)**.** *Fix a measure space (X, m, μ). Suppose $f : X \times (c, d) \to \mathbb{C}$ satisfies $\forall t \in (c, d)$, $x \mapsto f(x, t)$ is integrable, i.e., $x \mapsto f(x, t) \in L^1(\mu)$. Let $F(t) := \int f(x, t) d\mu(x)$.*

(a) If $\exists g \in L^1$ such that

$$\text{1) } \forall x, \forall t, |f(x, t)| \leq g(x)$$

and

$$\text{2) for fixed } t_0 \in (c, d), \forall x, \lim_{t\to\infty} f(x, t) = f(x, t_0) \text{ (continuity w.r.t } t)$$

then $\lim_t \to t_0 F(t) = F(t_0)$.

(b) If $\exists g \in L^1$ and if

$$\text{1) } \forall x, \forall t, \frac{\partial f}{\partial t}(x, t) \text{ exists,}$$

and

$$\text{2) } \left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x),$$

then

$$\text{1) } F'(t_0) \text{ exists and 2) } F'(t_0) = \int \frac{\partial f}{\partial t}(x, t) d\mu(x).$$

Proof. Note that the conditions of (a) satisfies the DCT's condition. Let $\{t_n\}$ be a sequence in (c, d) converging to t_0. Then,

$$\lim_{n\to\infty} F(t_n) = \lim_{n\to\infty} \int f(x, t_n) d\mu(x) \underbrace{=}_{\text{(DCT)}} \int \lim_{n\to\infty} f(x, t_n) d\mu(x) = \int f(x, t_0) d\mu(x) = F(t_0).$$

For the part (b), let t_n be a sequence in $(c, d) \setminus \{t_0\}$ such that $t_n \to t_0$. Then,

$$\lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \lim_{n\to\infty} \int \frac{f(x, t_n) - f(x, t_0)}{(t_n - t_0)} d\mu(x).$$

By the Mean Value Theorem, $\forall x, \forall n, \exists t^* \in (c, d)$ such that $\frac{f(x,t_n) - f(x,t_0)}{t_n - t_0} = \frac{\partial f}{\partial t}(x, t^*)$. Thus,

$$\left| \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0} \right| \leq \left| \frac{\partial f}{\partial t}(x, t^*) \right| \leq g(x).$$

Therefore, for all x and for each $n \in \mathbb{N}$, $\frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}$ satisfies each condition of DCT. Thus,

$$\lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \lim_{n\to\infty} \int \frac{f(x, t_n) - f(x, t_0)}{(t_n - t_0)} d\mu(x) \underbrace{=}_{\text{(DCT)}} \int \lim_{n\to\infty} \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0} d\mu(x) = \int \frac{\partial f}{\partial t}(x, t_0) d\mu(x).$$

Thus, since t_n was arbitrary,

$$F'(t_0) = \lim_{n\to\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \int \frac{\partial f}{\partial t}(x, t) d\mu(x)$$

as desired. $\square$

Remark 3.65 (Recall of Riemann Integral). *Let $[a,b]$ be a compact interval. By a* **partition** *of $[a,b]$ we shall mean a finite sequence $P = \{t_j\}_0^n$ such that*

$$a = t_0 < t_1 < \cdots < t_n = b.$$

Let f be an arbitrary bounded real-valued function on $[a,b]$. For each partition P we define

$$S_P f = \sum_{j=1}^{n} M_j (t_j - t_{j-1}), s_P f = \sum_{j=1}^{n} m_j (t_j - t_{j-1})$$

where M_j, m_j are supremum and infimum of f on $[t_{j-1}, t_j]$. Then define

$$\bar{I}_a^b = \inf_P S_P f, \underline{I}_a^b \sup_P s_P f,$$

where the infimum and supremum are taken over all partitions P. If $\bar{I}_a^b = \underline{I}_a^b$, there common value is **Riemann inegral** *$\int_a^b f(x)dx$ and f is called* **Riemann integrable.**

Theorem 3.66 (Theorem 2.28 in [1] p.57). *(a) If $f : [a,b] \to \mathbb{R}$ is Riemann integrable then $f \in L^1(m)$, where m is Lebesgue measure, and $\int_a^b f(t)dt = \int_{[a,b]} f dm$.*

(b) f is Riemann integrable if and only if $\{x \in [a,b] : f$ is discontinuous at $x\}$ has Lebesgue measure zero.

Proof. Suppose that f is Riemann integrable. For each partition P, let

$$G_P = \sum_{j=1}^{n} M_j 1_{(t_{j-1}, t_j]}, g_P = \sum_{j=1}^{n} m_j 1_{(t_{j-1}, t_j]}.$$

Thus, $S_P f = \int G_P dm$ and $s_P f = \int g_P dm$. Then, since f is Riemann integrable, there exists $\{P_k\}_{k=1}^{\infty}$, a sequence of partitions such that $P_k \subseteq P_{k+1}$ and $S_{P_k} f \to \int_a^b f$ and $s_P f \to \int_a^b f$ as $k \to \infty$. Now note that

$$G_{P_k} \geq G_{P_{k+1}} \geq f \text{ and } g_{P_k} \leq g_{P_{k+1}} \leq f.$$

Thus, it is monotonic and bounded, thus converge by the Monotone Convergence theorem. Hence, we can define

$$G := \lim_{k\to\infty} G_{P_k}, g := \lim_{k\to\infty} g_{P_k}.$$

Then, by definition,

$$g \leq f \leq G.$$

Thus, since G_{P_k} is bounded by G_{P_1}, and g_{P_k} is bounded by f, we can apply the Dominated Convergence theorem to conclude that

$$G, g \text{ are integrable}$$

and

$$\int G \underbrace{=}_{(DCT)} \lim_{k\to\infty} \int G_{P_k} \underbrace{=}_{Def.} \lim_{k\to\infty} S_{P_k} f \underbrace{=}_{\text{Riemann integrable}} \int_a^b f,$$

$$\int g \underbrace{=}_{(DCT)} \lim_{k\to\infty} \int g_{P_k} \underbrace{=}_{Def.} \lim_{k\to\infty} s_{P_k} f \underbrace{=}_{\text{Riemann integrable}} \int_a^b f.$$

Thus,

$$\int (G - g) dm = 0 \underbrace{\Longrightarrow}_{\text{Prop 2.16}} G = g \text{ a.e.} \Longrightarrow G = f \text{ a.e.}$$

Note that G is the limit of a sequence of simple function, by Proposition 2.11 (b), G is measurable. Also, since m is complete measure, thus by Proposition 2.11 (a), f is measurable. Thus,

$$\int_{[a,b]} f \underbrace{=}_{\text{from } f=g \text{ a.e.}} \int G dm = \int_a^b f.$$

Hence (a) is proved. □

Remark 3.67 (Remarks in book). 1. *the imporper Riemann integral can be also viewed as Lebesgue integral, using DCT.*

2. *Lebesgue integral gives more powerful convergence theorem.*

3. *Lebesgue integral is applied wider class of functions; this gives a complete metric for some important function spaces. For example, $L^1(\mu)$ is complete, and it can be seen when we deal with Thm 2.25 which is disguised form of theorem 5.1.*

Now, fix (X, m, μ)

Theorem 3.68 (Theorem 2.25 in [1] p.55). *If $\{f_n\}_{n=1}^\infty$ is a sequence in L^1 and $\sum_{n=1}^\infty \int |f_n| < +\infty$, then $\sum_{n=1}^\infty f_n$ converges a.e. to a function $f \in L^1$, and $\int f = \sum_{n=1}^\infty \int f_n$.*

Proof. Note that $\sum_{n=1}^\infty |f_n| : X \to [0, +\infty]$ is measurable and

$$\int \sum_{n=1}^\infty |f_n| = \sum_{n=1}^\infty \int |f_n| \underbrace{<}_{\text{Given condition}} \infty$$

by theorem 2.15. Thus, $\sum_{n=1}^\infty |f_n| \in L^1$, thus it is finite on a set of full measure. Since

$$\sum_{n=1}^\infty f_n \le \sum_{n=1}^\infty |f_n|,$$

$\sum_{n=1}^\infty f_n$ converges almost everywhere. Therefore, we can define

$$f = \sum_{n=1}^\infty f_n.$$

Note that f is measurable since it is limit of sequence of measurable function. Since $\forall N \in \mathbb{N}$,

$$|\sum_{n=1}^N f_n| \le \sum_{n=1}^N |f_n| \le \sum_{n=1}^\infty |f_n| \in L^1,$$

we can apply the Dominated Convergence Theorem to conclude that $f \in L^1$ and

$$\int f = \int \lim_{N\to\infty} \sum_{n=1}^N f_n \underbrace{=}_{\text{(DCT)}} \lim_{N\to\infty} \int \sum_{n=1}^N f_n \underbrace{=}_{\text{Thm 2.13}} \lim_{N\to\infty} \sum_{n=1}^N \int f_n = \sum_{n=1}^\infty \int f_n$$

□

Theorem 3.69 (Completeness of L^1). *$L^1(\mu)$ is complete with respect to $\rho(f,g) = \int |f-g| d\mu$.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in L^1. Let $(f_{n_k})_{k=1}^{\infty}$ be a subseqeunce such that

$$\forall k \in \mathbb{N}, \rho(f_{n_k}, f_{n_{k+1}}) \leq 2^{-k}.$$

Now we want to define

$$f := f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k}).$$

We have

$$\int |f_{n_1}| + \sum_{k=1}^{\infty} \int |f_{n_{k+1}} - f_{n_k}| < \int |f_{n_1}| + \sum_{k=1}^{\infty} 2^{-k} < \infty.$$

Thus, by the theorem 2.5, $f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_k})$ converges a.e. to a function in L^1. Say that function f. Then,

$$f \underbrace{=}_{\text{a.e.}} \lim_{p\to\infty} \left(f_{n_1} + \sum_{k=1}^{p} \left(f_{n_{k+1}} - f_{n_k}\right) \right) = \lim_{p\to\infty} f_{n_{p+1}}.$$

Hence,

$$\begin{aligned}
\rho(f, f_{n_{p+1}}) &= \int |f - f_{n_{p+1}}| d\mu \\
&= \int |f_{n_1} + \sum_{k=1}^{\infty}(f_{n_{k+1}} - f_{n_{p+1}}) - (f_{n_1} + \sum_{k=1}^{p}(f_{n_{k+1}} - f_{n_{p+1}}))| d\mu \\
&= \int | \sum_{k=p+1}^{\infty} (f_{n_{k+1}} - f_{n_k})| \\
&\leq \int \sum_{k=p+1}^{\infty} |f_{n_{k+1}} - f_{n_k}| \\
&\underbrace{=}_{\text{Thm 2.25}} \sum_{k=p+1}^{\infty} \int |f_{n_{k+1}} - f_{n_k}| \\
&= \sum_{k=p+1}^{\infty} \rho(f_{n_{k+1}}, f_{n_k}) \\
&< \sum_{k=p+1}^{\infty} 2^{-k} = 2^{-p} \to 0 \text{ as } p \to \infty.
\end{aligned}$$

Thus, $f_{n_k} \to f \in L^1$ in a metric ρ. ☐

Corollary 3.70 (Corollary of the proof)**.** *If $f_n \to f$ in L^1-metric, ρ, then $\exists$ a subseqeunce $(f_{n_k})_{k=1}^{\infty}$ such that $f_{n_k} \to f$ a.e. as $k \to \infty$.*

Proposition 3.71 (Proposition 1.20 in [1] p. 37, Exercise 1.26)**.** *Let μ be a Lebesgue-Stieltjes measure, i.e., $\mu((a,b]) = G(b) - G(a)$ for some $G : \mathbb{R} \to \mathbb{R}$, nondecreasing right continuous function, and let $E \in m_\mu$, a Borel σ-algebra with respect to μ, such that $\mu(E) < +\infty$, $\epsilon > 0$. Then, $\exists F \subseteq \mathbb{R}$ such that F is a finite union of bounded open intervals such that $\mu(E\Delta F) < \epsilon$.*

Proof. We proved theorem 1.18, i.e., $\exists U \subseteq \mathbb{R}$ such that U is open and $E \subseteq U$ and $\mu(U) < \mu(E) + \frac{\epsilon}{2}$. We have $U = \bigcup_{j=1}^{\infty} I_j$ for disjoint open intervals (or empty sets) I_j. Thus,

$$\mu(U) = \sum_{j=1}^{\infty} \mu(I_j).$$

Let $N \in \mathbb{N}$ such that
$$\sum_{j=1}^{N} \mu(I_j) > \mu(U) - \frac{\epsilon}{2}.$$
If I_j is unbounded, then $\exists$ a bounded open interval $I'_j \subseteq I_j$ such that
$$\mu(I'_j) > \mu(I_j) - \frac{\epsilon}{8}.$$
If I_j is bounded, take $I'_j = I_j$. Let $F = \bigcup_{j=1}^{N} I'_j$. Then, $F \subseteq U$, and
$$\mu(F) > \mu(U) - \frac{\epsilon}{4} - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \mu(U) - \frac{\epsilon}{2}$$
Note that since μ is arbitrary measure and $\mu(U) < \infty$, this measure gives finite value for unbounded interval conatined in U. Thus the above inequality is derived.

Thus,
$$E \Delta F = (E \setminus F) \cup (F \setminus E) \subseteq (U \setminus F) \cup (U \setminus E),$$
therefore,
$$\mu(E \Delta F) \leq \mu(U \setminus F) + \mu(U \setminus E) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
□

Theorem 3.72 (Theorem 2.26 in [1] p.55, Approximation of $f \in L^1$)**.** *Let (X, m, μ) be a measure space and let $f \in L^1(\mu)$, $\epsilon > 0$. Then, $\exists$ a simple function $\phi \in L^1(\mu)$ such that*
$$\int |f - \phi| d\mu < \epsilon.$$

If μ is a Lebesgue Stieltjes measure, then ϕ can be taken of the form $\phi = \sum_{j=1}^{n} \alpha_j 1_{E_j}$ where $E_1, \cdots, E_n$ are bounded open intervals in $\mathbb{R}$. Moreover, $\exists$ a continuous function $g : \mathbb{R} \to \mathbb{R}$ with bounded supports, such that $\int |f - g| d\mu < \epsilon$.

Proof. Suppose μ is any measure. By theorem 2.10, $\exists$ a sequence $(\phi_n)_{n=1}^{\infty}$ of simple functions such that $\phi_n \to f$ pointwise and
$$|\phi_1| \leq |\phi_2| \leq \cdots \leq |f|.$$
Thus, $|\phi_n - f| \to 0$ pointwisely and $|\phi_n - f| \leq 2|f|$. Therefore, by the Dominated Convergence Theorem,
$$\lim_{n\to\infty} \int |\phi_n - f| d\mu \underbrace{=}_{\text{DCT}} \int \lim_{n\to\infty} |f_n - f| d\mu = \int 0 d\mu = 0.$$

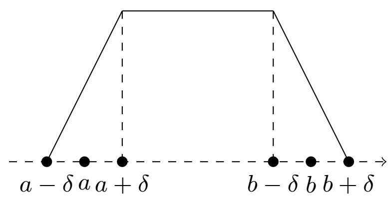
Now suppose μ is Lebesgue Stieltjes measure. Note that we can represent
$$\phi_n = \sum_{j=1}^{N_n} \alpha_j 1_{E_j}$$
for some disjoint intervals $E_j \in m$ and $\alpha_j \neq 0$ for any $j \in \mathbb{N}$. Also, we know that $\mu(E \Delta F) = \int |1_E \setminus 1_F|$ for any $E, F \in m$. Thus for ϕ_n, each 1_{E_j} can be approximated by 1_{F_j} with $\mu(E_j \Delta F_j) < \frac{1}{\alpha_j 2^j} \epsilon$, where F_j is finite union of bounded open intervals. Define $\phi'_n := \sum_{j=1}^{N_n} \alpha_j 1_{E_j}$. Then,
$$\begin{aligned} \int |\phi_n - \phi'_n| = \int \left| \sum_{j=1}^{N_n} \alpha_j (1_{E_j} - 1_{F_j}) \right| &\underbrace{\leq}_{\text{Trianlge Ineq.}} \int \sum_{j=1}^{N_n} |\alpha_j| |1_{E_j} - 1_{F_j}| \underbrace{=}_{\text{Thm 2.13}} \sum_{j=1}^{N_n} \alpha_j \int |1_{E_j} - 1_{F_j}| \\ = \sum_{j=1}^{N_n} \alpha_j \mu(E_j \Delta F_j) &\underbrace{=}_{\text{Def. of } \mu(E_j \Delta F_j)} \sum_{j=1}^{N_n} \frac{\epsilon}{2^j} < \epsilon. \end{aligned}$$

Thus, we can approximate f by $\phi'(n)$, since $\int |f - \phi_n'| \leq \int |f - \phi| + \int |\phi - \phi'| \leq 2\epsilon$ for suitable n.

To find a continuous function, it suffices to approximate simple functions ϕ_n, which means that it will suffices to approximate $1_{(a,b)}$ for some $a, b \in \mathbb{R}$ since by the argument every simple function is consists of $1_{(a,b)}$ for some $a, b \in \mathbb{R}$. Thus, construct a continuous function g_δ as follow for some $\delta > 0$;

$$g_\delta := \begin{cases} \frac{x-a+\delta}{2\delta} & \text{if } x \in (a-\delta, a+\delta] \\ 1 & \text{if } x \in (a+\delta, b-\delta] \\ -\frac{x-b-\delta}{2\delta} & \text{if } x \in (b-\delta, b+\delta) \\ 0 & \text{otherwise.} \end{cases}$$

It looks like below picture;



Thus,

$$\int |1_{(a,b)} - g_\delta| d\mu \leq \mu(a-\delta, a+\delta) + \mu(b-\delta, b+\delta) \to 0 \text{ as } \delta \to 0.$$

Thus, use this approximation to approximate the simple functions. □

3.4 Modes of Convergence

Definition 3.73 (Modes of Convergence)**.** *Fix (X, m, μ) and let $f_n : X \to \mathbb{C}$ for $n \in \mathbb{N}$, and $f : X \to \mathbb{C}$ be measurable function. We say that f_n* **converges to** *f (asn $\to \infty$)*

- **uniformly**;
$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall x \in X, \forall n \geq N, |f(x) - f_n(x)| < \epsilon.$$
- **pointwise**;
$$\forall \epsilon > 0, \forall x \in X, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, |f(x) - f_n(x)| < \epsilon.$$
- **almost everywhere**; *$\exists E \in m$ such that $\mu(E) = 0$ and*
$$\forall \epsilon > 0, \forall x \in E^c, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, |f(x) - f_n(x)| < \epsilon.$$
- **in measure**;
$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, |\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\})| \leq \epsilon,$$
which is equivalent to say that
$$\forall \epsilon > 0, \lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$
- **in L^1**;
$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \int |f_n - f| d\mu \leq \epsilon$$
which is equivalent to say that
$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0.$$

- **almost uniformly**;

$$\forall\delta > 0, \exists E \in m \text{ with } \mu(E) < \delta, \text{ s.t. } \forall\epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall x \in X, \forall n \geq N, |1_{E^c}f(x) - 1_{E^c}f_n(x)| < \epsilon$$

which is equivalent to say that $\forall\delta > 0, \exists E \in m$ *such that* $\mu(E) < \delta$ *and* $1_{E^c}f_n$ *converges uniformly to* $1_{E^c}f$.

Also, we say a sequence of measurable function $(f_n)_{n=1}^{\infty}$ *is* **Cauchy in measure** *if*

$$\forall\epsilon > 0, \mu(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}) \to 0 \text{ as } m, n \to \infty,$$

i.e., $\exists M \in \mathbb{N}$ *such that* $\forall n, m \geq M$, $\mu(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}) < \epsilon$. *Clearly, if* (f_n) *converges in measure, then it is Cauchy in measure.*

We can describe the relationship between modes of convergence as below.

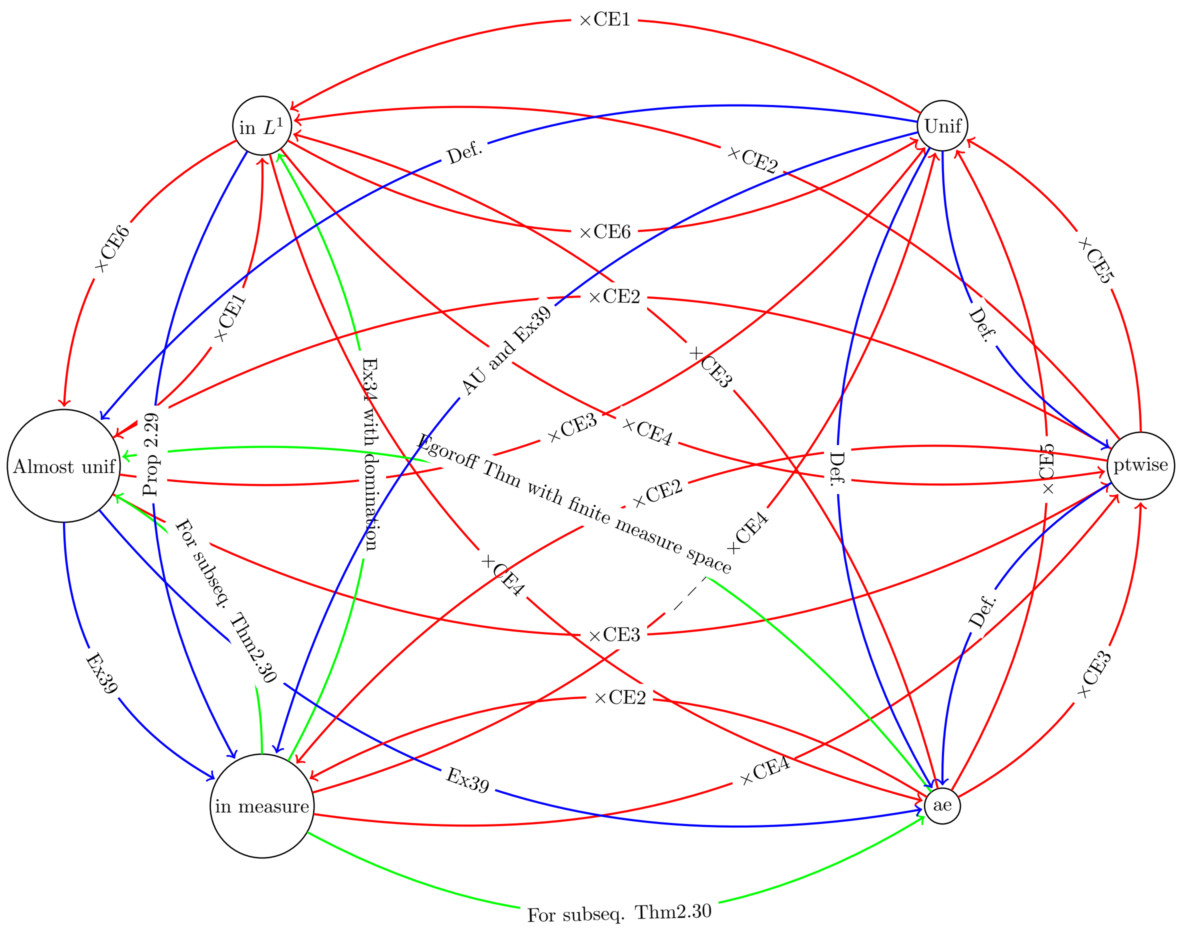


Figure 1: Modes of Convergence; Red represents counterexample. Green represents with some condition. Blue represents implying.

Now we can deal with the counter examples in the above complete digraph. For all example, we assume that $(\mathbb{R}, \mathcal{L}, m)$, Lebesgue measure space.

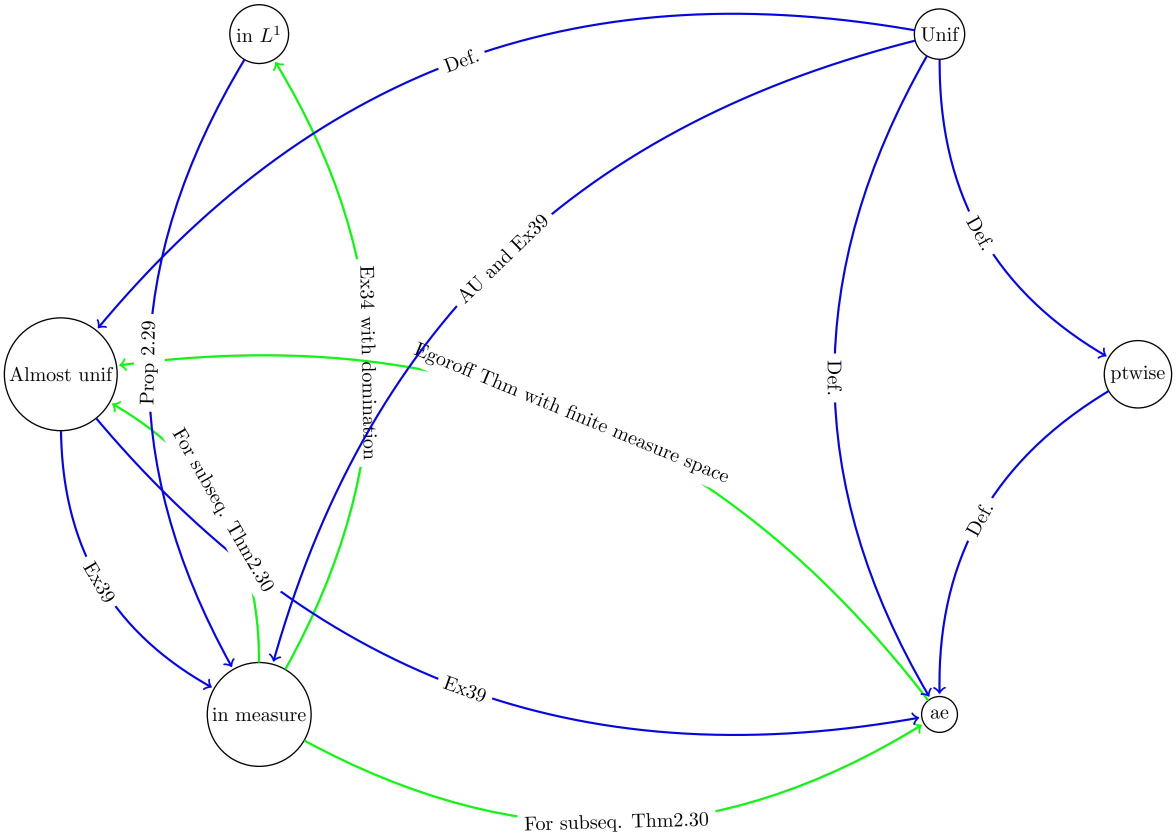


Figure 2: Modes of Convergence without counter example. Green represents with some condition. Blue represents implying. Non-edge implies there exists counter example.

Example 3.74 (CE1). *Let*

$$f_n = \frac{1}{n} 1_{(0,n)}.$$

Then, $f_n \to 0$ **uniformly** *(therefore* **almost uniformly***), but* $f_n \not\to 0$ **in** L^1*, since* $\forall n \in \mathbb{N}, \int |f_n| = 1$.

Example 3.75 (CE2). *Let*

$$f_n = \frac{1}{n} 1_{(n,n+1)}.$$

Then, $f_n \to 0$ **pointwisely***, but* $f_n \not\to 0$ **in** L^1*, since* $\forall n \in \mathbb{N}, \int |f_n| = 1$*. Also,* $\mu(\{x : |f_{n(x)-0|\geq \frac{1}{2}}\}) = 1$ *for all* $n \in \mathbb{N}$*. Hence,* $f_n \not\to 0$ **in measure.** *Also,* $f_n \not\to 0$ **almost uniformly** *since for* $\delta = \epsilon < 1$*, every* E *with* $\mu(E) < \delta$*,* $E^c \cap [n, n+1] \neq \emptyset$*, thus it gives*

$$|f_n 1_{E^c}(x)| = 1 > \epsilon \text{ at } x \in [n, n+1] \cap E^c.$$

Example 3.76 (CE3). *Let*

$$f_n = n 1_{[0, \frac{1}{n}]}$$

Then, $f_n \to 0$ **almost uniformly** *and* **a.e.** *(except* $\{\ 0\}$*), but* $f_n \not\to 0$ *in* L^1*, since* $\forall n \in \mathbb{N}, \int |f_n| = 1$*. Also,* $f_n \not\to 0$ **pointwisely** *since* $f_n(0) = 1$ *for any* $n \in \mathbb{N}$.

*Also note that $f_n \not\to 0$ **uniformly** or **pointwisely** since $f_n(0) = 1$ for any $n \in \mathbb{N}$.*

Example 3.77 (CE4)**.** *Let*

$$f_1 = 1_{[0,1]}, f_2 = 1_{[0,\frac{1}{2}]}, f_3 = 1_{[\frac{1}{2},1]}, f_4 = 1_{[0,\frac{1}{4}]}, f_5 = 1_{[\frac{1}{4},\frac{1}{2}]}, f_6 = 1_{[\frac{1}{2},\frac{3}{4}]}, f_7 = 1_{[\frac{3}{4},1]},$$

and in general,

$$f_n = 1_{[\frac{j}{2^k}, \frac{j+1}{2^k}]} \text{ where } n = 2^k + j \text{ with } 0 \leq j < 2^k.$$

*Then, $f_n \to 0$ **in** L^1. Also, for any $\epsilon > 0$, we set $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$, therefore,*

$$\mu(\{x \in \mathbb{R} : |f_n(x) - 0| \geq \epsilon\}) \leq \frac{1}{N} < \epsilon.$$

*Hence $f_n \to 0$ **in measure**. However, $\forall x \in [0,1], (f_n(x))_{n=1}^{\infty}$ diverges. Hence, $f_n \not\to 0$ **a.e.**, thus not **pointwisely.** (thus not **uniformly.**)*

Example 3.78 (CE5)**.** *Let*

$$f_n = \frac{x^2 + nx}{n}$$

Then, for any fixed $x \in \mathbb{R}$, and $\epsilon > 0$,

$$|f_n(x) - x| = |\frac{x^2}{n}| \to 0 \text{ as } n \to \infty.$$

*Thus, $f_n \to x$ **pointwisely** (hence **a.e.**). However, if x is not fixed, then for any $n \in \mathbb{N}$, there exists $x \in \mathbb{R}$ such that $x > n$, hence*

$$|f_n(x) - x| = |\frac{x^2}{n}| \geq |\frac{n^2}{n}| = |n| > \epsilon.$$

*Thus it does not **uniformly** converge.*

Example 3.79 (CE6)**.** *Let*

$$f_n = n1_{[0,\frac{1}{n^2}]}$$

*Then, $f_n \to 0$ **in** L^1, but $f_n \not\to 0$ not almost uniformly, since for any δ, we have $E = [0, \frac{1}{N^2}]$ with $\frac{1}{N^2} < \delta$, such that for any $n \geq N$,*

$$|1_{E^c} f_n - 0| = n \text{ for } x \in E^c \cap [0, \delta].$$

*Hence it is not **uniformly** convergent or **almost uniformly** convergent.*

Now we can deal with blue lines and green lines.

Proposition 3.80 (Proposition 2.29 in [1] p.61)**.** *If $f_n \to f$ **in** L^1 then $f_n \to f$ **in measure.***

Proof. Let $E_{n,\epsilon} = \{x : |f_n(x) - f(x)| \geq \epsilon\}$. We must show that $\forall \epsilon > 0$, $\mu(E_{n,\epsilon}) \to 0$. However,

$$\int |f_n - f| d\mu \geq \epsilon \mu(E_{n,\epsilon}).$$

Thus, by the sandwich lemma with L^1 convergence, we can conclude that $\mu(E_{n,\epsilon}) \to 0$ as $n \to \infty$. □

Theorem 3.81 (Revised Theorem 2.30 in [1] p.61)**.** *Suppose $(f_n)_{n=1}^{\infty}$ is **Cauchy in measure.** Then, $\exists$ a subsequence $(f_{n_k})_{k=1}^{\infty}$ that converges **almost uniformly** to some measurable function $f : X \to \mathbb{C}$. Moreover, $f_n \to f$ **in measure** and if $h : X \to \mathbb{C}$ is measurable and $f_n \to h$ in measure, then $h = f$ a.e.*

Note that the above theorem is more powerful than that in Folland, since almost uniform convergence implies a.e. convergence.

Proof. Given $j \geq 1$, let N_j be s.t. for any $n, m \geq N_j$,

$$\mu(\{x : |f_n(x) - f_m(x)| \geq 2^{-j}\}) < 2^{-j}.$$

Let $n_1 < n_2 < \cdots$ be a sequence in $\mathbb{N}$ such that $\forall j \in \mathbb{N}$, $n_j \geq N_j$. Write $g_j = f_{n_j}$. Let

$$E_j := \{x : |g_j(x) - g_{j+1}(x)| \geq 2^{-j}\}.$$

Then, $\mu(E_j) < 2^{-j}$ Now let $F_k = \bigcup_{j=k}^{\infty} E_j$. Then,

$$\mu(F_k) \leq \sum_{j=k}^{\infty} \mu(E_j) < 2^{1-k}.$$

Thus, if $x \in F_k^c$, then for any $l_1 > l_2 \geq k$,

$$|g_{l_1}(x) - g_{l_2}(x)| \underbrace{\leq}_{\text{Triangle ineq.}} \sum_{j=l_1+1}^{l_2-1} |g_j(x) - g_{j+1}(x)| < \sum_{j=l_1}^{l_2-1} 2^{-j} < 2^{1-l_1}. \tag{9}$$

This implies $(g_j)_{j=1}^{\infty}$ is Cauchy for all $x \in F_k^c$.

Now let $F = \bigcap_{k=1}^{\infty} F_k$. Then, $\mu(F) = 0$ and $\forall x \in F^c$, $(g_j(x))_{j=1}^{\infty}$ is Cauchy as a sequence of $\mathbb{C}$. Thus, we can well define $f(x)$ such that

$$f(x) = \begin{cases} \lim_{j\to\infty} g_j(x) & \text{if } x \in F^c \\ 0 & \text{otherwise.} \end{cases}$$

Claim 3.82. *$g_j \to f$ almost uniformly.*

Proof of the claim. From the equation (9), we can get $\forall x \in F_k^c, \forall l \geq k$,

$$|f(x) - g_l(x)| \leq 2^{1-l}$$

by taking $l_2 \to \infty$ in the equation (9). Thus, $g_l 1_{F_k^c} \to f 1_{F_k^c}$ uniformly, and from the fact that $\mu(F_k) < 2^{1-k}$ for any $k \in \mathbb{N}$, we can conclude that $g_l \to f$ almost uniformly. ☐

Claim 3.83. *$f_n \to f$ in measure.*

Proof of the claim. Define

$$G(n) := \{x : |f_n(x) - f(x)| \geq \epsilon\}, G_1(n,j) := \left\{x : |f_n(x) - g_j(x)| \geq \frac{\epsilon}{2}\right\}, \text{ and } G_2(j) := \left\{x : |g_j(x) - f(x)| \geq \frac{\epsilon}{2}\right\}.$$

Then, from the trianlge inequality, $\forall j \in \mathbb{N}$,

$$G(n) \subseteq G_1(n,j) \cup G_2(j). \implies \mu(G(n)) \leq \mu(G_1(n,j)) + \mu(G_2(j)).$$

Now it suffices to show that $\lim_{n\to\infty} \mu(G(n)) = 0$. Let $\delta > 0$. Since f_n is Cauchy in measure, for some $\frac{\epsilon}{2} < \delta$,$\exists N \in \mathbb{N}$ such that $\forall n \geq N$, which implies $n_j \geq N$, therefore

$$\mu(G_1(n,j)) = \mu(\left\{x : |f_n(x) - g_j(x)| \geq \frac{\epsilon}{2}\right\}) \leq \frac{\epsilon}{2} < \delta.$$

Since g_j converges in measure to f, for some $\frac{\epsilon}{2} < \delta$, $\exists J \in \mathbb{N}$ such that $\forall j \geq J$,

$$\mu(G_2(j)) = \mu(\left\{x : |g_j(x) - f(x)| \geq \frac{\epsilon}{2}\right\}) \leq \frac{\epsilon}{2} < \delta.$$

Thus, take j such that $n(j) \geq N$ and $j \geq J$, and take n such that $n \geq N$. Then, $\mu(G(n)) < 2\delta$, as desired. ☐

Claim 3.84. *if $h : X \to \mathbb{C}$ is measurable and $f_n \to h$ in measure, then $h = f$ a.e.*

Proof of the claim. If $f_n \to h$ in measure, then define

$$H := \{x : |f(x) - g(x)| \geq \epsilon\}, H_1(n) := \left\{x : |f(x) - f_n(x)| \geq \frac{\epsilon}{2}\right\}, \text{ and } H_2(n) := \left\{x : |f_n(x) - h(x)| \geq \frac{\epsilon}{2}\right\}.$$

Then, for all $n \in \mathbb{N}$,

$$H \subseteq H_1(n) \cup H_2(n) \implies \mu(H) \leq \mu(H_1(n)) + \mu(H_2(n)).$$

Since $f_n \to f$ in measure and $f_n \to h$ in measure, $\mu(H_1(n)) + \mu(H_2(n)) \to 0$ as $n \to \infty$. Thus, by the sandwich lemma, $\mu(H) \to 0$ as $n \to \infty$. This implies $f(x) = g(x)$ when $x \in H^c$, thus $f = g$ a.e. □

□

Corollary 3.85 (Corollary 2.32 in [1] p.62)**.** *If $f_n \to f$* **in** *L^1, then there exists a subseqeunce $(f_{n_j})_{j=1}^{\infty}$ such that $f_{n_j} \to f$ a.e.*

Proof. By theorem 2.29, $f_n \to f$ in measure, and by theorem 2.30, there exists a subseqeunce $(f_{n_j})_{j=1}^{\infty}$ such that $f_{n_j} \to f$ almost uniformly, which implies $f_{n_j} \to f$ a.e. □

Also note that [1] doesn't have 2.31 Statement. Weird.

Theorem 3.86 (Egoroff's theorem, Theorem 2.33 in [1] p.62)**.** *Suppose $\mu(X) < +\infty$ and $(f_n : X \to \mathbb{C})_{n=1}^{\infty}$ be a sequence of measurable functions such that $f_n \to f$ a.e. Then, $f_n \to f$ almost uniformly, i.e., $\forall \delta > 0$, $\exists E \in m$ such that $\mu(E) < \delta$ and $f_n 1_{E^c} \to f 1_{E^c}$ uniformly.*

Proof. Let $A \in m$ satisfy $\mu(A) = 0$ and $\forall x \in A^c$, $f_n(x) \to f(x)$ pointwisely. Let

$$E_n(k) = \left\{x \in X : \exists p \geq n \text{ with } |f_p(x) - f(x)| \geq \frac{1}{k}\right\}.$$

Then,

$$E_n(k) \supseteq E_{n+1}(k) \supseteq \cdots$$

and

$$\bigcap_{n=1}^{\infty} E_n(k) = \{x \in X : \text{ for infinitely many } p \in \mathbb{N}, |f_p(x) - f(x)| \geq \frac{1}{k}\} \subseteq A.$$

Thus, $\mu(\bigcap_{n=1}^{\infty} E_n(k)) = 0$. Since X is finite measure space, by continuity from above of μ,

$$\lim_{n\to\infty} \mu(E_n(k)) = 0.$$

Thus, given $\delta > 0$, $\forall k \in \mathbb{N}$, $\exists n_k$ such that

$$\mu(E_{n_k}(k)) < \delta 2^{-k}.$$

Let $E_\delta = \bigcup_{k=1}^{\infty} E_{n_k}(k)$. Then,

$$\mu(E_\delta) \leq \sum_{k=1}^{\infty} \mu(E_{n_k}(k)) < \delta \sum_{k=1}^{\infty} 2^{-k} = \delta.$$

Claim 3.87. *$f_n 1_{E_\delta^c} \to f 1_{E_\delta^c}$ uniformly.*

Proof of the claim. If $x \in E_\delta^c$ then $x \in E_{n_k}(k)^c$ for any $k \in \mathbb{N}$, thus $\forall p \geq n_k$,

$$|f_p(x) - f(x)| < \frac{1}{k},$$

as desired. □

□

Homework 3.88 (Exercise 34 in [1] p.63)**.** *Suppose $|f_n| \le g \in L^1$ and $f_n \to f$ in measure. Then,*

(a) $\int f = \lim_{n\to\infty} \int f_n$

(b) $f_n \to f$ *in* L^1.

Proof. 1. $\int f = \lim_{n\to\infty} \int f_n$

Proof. Suppose that f_n is real valued function first. Then, by the exercise 33's argument, there exists a subsequence of f_n, say $f_{n_{j_k}}$ which converges to f pointwisely almost everywhere. Since $f_{n_{j_k}} \in L^1$ for any $k \in \mathbb{N}$, and dominated by g, thus by the Dominated Convergence Theorem, $f \in L^1$.

Now note that $g - f_n \ge 0$ and $g + f_n \ge 0$ from the given condition that $|f_n| \le g$. Also, $g - f_n \to g - f$ in measure, since for any ϵ,

$$\{x \in X : |g - f_n - (g - f)| \ge \epsilon\} = \{x \in X : |f_n - f| \ge \epsilon\} = \{x \in X : |g + f_n - (g + f)| \ge \epsilon\}.$$

Thus, by the exercise 33, we have two inequalities such that

$$\begin{aligned} \int g + f \le \quad \liminf_{n\to\infty} \textstyle\int g + f_n \quad &= \int g + \liminf_{n\to\infty} \int f_n \\ \int g - f \le \quad \liminf_{n\to\infty} \textstyle\int g - f_n \quad &= \int g - \limsup_{n\to\infty} \int f_n. \end{aligned}$$

Thus,

$$\limsup_{n\to\infty} \int f_n \le \int f \le \liminf_{n\to\infty} \int f_n$$

implies $\lim_{n\to\infty} f_n$ exists and $\lim_{n\to\infty} f_n = \int f$. It is actually the same argument of proving Dominated Convergence Theorem, when sequence coverges in measure.

For the case of f is complex function, then $f_n = Ref_n + iImf_n$, thus $\int Ref = \lim_{n\to\infty} \int Ref_n$ and $\int Imf = \lim_{n\to\infty} \int Imf_n$. Therefore,

$$\int f = \int Ref + i \int Imf = \lim_{n\to\infty} \int (Ref_n + iImf_n) = \lim_{n\to\infty} \int f_n,$$

as desired. □

2. $f_n \to f$ ***in*** L^1**.**

Proof. First of all, $|f_n - f| \to 0$ in measure, since

$$\{x \in X : ||f_n(x) - f(x)| - 0| \ge \epsilon\} = \{x \in X : |f_n(x) - f(x)| \ge \epsilon\},$$

for any ϵ. Also, since $f_{n_{j_k}} \to f$ pointwisely a.e. and $|f_{n_{j_k}}| \le g$ for any $k \in \mathbb{N}$, thus $|f| \le g$. Hence, $|f_n - f| \le |f_n| + |f| \le 2g \in L^1$. Thus, by the part (a),

$$0 = \int 0 = \lim_{n\to\infty} \int |f_n - f|.$$

Thus, by the definition of L^1 convergence, $f_n \to f$ in L^1. □

□

Homework 3.89 (Exercise 39 in [1] p.63)**.** *If $f_n \to f$ almost uniformly, then $f_n \to f$ in a.e. and in measure.*

Proof. $f_n \to f$ almost uniformly implies that $\forall \delta > 0, \exists E \in m$ such that $\mu(E) < \delta$ and $1_{E^c} f_n$ converges uniformly to $1_{E^c} f$. Thus, let $\delta = \frac{1}{n}$ and E_n is corresponding set satisfying the condition of almost uniformly convergence. Then, let $E = \bigcap_{n=1}^{\infty} E_n$. Then,

$$\mu(\bigcap_{n=1}^{\infty} E_n) < \frac{1}{n}, \forall n \in \mathbb{N},$$

therefore $\mu(\bigcap_{n=1}^{\infty} E_n) = 0$ but $1_{E^c} f_n$ converges uniformly to $1_{E^c} f$. Since uniform convergence implies pointwise convergence, we can conclude that $f_n \to f$ a.e. except E, which is null set.

Also, for any $\epsilon > 0$, we know that there exists $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$E_n \supseteq \{x \in E_n^c : |f_n(x) - f(x)| \geq \epsilon\} = \emptyset.$$

Thus,

$$\{x \in X : |f_n(x) - f(x)| \geq \epsilon\} \subseteq E_n \implies \mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \leq \mu(E_n) \leq \frac{1}{n},$$

for any $n \geq N$. Thus take n such that $\epsilon > \frac{1}{n}$, then we can conclude that for any $m > n$,

$$\mu(\{x \in X : |f_m(x) - f(x)| \geq \epsilon\}) \leq \mu(E_m) \leq \frac{1}{m} < \frac{1}{n} < \epsilon.$$

Thus, $f_n \to f$ in measure. $\square$

3.5 Product Measures

Let (X, m, μ) and (Y, n, ν) be measure spaces. Recall that

Definition 3.90. *$m \otimes n$ is the σ-algebra of subsets of $X \times Y$ generated by $\{A \times Y : A \in m\} \cup \{X \times B : B \in n\}$.*

Definition 3.91 (Rectangle)**.** *A* **(measurable) rectangle** *in $X \times Y$ is $A \times B$ for $A \in m, B \in n$. Let $\mathfrak{a}$ be the algebra of sets consisting of finite disjoint unions of measurable rectangles.*

Check that $\mathfrak{a}$ is an algebra. Recall that $\mathfrak{a}$ is an ***algebra*** of subsets of X if

(i) $n \in \mathbb{N}, E_1, \cdots, E_n \in \mathfrak{a} \implies \cup_{j=1}^{n} E_j \in \mathfrak{a}$.

(ii) $E \in \mathfrak{a} \implies E^c = X \setminus E \in \mathfrak{a}$.

Note that

$$(A \times B) \cap (E \times F) = (A \cap E) \times (B \cap F).$$

So, for disjoint union of $(A_i \times B_i)_{i=1}^{n}$ and $(E_j \times F_j)_{j=1}^{m}$,

$$\left(\bigcup_{i=1}^{n} A_i \times B_i\right) \cap \left(\bigcup_{j=1}^{m} E_i \times E_j\right) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \cap E_j) \times (B_i \cap F_j),$$

and $(A_i \cap E_j) \times (B_i \cap F_j)$s are disjoint. So $\mathfrak{a}$ is closed under taking intersection. From the fact that

$$(A \times B)^c = (A^c \times B^c) \cup (A \times B^c) \cup (A^c \times B),$$

we know that

$$\left(\bigcup_{i=1}^{n} (A_i \times B_i)\right)^c = \bigcap_{i=1}^{n} (A_i \times B_i)^c \in \mathfrak{a},$$

thus $\mathfrak{a}$ is closed under complement.

For (i), let $n \in \mathbb{N}, E_1, \cdots, E_n \in \mathfrak{a}$. Then, define $F_k = E_k \setminus \bigcap_{j=1}^{k-1} E_j \in \mathfrak{a}$. Then, each F_k are disjoint, hence

$$\bigcup_{k=1}^{n} E_k = \bigcup_{k=1}^{n} F_k \in \mathfrak{a},$$

as desired. $\square$

Our goal is ***to find a measure on*** $m \otimes n$ ***such that***

$$A \times B \mapsto \mu(A)\mu(B).$$

We would like to define $\pi : \mathfrak{a} \to [0, +\infty]$ such that $\pi(\bigcup_{i=1}^{n} A_i \times B_i) = \sum_{i=1}^{n} \mu(A_i)\mu(B_i)$ for (finite) disjoint family of rectangles $A_i \times B_i$.

Claim 3.92. *Let $\pi : \mathfrak{a} \to [0, +\infty]$ by $\pi(\bigcup_{i=1}^{n} A_i \times B_i) = \sum_{i=1}^{n} \mu(A_i)\mu(B_i)$ for (finite) disjoint family of rectangles $A_i \times B_i$. Then, π is well-defined.*

Proof. Let $R \in \mathfrak{a}$. Then $R = \bigcup_{i=1}^{n} A_i \times B_i$ for some finite disjoint family rectangles, by construction of $\mathfrak{a}$. Then,

$$1_{R(x,y)} = \sum_{i=1}^{\infty} 1_{A_i(x)} \cdot 1_{B_i(x)}.$$

Thus,

$$\sum_{i=1}^{n} \mu(A_i)\nu(B_i) = \sum_{i=1}^{n} \int 1_{A_i(x)} d\mu(x) \int 1_{B_i(y)} d\nu(y) \underbrace{=}_{\text{Thm 2.13}} \int\int 1_{A_i(x)} 1_{B_i(y)} d\mu(x) d\nu(y) = \int\int 1_{R(x,y)} d\mu(x) d\nu(y).$$

It means that any finite family of disjoint rectangles representing R, π gives the same value. Thus, we can conclude that π doesn't depends on a rectangle, which means π is well-defined function from $\mathfrak{a} \to [0, +\infty]$. $\square$

Claim 3.93. *π is a premeasure.*

Proof. Suppose $R \in \mathfrak{a}$ such that $R = \bigcup_{p=1}^{\infty} S_p$ for some $S_p \in \mathfrak{a}$. We want to show that

$$\pi(R) = \sum_{p=1}^{\infty} \pi(S_p).$$

Note that

$$1_R(x, y) = \sum_{p=1}^{\infty} 1_{S_p}(x, y).$$

Thus, let $f_n = \sum_{p=1}^{n} 1_{S_p}(x, y)$. Then, $\lim_{n\to\infty} f_n = 1_R(x, y)$ and $f_n \leq f_{n+1}$ for any $n \in \mathbb{N}$. Thus, we can apply the Monotone Convergence Theorem. Hence,

$$\pi(R) = \int \left(\int 1_R(x, y) d\mu(x) \right) d\nu(y) = \int \left(\int \sum_{p=1}^{\infty} 1_{S_p}(x, y) d\mu(x) \right) d\nu(y) \underbrace{=}_{\text{MCT}} \int \sum_{p=1}^{\infty} \left(\int 1_{S_p}(x, y) d\mu(x) \right) d\nu(y).$$

And we need a lemma that each $\left(\int 1_{S_p}(x, y) d\mu(x) \right)$ is measurable by $d\nu(y)$.

Lemma 3.94. *If $S \in \mathfrak{a}$, then $y \mapsto \left(\int 1_S(x, y) d\mu(x) \right)$ is ν-measurable function.*

Proof of the Lemma. If $S = A \times B$, then

$$\int 1_S(x, y) d\mu(x) = \begin{cases} \mu(A) & \text{if } y \in B \\ 0 & \text{if } y \notin B. \end{cases}$$

In general, if $S \in \mathfrak{a}$, by the proposition 2.6 stating that sum of measurable function is also measurable, the given function is measurable. $\square$

Thus, let $g_n(y) := \sum_{p=1}^{\infty} \left(\int 1_{S_p}(x,y)d\mu(x)\right)$. Then by the Lemma, g_n is ν-measurable, and $\lim_{n\to\infty} g_n = \sum_{p=1}^{\infty} \left(\int 1_{S_p}(x,y)d\mu(x)\right)$ and $g_n(y) \le g_{n+1}(y)$ for any $n \in \mathbb{N}$. Thus, we can apply the Monotone Convergence Theorem, hence,

$$\pi(R) \underbrace{=}_{\text{MCT}} \int \sum_{p=1}^{\infty} \left(\int 1_{S_p}(x,y)d\mu(x)\right) d\nu(y) \underbrace{=}_{\text{MCT}} \sum_{p=1}^{\infty} \int \left(\int 1_{S_p}(x,y)d\mu(x)\right) d\nu(y) = \sum_{p=1}^{\infty} \pi(S_p),$$

as required. Since $\pi(\emptyset) = 0$ clearly, thus π is a premeasure on $\mathfrak{a}$. □

Claim 3.95. *$m \otimes n = \sigma$-alg$(\mathfrak{a})$*

Proof. Apply the Proposition 1.3 in [1][p.23]. □

Claim 3.96. *There exists a measure $\mu \times \nu$ on $m \otimes n$ that extends π. If μ and ν are σ-finite, then π is also σ-finite, thus $\mu \times \nu$ is the unique measure on $m \otimes n$ satisfying $A \times B \mapsto \mu(A) \cdot \mu(B)$ for any $A \in m, B \in n$.*

Proof. Apply the theorem 1.14 in [1][p.31]. □

Thus, $\mu \times \nu(A \times B) = \mu(A) \times \nu(B)$ for any $A \in m, B \in n$.

Definition 3.97. *We call $\mu \times \nu$ obtained by the above claim is* **the product measure** *of μ and ν.*

Definition 3.98. *Let $E \in m \otimes n$. Then we denote E_x be* **x-section** *and E^y be* **y-section** *of E as below;*

$$\text{For } x \in X, E_x := \{y \in Y : (x,y) \in E\} \text{ and For } y \in X, E^y := \{x \in X : (x,y) \in E\}.$$

Also, given $f : X \times Y \to S$, for any set S, we define its x-section and y-sections by

$$\text{For } x \in X, f_x(y) : X \to S \text{ by } f_x(y) := f(x,y) \text{ and For } y \in X, f^y(x) : Y \to S \text{ by } f^y(x) := f(x,y).$$

Theorem 3.99 (Proposition 2.34 in [1] p.65). *(i) If $E \in m \otimes n$, then $\forall x \in X, E_x \in n$ and $\forall y \in Y, E^y \in m$*

(ii) If $(S, \mathcal{S})$ is any measurable space and if $f : X \times Y \to S$ is $(m \otimes n, \mathcal{S})$-measurable function, then $\forall x \in X, f_x$ is $(n, \mathcal{S})$-measurable and $\forall y \in Y, f^y$ is $(m, \mathcal{S})$-measurable.

Proof. Let

$$R = \{E \subseteq X \times Y : \forall x \in X, E_x \in n \text{ and } \forall y \in Y, E^y \in m\}.$$

Since $\forall A \in m, B \in n$,

$$(A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \text{ and } (A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B \end{cases}$$

Thus, in any case, $(A \times B)_x \in n, (A \times B)^y \in m$. Thus, $A \times B \in R$. Then, it suffices to show that R is σ-algebra; if we show this, then $m \otimes n \subseteq R$ since $m \times n \in R$, thus the statement holds.

Let $E \in R$. Then,

$$(E^c)_x = \{y : (x,y) \in E^c\} = \{y : (x,y) \notin E\} = \{y : (x,y) \in E\}^c = (E_x)^c \in n$$

$$(E^c)^y = \{x : (x,y) \in E^c\} = \{x : (x,y) \notin E\} = \{x : (x,y) \in E\}^c = (E^y)^c \in m$$

Thus, $E^c \in R$. Also, let $E_1, E_2, \cdots \in R$ which is disjoint sequence. Then, we want to show $E := \bigcup_{n=1}^{\infty} E_n \in R$. Note that

$$E_x = \{y : (x,y) \in \bigcup_{n=1}^{\infty} E_n\} = \bigcup_{n=1}^{\infty} \{y : (x,y) \in E_n\} = \bigcup_{n=1}^{\infty} (E_n)_x \in n$$

$$E_y = \{x : (x,y) \in \bigcup_{n=1}^{\infty} E_n\} = \bigcup_{n=1}^{\infty} \{x : (x,y) \in E_n\} = \bigcup_{n=1}^{\infty} (E_n)^y \in m$$

Thus R is a σ-algebra. Hence the statement holds as mentioned above.

For (ii), suppose $f : X \times Y \to S$ is $(m \otimes n, \mathcal{S})$-measurable function. Let $x \in X$, $G \in \mathcal{S}$. Then it suffices to show that $f_x^{-1}(G) \in n$ and $(f^y)^{-1}(G) \in m$. Note that

$$f_x^{-1}(G) = \{y : f_x(x,y) \in G\} = \{y : (x,y) \in f^{-1}(G)\} = (f^{-1}(G))_x$$

$$(f^y)^{-1}(G) = \{x : f^y(x,y) \in G\} = \{x : (x,y) \in f^{-1}(G)\} = (f^{-1}(G))^y$$

Since f is $(m \otimes n, \mathcal{S})$-measurable, $f^{-1}(G) \in m \otimes n$ and by part (i), $(f^{-1}(G))_x \in n$ and $(f^{-1}(G))^y \in m$. Thus, the conclusion holds. $\square$

Definition 3.100. *A* **monotone class on** X *is a subset* $\mathcal{C}$ *of* $\mathcal{P}(X)$ *that is closed under taking*

- *countable increasing unions, i.e., if there exist* $E_1, E_2, \cdots \in \mathcal{C}$ *with* $E_1 \subseteq E_2 \subseteq \cdots$ *then* $\bigcup_{n=1}^{\infty} E_n \in \mathcal{C}$.
- *countable decreasing intersections, i.e., if there exist* $E_1, E_2, \cdots \in \mathcal{C}$ *with* $E_1 \supseteq E_2 \supseteq \cdots$ *then* $\bigcap_{n=1}^{\infty} E_n \in \mathcal{C}$.

Remark 3.101. *(i) Every* σ*-algebra is a monotone class.*

(ii) If Λ *is a set and* $\forall \lambda \in \Lambda$, $\mathcal{C}_\lambda \subseteq \mathcal{P}(X)$ *is a monotone class, then* $\bigcap_{\lambda \in \Lambda} \mathcal{C}_\lambda$ *is a monotone class.*

(iii) Given $\mathcal{F} \subseteq \mathcal{P}(X), \exists$ *the smallest monotone class on* X *containing* $\mathcal{F}$*. This is called* **the monotone class generated by** $\mathcal{F}$**.**

Proof. For (ii), note that if $E_1, \cdots$ is countable increasing sequence (or decreasing sequence) in $\bigcap_{\lambda \in \Lambda} \mathcal{C}_\lambda$, then they are an countable increasing sequence in each $\mathcal{C}_\lambda$, thus their union (or intersection) is in each $\mathcal{C}_\lambda$, hence the union (or intersection) is in $\bigcap_{\lambda \in \Lambda} \mathcal{C}_\lambda$.

For (iii), just define $A = \bigcap_{F \in \mathcal{C}} \mathcal{C}$. Then by part (ii), it is also a monotone class, and it is the smallest one. $\square$

Lemma 3.102 (The monotone class lemma, theorem 2.35 in [1] p.66)**.** *If* $\mathfrak{a}$ *is an algebra of subsets of* X*, and* $\mathcal{C}$ *be a monotone class generated by* $\mathfrak{a}$*, then* $\mathcal{C} = \sigma\text{-}alg\,(\mathfrak{a})$.

Proof. By definition of σ-algebra, $\mathcal{C} \subseteq \sigma\text{-alg}\,(\mathfrak{a})$ is clear. To show reversed inclusion, it suffices to show that $\mathcal{C}$ is a σ-algebra.

Given $E \subseteq X$, let

$$\mathcal{C}(E) = \{F \in \mathcal{C} : F \setminus E \in \mathcal{C}, E \setminus F \in \mathcal{C}, E \cap F \in \mathcal{C}\}.$$

Claim 3.103. $\mathcal{C}(E)$ *is a monotone class.*

Proof of the claim. If $F_j \in \mathcal{C}(E)$ with $F_j \subseteq F_{j+1}$ for all $j \in \mathbb{N}$, then let $F := \bigcup_{j=1}^{\infty} F_j$. We want to show $F \in \mathcal{C}(E)$ to show that $\mathcal{C}(E)$ satisfies closed under countable increasing union. Note that

$$F \setminus E = \bigcup_{j=1}^{\infty} (F_j \setminus E) \text{ and } E \setminus F = \bigcap_{j=1}^{\infty} (E \setminus F_j), \text{ and } E \cap F = \bigcup_{j=1}^{\infty} (E \cap F_j).$$

Since $F_j \setminus E$ and $E \cap F_j$ are countable increasing in $\mathcal{C}$ since $F_j \in \mathcal{C}(E)$ for all $j \in \mathbb{N}$, its union is in $\mathcal{C}$. Also, since $E \setminus F_j$ is countable decreasing in $\mathcal{C}$ since $F_j \in \mathcal{C}(E)$ for all $j \in \mathbb{N}$, its intersection is in $\mathcal{C}$. Thus,

$$F \setminus E = \bigcup_{j=1}^{\infty} F_j \setminus E \in \mathcal{C} \text{ and } E \setminus F = \bigcap_{j=1}^{\infty} E \setminus F_j \in \mathcal{C}, \text{ and } E \cap F = \bigcup_{j=1}^{\infty} E \cap F_j \in \mathcal{C}.$$

Thus, $\mathcal{C}(E)$ satisfies countable increasing union condition.

Similarly, let $F_j \in \mathcal{C}(E)$ with $F_j \supseteq F_{j+1}$ for all $j \in \mathbb{N}$, then let $F := \bigcap_{j=1}^{\infty} F_j$. We want to show $F \in \mathcal{C}(E)$ to show that $\mathcal{C}(E)$ satisfies closed under countable decreasing intersection. Note that

$$F \setminus E = \bigcap_{j=1}^{\infty}(F_j \setminus E) \text{ and } E \setminus F = \bigcup_{j=1}^{\infty}(E \setminus F_j), \text{ and } E \cap F = \bigcap_{j=1}^{\infty}(E \cap F_j).$$

Since $F_j \setminus E$ and $E \cap F_j$ are countable decreasing in $\mathcal{C}$ since $F_j \in \mathcal{C}(E)$ for all $j \in \mathbb{N}$, its intersection is in $\mathcal{C}$. Also, since $E \setminus F_j$ is countable increasing in $\mathcal{C}$ since $F_j \in \mathcal{C}(E)$ for all $j \in \mathbb{N}$, its union is in $\mathcal{C}$. Thus,

$$F \setminus E = \bigcap_{j=1}^{\infty}(F_j \setminus E) \in \mathcal{C} \text{ and } E \setminus F = \bigcup_{j=1}^{\infty}(E \setminus F_j) \in \mathcal{C}, \text{ and } E \cap F = \bigcap_{j=1}^{\infty}(E \cap F_j) \in \mathcal{C}.$$

Thus, $\mathcal{C}(E)$ also satisfies countable decreasing intersection condition. Hence it is a monotone class. □

Claim 3.104. *Let $E, F \subset X$, then $F \in \mathcal{C}(E) \iff E \in \mathcal{C}(F)$*

Proof of the claim.

$$F \in \mathcal{C}(E) \iff F \setminus E \in \mathcal{C}, E \setminus F \in \mathcal{C}, \text{ and } E \cap F \in \mathcal{C} \iff E \in \mathcal{C}(F).$$

□

Claim 3.105. *If $B \in \mathfrak{a}$ then $\mathcal{C} \subseteq \mathcal{C}(B)$.*

Proof of the claim. Since $\mathcal{C}(B)$ is a monotone class by claim 3.103, it suffices to show that $\mathfrak{a} \subseteq \mathcal{C}(B)$. Let $A \in \mathfrak{a}$ then since $\mathfrak{a}$ is an algebra,

$$A \setminus B, B \setminus A, A \cap B \in \mathfrak{a} \subseteq \mathcal{C}.$$

Thus, $A \in \mathcal{C}(B)$ by construction of $\mathcal{C}(B)$. Since A was arbitrarily chosen, $\mathcal{C} \subseteq \mathcal{C}(B)$. □

Claim 3.106. *For all $E \in \mathcal{C}, \mathcal{C} = \mathcal{C}(E)$.*

Proof of the claim. By construction of $\mathcal{C}(E)$, $\mathcal{C} \supseteq \mathcal{C}(E)$. To show the other direction, it suffices to show that $\mathfrak{a} \subseteq \mathcal{C}(E)$, then the monotone class generated by $\mathfrak{a}$, which is $\mathcal{C}$ is contained in $\mathcal{C}(E)$.

Let $B \in \mathfrak{a}$. By the claim 3.105,

$$E \in \mathcal{C} \subseteq \mathcal{C}(B).$$

Then by the claim 3.104,

$$E \in \mathcal{C}(B) \implies B \in \mathcal{C}(E).$$

Since B was arbitrarily chosen, $\mathfrak{a} \subseteq \mathcal{C}(E)$, as desired. □

Claim 3.107. *$\mathcal{C}$ is an algebra.*

Proof of the claim. Let $E, F \in \mathcal{C}$. Then, by the claim 3.106, $E \in \mathcal{C} = \mathcal{C}(F)$. Thus, $E \setminus F, F \setminus E$, and $E \cap F \in \mathcal{C}$. Since $X \in \mathfrak{a} \subseteq \mathcal{C}$, we have $E^c \in \mathcal{C}$, by taking $F = X$. Thus, $E \cup F = (E^c \cap F^c) \in \mathcal{C}$ for any $E, F \in \mathcal{C}$, since $E^c \in \mathcal{C} = \mathcal{C}(F^c)$ by the claim 3.106. Thus $\mathcal{C}$ closed under complements and finite union and intersection, thus it is an algebra. □

Claim 3.108. *$\mathcal{C}$ is a σ-algebra.*

Proof of the claim. If $E_1, E_2, \dots \in \mathcal{C}$, then $\forall n \in \mathbb{N}$, let

$$G_n := \bigcup_{j=1}^{n} E_j \in \mathcal{C}$$

since $\mathcal{C}$ is an algebra. Since $\mathcal{C}$ is also a monotone class, and G_n forms a countable increasing sequence,

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{n=1}^{\infty} G_n \in \mathcal{C}.$$

$\square$

$\square$

Theorem 3.109 (Theorem 2.36 in [1] p.66)**.** *Suppose $(X, m, \mu)$and(Y, n, ν) are σ-finite. Let $E \in m \otimes n$. Then the functions $x \mapsto \nu(E_x), y \mapsto \mu(E^y)$ are measurable and*

$$\int \nu(E_x) d\mu(x) = \mu \times \nu(E) = \int \mu(E^y) d\nu(y).$$

Proof. Suppose $\mu(X) < +\infty$ and $\nu(Y) < +\infty$. Let $\mathcal{D}$ be the set of all $E \in m \otimes n$ such that the conclusion holds.

Claim 3.110. *If $A \in m$ and $B \in n$, then $E = A \times B \in \mathcal{D}$.*

Proof of the claim. Note that

$$E_x := \{y : (x, y) \in E\} = \begin{cases} \emptyset & \text{if } x \notin A \\ B & \text{if } x \in A \end{cases} \text{ and } E^y := \{x : (x, y) \in E\} = \begin{cases} \emptyset & \text{if } y \notin B \\ A & \text{if } y \in B. \end{cases}$$

Thus,

$$\nu(E_x) = \begin{cases} 0 & \text{if } x \notin A \\ \nu(B) & \text{if } x \in A \end{cases} = \nu(B) 1_A(x) \text{ and } \mu(E^y) = \begin{cases} 0 & \text{if } y \notin B \\ \mu(A) & \text{if } y \in B \end{cases} = \mu(A) 1_B(y).$$

Thus, $x \mapsto \nu(E_x) = \nu(B) 1_A(x)$ is clearly μ-measurable and $y \mapsto \mu(E^y) = \mu(A) 1_B(y)$ is ν-measurable. Hence,

$$\int \nu(E_x) d\mu(x) = \nu(B)\mu(A) = \mu \times \nu(E) = \nu(B)\mu(A) = \int \mu(E^y) d\nu(y).$$

Hence $E \in \mathcal{D}$. $\square$

Claim 3.111. *If $E = \bigcup_{i=1}^{n} A_i \times B_i$ is a finite disjoint union where $A_i \in m, B_i \in n$, then $E \in \mathcal{D}$.*

Proof. Note that

$$E_x := \{y : (x, y) \in E\} = \begin{cases} \emptyset & \text{if } x \notin \bigcup_{i=1}^{n} A_i \\ B_i & \text{if } x \in A_i \end{cases} \text{ and } E^y := \{x : (x, y) \in E\} = \begin{cases} \emptyset & \text{if } y \notin \bigcup_{i=1}^{n} B_i \\ A_i & \text{if } y \in B_i. \end{cases}$$

Thus,

$$\nu(E_x) = \begin{cases} 0 & \text{if } x \notin \bigcup_{i=1}^{n} A_i \\ \nu(B_i) & \text{if } x \in A_i \end{cases} = \sum_{i=1}^{n} \nu(B_i) 1_{A_i}(x) \text{ and } \mu(E^y) = \begin{cases} 0 & \text{if } y \notin \bigcup_{i=1}^{n} B_i \\ \mu(A_i) & \text{if } y \in B_i \end{cases} = \sum_{i=1}^{n} \mu(A_i) 1_{B_i}(y).$$

Thus, $x \mapsto \nu(E_x) = \sum_{i=1}^n \nu(B_i) 1_{A_i}(x)$ is clearly μ-measurable since it is a simple fuction, and by the same reason, $y \mapsto \mu(E^y) = \sum_{i=1}^n \mu(A_i) 1_{B_i}(y)$ is ν-measurable. Hence,

$$\int \nu(E_x) d\mu(x) = \sum_{i=1}^n \nu(B_i)\mu(A_i) = \mu \times \nu(E) = \sum_{i=1}^n \nu(B_i)\mu(A_i) = \int \mu(E^y) d\nu(y).$$

Hence $E \in \mathcal{D}$. $\square$

Claim 3.112. *$\mathcal{D}$ is a monotone class.*

Proof. Suppose $E_n \in \mathcal{D}$ with $E_n \subseteq E_{n+1}$ for each $n \in \mathbb{N}$, and let $E := \bigcup_{n=1}^\infty E_n$. Then,

$$E_x = \{y : (x,y) \in E\} = \bigcup_{n=1}^\infty \{y : (x,y) \in E_n\} = \bigcup_{n=1}^\infty (E_n)_x.$$
$$E^y = \{x : (x,y) \in E\} = \bigcup_{n=1}^\infty \{x : (x,y) \in E_n\} = \bigcup_{n=1}^\infty (E_n)^y.$$

By continuity from below, we know that

$$\lim_{n\to\infty} \nu((E_n)_x) = \nu\left(\bigcup_{n=1}^\infty (E_n)_x\right) = \nu(E_x) \text{ and } \lim_{n\to\infty} \mu((E_n)^y) = \mu\left(\bigcup_{n=1}^\infty (E_n)^y\right) = \nu(E^y).$$

Thus $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ is the limit of measurable functions, therefore by the corollary 2.9, it is measurable.

Also, let $f_n := \nu((E_n)_x)$ and $g_n := \mu((E_j)^y)$. Then, $f_n \le f_{n+1}, g_n \le g_{n+1}$ for any $n \in \mathbb{N}$, since $E_n \subseteq E_{n+1}$. And as shown above,

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \nu((E_n)_x) = \nu(E_x) \text{ and } \lim_{n\to\infty} g_n = \lim_{n\to\infty} \mu((E_n)^y) = \mu(E^y).$$

Thus we can apply the Monotone Convergence Theorem on the sequence of f_n and g_n. Thus,

$$\int \nu(E_x) d\mu(x) \underbrace{=}_{\text{MCT}} \lim_{n\to\infty} \int \nu((E_n)_x) d\mu(x) \underbrace{=}_{E_n \in \mathcal{D}, \forall n \in \mathbb{N}} \lim_{n\to\infty} \mu \times \nu(E_n) \underbrace{=}_{\text{continuity from below}} \mu \times \nu(E)$$

and

$$\int \mu(E^y) d\nu(y) \underbrace{=}_{\text{MCT}} \lim_{n\to\infty} \int \mu((E_n)^y d\nu(y) \underbrace{=}_{E_n \in \mathcal{D}, \forall n \in \mathbb{N}} \lim_{n\to\infty} \mu \times \nu(E_n) \underbrace{=}_{\text{continuity from below}} \mu \times \nu(E)$$

Thus, $E \in \mathcal{D}$.

Conversely, suppose $E_n \in \mathcal{D}$ with $E_n \supseteq E_{n+1}$ for each $n \in \mathbb{N}$, and let $E := \bigcap_{n=1}^\infty E_n$. Then,

$$E_x = \{y : (x,y) \in E\} = \bigcap_{n=1}^\infty \{y : (x,y) \in E_n\} = \bigcap_{n=1}^\infty (E_n)_x.$$
$$E^y = \{x : (x,y) \in E\} = \bigcap_{n=1}^\infty \{x : (x,y) \in E_n\} = \bigcap_{n=1}^\infty (E_n)^y.$$

By continuity from above, we know that

$$\lim_{n\to\infty} \nu((E_n)_x) = \nu\left(\bigcap_{n=1}^\infty (E_n)_x\right) = \nu(E_x) \text{ and } \lim_{n\to\infty} \mu((E_n)^y) = \mu\left(\bigcap_{n=1}^\infty (E_n)^y\right) = \nu(E^y).$$

Thus $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ is the limit of measurable functions, therefore by the corollary 2.9, it is measurable.

Since μ, ν are finite measures, the function $x \mapsto \nu((E_1)_x), y \mapsto \mu(E_1^y)$ is integrable. Since $\nu((E_n)_x) \leq \nu((E_1)_x), \mu(E_n^y) \leq \mu(E_1^y)$ for any $n \in \mathbb{N}$, we can apply the Dominated Convergence Theorem. Thus,

$$\int \nu(E_x)d\mu(x) \underbrace{=}_{\text{DCT}} \lim_{n\to\infty} \int \nu((E_n)_x)d\mu(x) \underbrace{=}_{E_n\in\mathcal{D},\forall n\in\mathbb{N}} \lim_{n\to\infty} \mu\times\nu(E_n) \underbrace{=}_{\text{continuity from above}} \mu\times\nu(E)$$

and

$$\int \mu(E^y)d\nu(y) \underbrace{=}_{\text{DCT}} \lim_{n\to\infty} \int \mu((E_n)^y d\nu(y) \underbrace{=}_{E_n\in\mathcal{D},\forall n\in\mathbb{N}} \lim_{n\to\infty} \mu\times\nu(E_n) \underbrace{=}_{\text{continuity from above}} \mu\times\nu(E)$$

Thus, $E \in \mathcal{D}$. Hence, $\mathcal{D}$ is a monotone class. □

Let $\mathfrak{a}$ be a set of finite disjoint unions of measurable rectangles. Then by the argument in [1] p.64 with the Proposition 1.7 in [1]m $\mathfrak{a}$ is an algebra. By claim 3.111, $\mathfrak{a} \subseteq \mathcal{D}$. And since $\mathcal{D}$ is a monotone class by the claim 3.112, by the monotone class lemma,

$$m \otimes n = \sigma\text{-alg}\,(\mathfrak{a}) \subseteq \mathcal{D}.$$

Thus every statement holds for any set in $m \otimes n$.

Now suppose μ and ν are σ-finite measure. Take $A_1 \subseteq A_2 \subseteq \cdots X$, such that $A_n \in m$ and $\bigcup_{n=1}^{\infty} A_n = X$, and $\mu(A_n) < +\infty$, $\forall n \in \mathbb{N}$,. Similarly, take $B_1 \subseteq B_2 \subseteq \cdots \subseteq Y$ such that $B_n \in n, \bigcup_{n=1}^{\infty} B_n = Y$ and $\nu(A_n) < +\infty$, $\forall n \in \mathbb{N}$.

Now fix $n \in \mathbb{N}$. Let $\bar{\mu}, \bar{\nu}$ are measures on (X, m) and (Y, n) such that

$$\bar{\mu}(C) := \mu(C \cap A_n), \bar{\nu}(D) = \nu(D \cap B_n),$$

for $C \in m, D \in n$. Then these $\bar{\nu}, \bar{\mu}$ are finite measures, thus the statement holds as shown above.

Claim 3.113. $\forall F \in m \otimes n, \bar{\mu} \times \bar{\nu}(F) = \mu \times \nu(F \cap (A_n \times B_n)$

Proof. Let $A \in m, B \in n$. Then, by construction,

$$\bar{\mu} \times \bar{\nu}(A \times B) = \bar{\mu}(A)\bar{\nu}(B) = \mu(A \cap A_n)\nu(B \cap B_n) = \nu(A \times B \cap A_n \times B_n).$$

Let $F \in m\otimes n$. Then, since $m \otimes n$ is σ-algebra generated by measurable rectangles in $m \times n$, $F = \bigcup_{j=1}^{\infty} C_j \times D_j$ for some disjoint $C_j \in m, D_j \in n$ for any $j \in \mathbb{N}$. Then,

$$\begin{aligned}
\bar{\mu} \times \bar{\nu}(F) = \bar{\mu} \times \bar{\nu}\left(\bigcup_{j=1}^{\infty} C_j \times D_j\right) &\underbrace{=}_{\text{Ctbl additivity}} \sum_{j=1}^{\infty} \bar{\mu} \times \bar{\nu}(C_j \times D_j) \\
&= \sum_{j=1}^{\infty} \mu \times \nu(C_j \times D_j \cap A_n \times B_n) \underbrace{=}_{\text{Ctbl additivity}} \mu \times \nu\left(\bigcup_{j=1}^{n} (C_j \times D_j \cap A_n \times B_n)\right) \\
&= \mu \times \nu\left(\left(\bigcup_{j=1}^{n} C_j \times D_n\right) \cap A_n \times B_n\right) = \mu \times \nu\left(F \cap A_n \times B_n\right)
\end{aligned}$$

as desired. □

Claim 3.114. *$\forall f \in L^+(X, m)$, $\int f(x)d\bar{\mu}(x) = \int f(x)1_{A_n}(x)d\mu(x)$. Similarly, $\forall f \in L^+(Y, n)$, $\int f(x)d\bar{\nu}(x) = \int f(x)1_{B_n}(x)d\nu(x)$.*

Proof. It is just machinery argument; let f be a simple function, then $f = \sum_{j=1}^{k} a_j 1_{C_j}$ for some $k \in \mathbb{N}$, $C_j \in m$ for all $j \in [k]$, and C_j's are disjoint. Then,

$$\int f(x)d\bar{\mu}(x) \underbrace{=}_{\text{Prop 2.13}} \sum_{j=1}^{k} a_j \int 1_{C_j} = \sum_{j=1}^{k} a_j \bar{\mu}(C_j) = \sum_{j=1}^{k} a_j \mu(C_j \cap A_n) = \sum_{j=1}^{k} a_j \int 1_{C_j} 1_{A_n} \underbrace{=}_{\text{Prop 2.13}} = \int f(x) 1_{A_n} d\mu(x),$$

as desired. Now let f be any measurable function. Then, by Theorem 2.10 (a) in [1], there exists a sequence of simple functions such that $0 \le \phi_1 \le \phi_2 \le \cdots \le f$ and $\phi_j \to f$ pointwisely as $j \to \infty$. Then, since this sequence is also in L^+ we can apply the Monotonce Convergence Theorem to conclude that

$$\int f(x)d\bar{\mu}(x) = \int \lim_{j\to\infty} \phi_j d\bar{\mu}(x) \underbrace{=}_{\text{MCT}} \lim_{j\to\infty} \int \phi_j(x) d\bar{\mu}(x) = \lim_{j\to\infty} \int \phi_j 1_{A_n}(x) d\mu(x) \underbrace{=}_{\text{MCT}} \int f 1_{A_n}(x) d\mu(x),$$

since $\phi_j 1_{A_n} \to f 1_{A_n}$ pointwisely and $\phi_j 1_{A_n} \le \phi_j 1_{A_n}$. Hence, the first statement holds for any $f \in L^+(X, m)$ For the second statement, it is the same argument just when we change μ to ν and A_n to B_n. Hence I omit. $\square$

Since each $\bar{\nu}$ and $\bar{\mu}$ are finite measure, by the case just proved, $x \mapsto \bar{\nu}(E_x)$ and $y \mapsto \int \bar{\mu}(E^y)$ are measurable, and

$$\int \bar{\mu}(E_x) d\bar{\mu}(x) = \bar{\mu} \times \bar{\nu}(E) = \int \bar{\nu}(E^y) d\bar{\nu} \tag{10}$$

for any $E \in m \otimes n$. However, note that

$\bar{\nu}(E_x) = \nu(E_x \cap B_n) = \nu((E \cap A_n \times B_n)_x)$ when $x \in A_n$ and $\bar{\mu}(E^y) = \mu(E_y \cap A_n) = \mu((E \cap A_n \times B_n)^y)$ when $y \in B_n$

since

$$(E \cap (A_n \times B_n))_x = \{y : (x, y) \in E \cap (A_n \times B_n)\} = \begin{cases} \emptyset & \text{if } x \notin A_n \\ \{y : (x, y) \in E, y \in B_n\} & \text{if } x \in A_n \end{cases} = \begin{cases} \emptyset & \text{if } x \notin A_n \\ E_x \cap B_n & \text{if } x \in A_n \end{cases}$$

$$(E \cap (A_n \times B_n))^y = \{x : (x, y) \in E \cap (A_n \times B_n)\} = \begin{cases} \emptyset & \text{if } y \notin B_n \\ \{x : (x, y) \in E, x \in A_n\} & \text{if } y \in B_n \end{cases} = \begin{cases} \emptyset & \text{if } x \notin A_n \\ E^x \cap A_n & \text{if } y \in B_n \end{cases}$$

Thus,

$\nu((E \cap A_n \times B_n)_x) = 1_{A_n}(x) \nu(E_x \cap B_n) = \bar{\nu}(E_x) 1_{A_n}(x)$ and $\mu((E \cap A_n \times B_n)^y) = 1_{B_n}(y) \nu(E^y \cap A_n) = \bar{\mu}(E^y) 1_{B_n}(y)$.

Thus, since $x \mapsto \bar{\nu}(E_x)$ and $y \mapsto \int \bar{\mu}(E^y)$ are measurable, by the proposition 2.6, $x \mapsto \bar{\nu}(E_x) 1_{A_n}(x)$ and $y \mapsto \bar{\mu}(E^y) 1_{B_n}(y)$ are measurable. Thus,

$$\int \nu((E \cap A_n \times B_n)_x) d\mu(x) = \int \bar{\nu}(E_x) 1_{A_n} d\mu(x) \underbrace{=}_{\text{Claim 3.114}} \int \bar{\nu}(E_x) d\bar{\mu}(x) \underbrace{=}_{(10)} \bar{\mu} \times \bar{\nu}(E) \underbrace{=}_{\text{Claim 3.113}} \mu \times \nu(E \cap A_n \times B_n).$$

and

$$\int \mu((E \cap A_n \times B_n)^y) d\nu(x) = \int \bar{\mu}(E^y) 1_{B_n} d\nu(y) \underbrace{=}_{\text{Claim 3.114}} \int \bar{\nu}(E^y) d\bar{\nu}(y) \underbrace{=}_{(10)} \bar{\mu} \times \bar{\nu}(E) \underbrace{=}_{\text{Claim 3.113}} \mu \times \nu(E \cap A_n \times B_n).$$

Now let $n \to \infty$. Then, $E \cap A_n \times B_n$ is increasing and $E = \bigcup_{n=1}^{\infty} E \cap (A_n \times B_n)$, thus by continuity from below,

$$\nu(E \cap (A_n \times B_n)) \le \nu(E \cap (A_{n+1} \times B_{n+1})) \tag{11}$$

and

$$\mu \times \nu(E) = \lim_{n\to\infty} \mu \times \nu(E \cap A_n \times B_n) \tag{12}$$

Also note that $(E \cap A_n \times B_n)_x \subseteq (E \cap A_{n+1} \times B_{n+1})_x$ and $E_x = \bigcup_{n=1}^{\infty} (E \cap (A_{n+1} \times B_{n+1}))_x$. Thus, as a function, $\nu((E \cap A_n \times B_n)_x)$the condition of the Monotone convergence theorem. Thus, $x \mapsto \nu(E_x)d\mu(x)$ is measurable by the Corollary 2.9. and

$$\lim_{n\to\infty} \int \nu((E \cap A_n \times B_n)_x)\, d\mu(x) \underbrace{=}_{\text{MCT}} \int \nu(E_x) d\mu(x). \tag{13}$$

Similarly, note that $(E \cap A_n \times B_n)^y \subseteq (E \cap A_{n+1} \times B_{n+1})^y$ and $E^y = \bigcup_{n=1}^{\infty} (E \cap (A_{n+1} \times B_{n+1}))^y$. Thus, as a function, $\mu((E \cap A_n \times B_n)^y)$ satisfies the condition of the Monotone convergence theorem. Thus, $y \mapsto \mu(E^y)d\nu(y)$ is measurable by the Corollary 2.9. and

$$\lim_{n\to\infty} \int \mu((E \cap A_n \times B_n)^y)\, d\nu(y) \underbrace{=}_{\text{MCT}} \int \mu(E^y) d\nu(y). \tag{14}$$

From (11), (12), (13), and (14) we show that

$$\mu(E \cap A_n \times B_n) \nearrow \mu \times \nu(E)$$

and

$$\int \nu((E \cap A_n \times B_n)_x) d\mu(x) \nearrow \int \nu(E_x) d\mu(x) \text{ and } \int \mu((E \cap A_n \times B_n)^y) d\nu(y) \nearrow \int \mu(E_x) d\nu(y).$$

And since

$$\int \nu((E \cap A_n \times B_n)_x) d\mu(x) = \mu(E \cap A_n \times B_n) = \int \mu((E \cap A_n \times B_n)^y) d\nu(y)$$

for each $n \in \mathbb{N}$, we can conclude that

$$\int \nu(E_x) d\mu(x) = \mu \times \nu(E) = \int \mu(E_x) d\nu(y),$$

as desired. □

Theorem 3.115 (Tonelli's Theorem, Theorem 2.37(a) in [1] p.67)**.** *Let (X, m, μ) and (Y, n, ν) be σ-finite measure spaces. Let $f \in L^+(X \times Y, m \otimes n)$ and*

$$g(x) := \int f_x(y) d\nu(y) \in [0, \infty] \text{ and } h(y) := \int f^y(x) d\mu(x) \in [0, \infty].$$

Then, $g \in L^+(X, m), h \in L^+(Y, n)$, and

$$\int g(x) d\mu(x) = \int h(y) d\nu(y) = \int f(x, y) d(\mu \times \nu)(x, y) \in [0, \infty]$$

Note that f_x, f^y are measurable by theorem 2.34 in [1].

Proof. If $E \in m \otimes n$, let $f = 1_E$ Then

$$f_x = 1_{E_x}, g(x) = \nu(E_x), f^y = 1_{E^y}, h(y) = \mu(E^y).$$

By theorem 2.36 in [1], g and h are measurable and

$$\int g(x) d\mu(x) = (\mu \times \nu)(E) = \int f d(\mu \times \nu) = (\mu \times \nu)(E) = \int h(y) d\mu(y).$$

By linearlity and using the proposition 2.13 in [1], the conclusion holds for every simple function with $f \geq 0$.

Now let $f \in L^+(m \otimes n)$. Then by theorem 2.10, there exists a sequence of simple functions ϕ_n such that

$$0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f \text{ s.t. } \phi_n \to f \text{ pointwise.}$$

Fix $x \in X$. Then, $(\phi_n)_x(y) \leq (\phi_{n+1})_x(y)$ and $(\phi_n)_x \to f_x$ pointwise. Thus, we can apply the Monotone convergence theorem to conclude that

$$\lim_{n\to\infty} \int (\phi_n)_x(y) d\nu(y) \underbrace{=}_{\text{MCT}} \int f_x d\nu = g(x).$$

Now let $g_n := \int (\phi_n)_x(y) d\nu(y)$. Then since $\phi_n \leq \phi_{n+1}$ for any $n \in \mathbb{N}$, $g_n \leq g_{n+1}$, and by the above equation, $g_n \to g$ pointwise. Thus,

$$\lim_{n\to\infty} \int g_n(x) d\mu(x) \underbrace{=}_{\text{MCT}} \int g(x) d\mu(x).$$

Also, the lefthandside of above equation is

$$\lim_{n\to\infty} \int g_n(x) d\mu(x) \underbrace{=}_{\text{Def.}} \lim_{n\to\infty} \int \left(\int (\phi_n)_x(y) d\nu(y) \right) d\mu(x) = \lim_{n\to\infty} \int \phi_n(x,y) d(\mu \times \nu)(x,y)$$

where the last equality comes from the fact we proved that Tonelli's theorem holds for simple function. Thus, we can conclude that

$$\lim_{n\to\infty} \int \phi_n(x,y) d(\mu \times \nu)(x,y) = \int g(x) d\mu(x)$$

And the lefthandside with the MCT gives the conclusion that

$$\int g(x) d\mu(x) = \lim_{n\to\infty} \int \phi_n(x,y) d(\mu \times \nu)(x,y) \underbrace{=}_{\text{MCT}} \int f d(\mu \times \nu). \tag{15}$$

Similarly, fix $y \in Y$. Then, $(\phi_n)^y(x) \leq (\phi_{n+1})^y(x)$ and $(\phi_n)^y \to f^y$ pointwise. Thus, we can apply the Monotone convergence theorem to conclude that

$$\lim_{n\to\infty} \int (\phi_n)^y(x) d\mu(x) \underbrace{=}_{\text{MCT}} \int f^y d\mu = h(x).$$

Now let $h_n := \int (\phi_n)^y(x) d\mu(x)$. Then since $\phi_n \leq \phi_{n+1}$ for any $n \in \mathbb{N}$, $h_n \leq h_{n+1}$, and by the above equation, $h_n \to g$ pointwise. Thus,

$$\lim_{n\to\infty} \int h_n(y) d\nu(y) \underbrace{=}_{\text{MCT}} \int g(y) d\nu(y).$$

Also, the lefthandside of above equation is

$$\lim_{n\to\infty} \int g_n(y) d\nu(y) \underbrace{=}_{\text{Def.}} \lim_{n\to\infty} \int \left(\int (\phi_n)^y(x) d\mu(x) \right) d\nu(y) = \lim_{n\to\infty} \int \phi_n(x,y) d(\mu \times \nu)(x,y)$$

where the last equality comes from the fact we proved that Tonelli's theorem holds for simple function. Thus, we can conclude that

$$\lim_{n\to\infty} \int \phi_n(x,y) d(\mu \times \nu)(x,y) = \int h(y) d\nu(y)$$

And the lefthandside with the MCT gives the conclusion that

$$\int h(y) d\nu(y) = \lim_{n\to\infty} \int \phi_n(x,y) d(\mu \times \nu)(x,y) \underbrace{=}_{\text{MCT}} \int f d(\mu \times \nu). \tag{16}$$

Hence, by (15) and (16), we can conclude that

$$\int g(x)d\mu(x) = \int fd(\mu \times \nu) = \int h(y)d\nu(y).$$

$\square$

Theorem 3.116 (Fubini's theorem, theorem 2.37(b) in [1] p.67)**.** *Let (X, m, μ) and (Y, n, ν) be σ-finite measure spaces. Let $f \in L^1(X \times Y, m \otimes n)$ Then $f_x \in L^1(\nu)$ for a.e. $x \in X$, and $f^y\ in L^1(\mu)$ for a.e. $y \in Y$.*

Consider the almost everywhere defined functions

$$g(x) := \int f_x(y)d\nu(y) = \int f(x, y)d\nu(y) \text{ and } h(y) := \int f^y(x)d\mu(x) = \int f(x, y)d\mu(x).$$

Then, $g \in L^1(\mu), h \in L^1(\nu)$, and

$$\int g(x)d\mu(x) = \int f(x, y)d(\mu \times \nu)(x, y) = \int h(y)d\nu(y)$$

i.e.,

$$\int \left(\int f(x, y)d\nu(y)\right) d\mu(x) = \int f(x, y)d(\mu \times \nu)(x, y) = \int \left(\int f(x, y)d\mu(x)\right) d\nu(y).$$

Proof. If $f \geq 0$ then $f \in L^+$, thus by Tonelli's theorem, g and h are defined everywhere, taking values in $[0, +\infty]$. Also, since $f \in L^1$, thus by tonelli's theorem

$$\int g\mu = \int h\nu = \int fd(\mu \times \nu) < \infty$$

holds. From this result, with the proposition 2.20, we can conclude that $g(x) < \infty$ a.e. and $h < +\infty$ a.e. Thus we proved Fubini's theorem when $f \in L^1 \cap L^+$.

Now assume $f \in L^1_{\mathbb{R}}(\mu \times \nu)$, means that range of f is $\mathbb{R}$. Then,

$$f = f_+ - f_- \text{ for } f_+, f_- \geq 0 \text{ and } |f| = f_+ + f_-.$$

Since $f_+, f_- \in L^1 \cap L^+$, Fubini's theorem holds for each f_+ and f_-.

Note that

$$f_x = (f_+)_x - (f_-)_x \text{ and } f^y = (f_+)^y - (f_-)^y,$$

and by Fubini's theorem in case of f_+, f_-, $(f_+)_x, (f_-)_x \in L^1(\nu)$ for a.e.x and $(f_+)^y, (f_-)^y \in L^1(\mu)$ for a.e. y. Thus,

$$g(x) = \int f_x d\nu = \int (f_+)_x d\nu - \int (f_-)_x d\nu$$

and

$$h(y) = \int f^y d\mu = \int (f_+)^y d\mu - \int (f_-)^y d\mu$$

and by Fubini's theorem in case of f_+, f_-, $\int (f_+)_x, \int (f_-)_x d\nu \in L^1(\nu)$ and $(f_+)^y, (f_-)^y \in L^1(\mu)$. Thus, $g \in L^1(\mu), h \in L^1(\nu)$. Also, by Fubini's theorem in case of f_+, f_- we know

$$\int (f_+)_x d\nu - \int (f_-)_x d\nu = \int f_+ d(\mu \times \nu) - \int f_- d(\mu \times \nu) = \int (f_+)^y d\mu - \int (f_-)^y d\mu.$$

Thus,

$$\int g(x)d\mu(x) = \int (f_+)_x d\nu - \int (f_-)_x d\nu = \int f_+ d(\mu \times \nu) - \int f_- d(\mu \times \nu) = \int (f_+)^y d\mu - \int (f_-)^y d\mu = \int h(y)d\nu(y).$$

And by proposition 2.13 in [1],

$$\int f_+ d(\mu\times\nu) - \int f_- d(\mu\times\nu) = \int (f_+ - f_-) d(\mu\times\nu) = \int f d(\mu\times\nu).$$

Thus, we have

$$\int g(x) d\mu(x) = \int f d(\mu\times\nu) = \int h(y) d\nu(y),$$

as desired. Thus we can say that Fubini's theorem holds for any function having codomain $\mathbb{R}$.

Now let $f \in L^1$. Then $f = Ref + iImf$, and each $Ref, Imf \in L^1_{\mathbb{R}}(\mu\times\nu)$. Thus, Fubini's theorem holds for Ref and Imf. Now note that

$$f_x = (Ref)_x + i(Imf)_x \text{ and } f_x = (Ref)^y + i(Imf)^y.$$

Hence, $(Ref)_x, (Imf)_x \in L^1(\nu)$ a.e. y and $(Ref)^y, (Imf)^y \in L^1(\mu)$ a.e. x, thus for almost every x,

$$\left|\int f_x d\nu\right| = \left|\int (Ref)_x d\nu + i\int (Imf)_x d\nu\right| \leq \left|\int (Ref)_x d\nu\right| + \left|\int (Imf)_x d\nu\right| < \infty,$$

and for almost every y,

$$\left|\int f^y d\mu\right| = \left|\int (Ref)^y d\mu + i\int (Imf)^y d\mu\right| \leq \left|\int (Ref)^y d\mu\right| + \left|\int (Imf)^y d\mu\right| < \infty,$$

Hence $f_x \in L^1(\nu)$ for a.e. y and $f^y \in L^1(\mu)$ for a.e. x.

Also,

$$g(x) = \int f_x d\nu = \int (Ref)_x d\nu + i\int (Imf)_x d\nu \text{ and } h(y) = \int f^y d\mu = \int (Ref)^y d\mu + i\int (Imf)^y d\mu$$

By Fubini's theorem on real function, $\int (Ref)_x d\nu, \int (Imf)_x d\nu \in L^1(\mu)$ and $\int (Ref)^y d\mu, \int (Imf)^y d\mu \in L^1(\nu)$ we can conclude that

$$\left|\int g d\mu\right| = \left|\int\int (Ref)_x d\nu d\mu + i\int\int (Imf)_x d\nu d\mu\right| \leq \left|\int\int (Ref)_x d\nu d\mu\right| + \left|\int\int (Imf)_x d\nu d\mu\right| < \infty$$

and

$$\left|\int h d\nu\right| = \left|\int\int (Ref)^y d\mu d\nu + i\int\int (Imf)^y d\mu d\nu\right| \leq \left|\int\int (Ref)^y d\mu d\nu\right| + \left|\int\int (Imf)^y d\mu d\nu\right| < \infty$$

Thus $g \in L^1(\mu), h \in L^1(\nu)$.

And since Fubini's theorem holds for Ref, Imf,

$$\int g d\mu = \int\int (Ref)_x d\nu d\mu + i\int\int (Imf)_x d\nu d\mu = \int Ref d(\mu\times\nu) + i\int Imf d(\mu\times\nu) = \int f d(\mu\times\nu)$$

and

$$\int h d\nu = \int\int (Ref)^y d\mu d\nu + i\int\int (Imf)^y d\mu d\nu = \int Ref d(\mu\times\nu) + i\int Imf d(\mu\times\nu) = \int f d(\mu\times\nu).$$

Thus,

$$\int g d\mu = \int f d(\mu\times\nu) = \int h d\nu,$$

as desired. Thus Fubini's theorem holds for any function in $L^1(\mu\times\nu)$. □

Corollary 3.117 (Typical Application; Change of order). *Given $f : X \times Y \to \mathbb{C}$, if $|f| \in L^1$, then*

$$\int \left(\int f(x,y) d\mu(x) \right) d\nu(y) = \int \left(\int f(x,y) d\nu(y) \right) \mu(x).$$

Proof. First we have to show

$$\int \left(\int |f(x,y)| d\mu(x) \right) d\nu(y) < \infty \text{ or } \int \left(\int |f(x,y)| d\nu(y) \right) \mu(x) < \infty.$$

Then, since $|f| \in L^+(X \times Y, m \otimes n)$, by Tonelli's theorem,

$$\int \left(\int |f(x,y)| d\mu(x) \right) d\nu(y) = \int |f| d(\mu \times \nu) = \left(\int |f(x,y)| d\nu(y) \right) \mu(x) < \infty$$

thus $f \in L^1(\mu \times \nu)$. Then by the Fubini's theorem, we can conclude that

$$\int \left(\int f(x,y) d\mu(x) \right) d\nu(y) = \int f(x,y) d(\mu \times \nu) = \int \left(\int f(x,y) d\nu(y) \right) \mu(x).$$

□

3.6 The n-Dimensional Lebesgue Integral

Fix $n \geq 2$. Let m^n denotes the completion of $m \times m \times \cdots \times m$ on $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}$, i.e., the completion of $\mathcal{L} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}$. Note that the completion is not the same as $\mathcal{L} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}$.

Proposition 3.118. $\mathcal{L} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L} \subsetneq \mathcal{L}^n$

Proof. It suffices to show that there exists a set in $\mathcal{L}^n$ but not in $\mathcal{L} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}$. For $n = 2$, let $E \in \mathcal{L} \otimes \mathcal{L}$, $A \subseteq \mathbb{R}$ such that $A \notin \mathcal{L}$ and $x \in \mathbb{R}$. Now, let

$$F := (E \setminus (\{x\} \times \mathbb{R})) \cup (\{x\} \times A).$$

Then, $m^2(F \Delta E) = 0$. By completeness of $\mathcal{L}$, $F \Delta E \in \mathcal{L}^2$, which implies $F \setminus E \in \mathcal{L}^2$ since it is subset of $F \Delta E$, thus $F = F \setminus E \cup E \in \mathcal{L}^2$.

However, $F_x = A$ by the Proposition 2.34. If $F \in \mathcal{L} \otimes \mathcal{L}$ then $F_x = A \in \mathcal{L}$, contradiction. Thus, $F \in \mathcal{L}^2 \setminus \mathcal{L} \otimes \mathcal{L}$. □

Theorem 3.119 (Theorem 2.40 in [1] p.70. Approximation of sets.). *(a)*

$$m^n(E) = \inf\{m^n(U) : U \text{ open in } \mathbb{R}^n, E \subseteq U\} = \sup\{m^n(K) : K \subseteq E, K \text{ compact }\}.$$

(b) $E = A_1 \cup N_1 = A_2 \setminus N_2$ where A_1 is F_σ set in $\mathbb{R}^n$, and A_2 is G_δ set in $\mathbb{R}^n$, and N_1, N_2 are null sets.

(c) If $m(E) < \infty$, for any $\epsilon > 0$ there exists a finite collection $(R_j)_{j=1}^N$ of disjoint rectangles whose sides are intervals such that $m(E \Delta \bigcup_{j=1}^N R_j) < \epsilon$.

Proof. Let $E \in \mathcal{L}^n$. Note that m^n induces outer measure on $\mathbb{R}^n$ with respect to an algebra $\times^n \mathcal{L}$. Thus, by definition of outer measure, we have disjoint sequence of measurable rectangles $T_1, T_2, \cdots \in \mathcal{L} \times \mathcal{L} \times \cdots \times \mathcal{L}$ such that

$$E \subseteq \bigcup_{i=1}^\infty T_i \text{ and for some } \epsilon > 0, m^n \left(\bigcup_{i=1}^\infty T_i \right) \leq m^n(E) + \epsilon.$$

(Note that $E = E' \cup F$ for some E' in $\mathcal{B}_{\mathbb{R}^n}$ and $F \subset N, N \in \mathcal{B}_{\mathbb{R}^n}$. Thus, we can choose such T_i using $E \subseteq E' \cup N \in \mathcal{B}_{\mathbb{R}^n}$.) For each j, by applying theorem 1.18 for each side of the rectangle T_j with $\epsilon' =$

$(\frac{\epsilon^2}{n^2 2^{2n} \prod_{k=1}^n m((T_j)_{x_k})})$ we get a rectangle U_j having each side as union of disjoint open intervals such that $U_j \supset T_j$ and for each $k \in [n]$,

$$m((U_j)_{x_k}) \leq m((T_j)_{x_k} + \epsilon',$$

thus

$$m^n(U_j) \leq \prod_{k=1}^n (m((T_j)_k + \epsilon') = m^n(T_j) + \sum_{k=0}^{n-1} \binom{n}{k} \sum_{\sigma \in S_k} \prod_{l=1}^k m((T_j)_{\sigma(l)})(\epsilon')^{n-k} \leq m^n(T_j) + \sum_{k=0}^{n-1} \frac{\binom{n}{k}}{2^n} \frac{(\frac{\epsilon}{2^n})^{n-k}}{n^2}$$

since $\prod_{l=1}^k m((T_j)_{\sigma(l)}) < \prod_{l=1}^n m((T_j)_l)$. Hence,

$$m^n(U_j) \leq m^n(T_j) + \sum_{k=0}^{n-1} \frac{\binom{n}{k}}{2^n} \frac{(\frac{\epsilon}{2^n})^{n-k}}{n^2} \leq m^n(T_j) + \epsilon 2^{-n}.$$

Now let $U = \bigcup_{j=1}^\infty U_j$ then U is open and

$$m^n(U) \leq \sum_{n=1}^\infty m^n(U_j) \leq \sum_{n=1}^\infty m^n(T_j) + \epsilon \leq m^n(E) + 2\epsilon.$$

Since ϵ was arbitrarily chosen, $m(E) = \inf\{m(U) : U \supset E, U \text{ is open.}\}$ The second one and part (b) follows from the exact same argument in Theorem 1.18 and 1.19 in [1][p.36-37].

For part (c), if $m^n(E) < \infty$, then $m^n(U_j) < \infty$. Note that each side of U_j consists of countable open intervals, thus, numbering each disjoint intervals of each sides of U_j and let $V_{j,k}$ be rectangle generated by subunion of intervals of U_j in each section from number 1 interval to number k interval. Then, $\lim_{k\to\infty} m^n(V_{j,k}) = m^n(U_j)$ Thus, $\exists N_j$ such that

$$\forall k \geq N_j, m^n(V_{j,k}) \geq m^n(U_j) - \epsilon 2^{-j}.$$

Then let $V_j = V_{j,N_j}$. Then since $m^n(\bigcup_{j=n}^\infty U_j) \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $m^n(\bigcup_{j=N+1}^\infty U_j) < \epsilon$. Thus,

$$m^n(E \setminus \bigcup_{j=N}^\infty V_j) = \bigcup_{j=1}^N U_j \setminus V_j + m^n(\bigcup_{j=N+1}^\infty U_j) < 2\epsilon.$$

and

$$m^n(\left(\bigcup_{j=1}^N V_j\right) \setminus E) \leq m^n(\left(\bigcup_{j=1}^\infty E_j\right) \setminus E) \leq \epsilon,$$

which is derived from the construction of U_j. Thus $V := \bigcup_{j=1}^N V_j$ gives

$$m^n(E \Delta V) = 2\epsilon + \epsilon = 3\epsilon,$$

as desired for (c). ∎

Theorem 3.120 (Theorem 2.41 in citefo p. 71, Approximation in L^1.)**.** *If $f \in L^1(m^n)$ and $\epsilon > 0$, then*

(a) $\exists$ a simple function $\phi = \sum_{j=1}^n a_j 1_{R_j}$ where each R_j is a product of bounded intervals, such that

$$\int |f - \phi| dm^n < \epsilon.$$

(b) $\exists$ a continuous function $g : \mathbb{R}^n \to \mathbb{C}$ of bounded supports such that

$$\int |f - g| dm^n < \epsilon.$$

Proof. By the argument in Thoerem 2.26, approximate f by a simple function, and approximate each preimage in the simple function using Theorem 2.40(c), and change those preimage to the approximated rectangle. Now, by using the argument in 2.26(b), get a continuous function. $\square$

Theorem 3.121 (Theorem 2.42 in [1] p. 72, Translation invariant of m^n.)**.** *If $a \in \mathbb{R}^n$, letting $\tau_a(x) = x + a$. Then*

(a) $E \in \mathcal{L}^n \implies \tau_a(E) \in \mathcal{L}^n$, and $m(\tau_a(E)) = m(E)$.

(b) If $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable, then so is $f \circ \tau_a$.

(c) If $f \in L^+(m^n)$ then $\int f \circ \tau_a dm^n = \int f dm^n$

(d) If $f \in L^1(m^n)$ then $f \circ \tau_a \in L^1(m^n)$.

Proof. Since τ_a and τ_{-a} are continuous, $\tau_a(E) \in \mathcal{B}_{\mathbb{R}^n}$ if $E \in \mathcal{B} + \mathbb{R}^n$. If $E \in \times^n \mathcal{B}_{\mathbb{R}}$, then

$$m^n(\tau_a(E)) = \prod_{i=1}^{n} m(\tau_{a_i}(E_{x_i})) \underbrace{=}_{\text{Thm 1.21}} \prod_{i=1}^{n} m(E_{x_i}) = m^n(E).$$

If $E \in \mathcal{B}_{\mathbb{R}^n}$, then $E = \bigcup_{j=1}^{\infty} E_j$ for some disjoint $E_j \in \times^n \mathcal{B}_{\mathbb{R}}$, hence

$$m^n(\tau_a(E)) = m^n(\bigcup_{j=1}^{\infty} \tau_a(E_j)) = \sum_{j=1}^{\infty} m(\tau_a(E_j)) = \sum_{j=1}^{\infty} m(E_j) = m(E).$$

If $E \in \mathcal{L}^n$, then $E = E' \cup F$ where $E', N \in \mathcal{B}_{\mathbb{R}^n}$ and $F \subseteq N$ and $m^n(N) = 0$. Thus,

$$m^n(\tau_a(E)) = m^n(\tau_a(E' \cup F)) = m^n(\tau_a(E') \cup \tau_a(F)) = m^n(\tau_a(E')) + m^n(\tau_a(F)) = m^n(E') + m^n(\tau_a(F))$$

and since $\tau_a(F) \subseteq \tau_a(N)$ and $m^n(\tau_a(N)) = m^n(N) = 0$, $m^n(\tau_a(F)) = 0$. Thus,

$$m^n(\tau_a(E)) = m^n(E') + m^n(\tau_a(F)) = m^n(E') = m^n(E).$$

Thus part (a) is proved.

For part (b), let $E \in \mathcal{B}_{\mathbb{C}}$. Then, $f^{-1}(E) \in \mathcal{L}^n$, thus $\tau_a^{-1}(f^{-1}(E)) = \tau_{-a}(f^{-1}(E)) \in \mathcal{L}^n$ by part (a). Hence $(f \circ \tau_a)^{-1}(E) \in \mathcal{L}^n$, thus $f \circ \tau_a$ is also Lebesgue measurable.

For part (c), the statement holds when $f = 1_E$, since then $f \circ \tau_a = 1_{\tau_a(E)}$, hence

$$\int f dm^n = m(E) = m(\tau_a(E)) = \int 1_{\tau_a(E)} = \int f \circ \tau_{a]} dm^n.$$

Thus, by linearlity the statement holds for any simple function. And by Theorem 2.41, any Lebesgue measurable function can be approximated by the sequence of simple functions, thus using the Monotone Convergence Theorem, we can conclude that the result holds for any measurable functions. Thus (d) also follows from the result of (c). $\square$

Definition 3.122 (Cube, Jordan Content, etc.)**.** *Define* **cube** *in $\mathbb{R}^n$ is a Cartesian product of n closed intervals whose side lengths are all equal. For $k \in \mathbb{Z}$, let Q_k be the collection of cubes whose side length is 2^{-k} and whose vertices are in the lattice $(2^{-k}\mathbb{Z})^n$. Note that any two cubes in Q_k have disjoint interiors, and that the cube in Q_{k+1} are obtained from the cubes in Q_k by bisecting the sides.*

If $E \subset \mathbb{R}^n$, we define the **inner and outer approximations to** *E by the grid of cubes Q_k to be*

$$\underline{A}(E,k) := \bigcup \{Q \in Q_k : Q \subset E\} \qquad \overline{A}(E,k) := \bigcup \{Q \in Q_k : Q \cap E \neq \emptyset\}$$

Then since cubes are disjoint, and have measure 2^{-nk}, so

$$m(\underline{A}(E,k)) = |\{Q \in Q_k : Q \subset E\}| \cdot 2^{-nk} \qquad\qquad m(\overline{A}(E,k)) = |\{Q \in Q_k : Q \cap E \neq \emptyset\}| \cdot 2^{-nk}.$$

Also, $\underline{A}(E,k)$ increase with k while $\overline{A}(E,k)$ decrease when k increase, since each cube in Q_k is a union of cubes in Q_{k+1}. And since they are boudned, so the limits

$$\underline{\kappa}(E) := \lim_{k\to\infty} m(\underline{A}(E,k)) \qquad\qquad \overline{\kappa}(E) := \lim_{k\to\infty} m(\overline{A}(E,k))$$

exists. They are called **inner and outer content** *of E. if they are equal, then the common value $\kappa(E)$ is called the* **Jordan Content** *of E.*

Note that Jordan content is meaningful only when E is bounded; otherwise $\kappa(E) = \infty$.

Lemma 3.123 (Lemma 2.43 in [1] p.72)**.** *Let*

$$\underline{A}(E) \bigcup_{k=1}^{\infty} \underline{A}(E,k) \text{ and } \underline{A}(E) \bigcap_{k=1}^{\infty} \underline{A}(E,k).$$

Then, by definition of inner and outer approximation,

$$\underline{A}(E) \subset E \subset \overline{A}(E),$$

and $\underline{A}(E)$ and $\overline{A}$ are Borel sets, and

$$\underline{\kappa}(E) = m(\underline{A}(E)) \text{ and } \overline{\kappa}(E) = m(\overline{A}(E)).$$

Thus, the jordan content of E exists if and only if $m(\overline{A}(E) \setminus \underline{A}(E)) = 0$, which implies $\kappa(E) = m(E)$.

Then, if $U \subseteq \mathbb{R}^n$ is open, then $U = \underline{A}(U)$. Moreover, U is a countable union of cubes with disjoint interiors.

Proof. Since we know $\underline{A}(U) \subset U$, it suffices to show that $\underline{A}(U) \supset U$. Let $x \in U$, and $\delta = \inf\{|y-x| : y \notin U\}$. Then, since U is open set, $\delta > 0$. So, if there exists a cube $Q \in Q_k$ containing x, then $\forall y \in Q$, $|y-x| \leq 2^{-k}\sqrt{n}$, thus $Q \subset U$ iff $2^{-k}\sqrt{n} < \delta$. Since $2^{-k}\sqrt{n} \to 0$ as $k \to \infty$, we can take a large enough k, such that $k > \log \frac{\sqrt{n}}{\delta}$, to get $Q \subset U$. Then, $x \in \underline{A}(U,k) \subset \underline{A}(U)$. Since x was arbitrary, $U \subset \underline{A}(U)$, as desired.

For the second statement, we can rewrite $\underline{A}(U)$ as below;

$$\underline{A}(U) = \underline{A}(U,0) \cup \bigcup_{k=1}^{\infty} \left(\underline{A}(U,k) \setminus \underline{A}(U,k-1)\right).$$

Since closure of $\underline{A}(U,k) \setminus \underline{A}(U,k-1)$ is disjoint union of cubes in Q_k and $\underline{A}(U,0)$ is a countable union of cubes, so does $\underline{A}(U)$ and by construction their interiors are pairwise disjoint. □

Remark 3.124. *The above lemma implies that Lebesgue measure of any open set is equal to its inner content. Also, it implies that Lebesgue measure of any compact set is equal to its outer content.*

Proof. First statement is just result of the lemma. For the second statement, let $F \subset \mathbb{R}^n$ be compact. Let $Q^0 = \{x : \max|x_j| \leq 2^M\}$, a rectangle centered at the origin, whose interior $int(Q_0)$ contains F. If $Q \in Q_k$ such that $Q \subset Q_0$, then either $Q \cap F \neq \emptyset$ or $Q \subset (Q_0 \setminus F)$, thus

$$m(\overline{A}(F,k)) + m(\underline{A}(Q_0 \setminus F, k)) = m(Q_0).$$

Letting $k \to \infty$, we get

$$\overline{\kappa}(F) + \underline{\kappa}(Q_0 \setminus F) = m(Q_0).$$

Note that $Q_0 \setminus F$ is the union of oepn set $int(Q_0) \setminus F$ and the boundary of Q_0, which has the content zero. Thus,

$$\underline{\kappa}(Q_0 \setminus F) = \underline{\kappa}(intQ_0 \setminus F) = m(Q_0 \setminus F)$$

where the last equality comes from the lemma 2.43. Hence,we have desired result. □

Theorem 3.125 (Theorem 2.44 in [1] p.73). *Let $T \in GL(n, \mathbb{R})$*

(a) $E \in \mathcal{L}^n \implies T(E) \in \mathcal{L}^n$ and $m(T(E)) = |\det(T)| m(E)$.

(b) $f \in L^+(\mathcal{L}^n) \implies f \circ T \in L^+(\mathcal{L}^n)$ and $\int f dm^n = |\det T| \int f \circ T dm^n$

(c) $f \in L^1(\mathcal{L}^n) \implies f \circ T \in L^1(m^n)$

To prove this result, we use the fact from the linear algebra that every $T \in GL(n, \mathbb{R})$ can be written as the product of finitely many transformations of three elementary types, such that

$$\begin{aligned} T_1(x_1, \cdots, x_j, \cdots, x_n) &= (x_1, \cdots, cx_j, \cdots, x_n) && (c \neq 0), \\ T_2(x_1, \cdots, x_j, \cdots, x_n) &= (x_1, \cdots, x_j + cx_k, \cdots, x_n) && (k \neq j), \\ T_3(x_1, \cdots, x_j, \cdots, x_j, \cdots, x_n) &= (x_1, \cdots, x_k, \cdots, x_j, \cdots, x_n) && (k \neq j). \end{aligned}$$

Also note that every $T \in GL(n, \mathbb{R})$ is continuous.

Proof. If f is Borel measurable, then $f \circ T$ is also Borel measurable since T is continuous.

Also, for T_3, $|\det T_3| = |-1| = 1$, and part (a),(b) and (c) holds because T_3 is just interchange of variables and Tonelli theorem gives part (a) and (b) and Fubini theorem gives part (c). (Note that part (a) is a special case of part (b) when $f = 1_E$ for some borel set E.) For T_2, using Tonelli's theorem (for (a),(b)) or Fubini's theorem (for (c)) we integrate first with respect to x_j, and by theorem 1.21 in [1], we know that

$$\int f(t + a) dt = \int f(t) dt$$

for one variable integration. Thus, this gives the same integration, i.e.,

$$\int f dm^n = \int f \circ T dm^n = |\det T_2| \int f \circ T dm^n.$$

where the last equality holds since $\det T_2 = 1$. Thus (a),(b), and (c) holds for T_2. For T_1, using Tonelli's theorem (for (a),(b)) or Fubini's theorem (for (c)) we integrate first with respect to x_j, and by theorem 1.21 in [1], we know that

$$\int f(ct) dt = |c| \int f(t) dt$$

for one variable integration. Thus, this gives

$$\int f dm^n = |c| \int f \circ T dm^n = |\det T_1| \int f \circ T dm^n,$$

since $|\det T_1| = |c|$.

Now if $T, S \in GL(n, \mathbb{R})$ are matrices which the theorem holds for, then

$$\int f dx = |\det T| \int f \circ T dx = |\det T| |\det S| \int (f \circ T) \circ S dx = |\det T \circ S| \int f \circ (T \circ S) dx,$$

this implies that this thoerem holds for $T \circ S$, too. Since every matrix in $GL(n, \mathbb{R})$ is generated by composition of T_1, T_2, and T_3, the theorem holds for every matrix in $GL(n, \mathbb{R})$.

For f is Lebesgue measurable, then for any Borel set $E \in \mathbb{R}^n$, $f^{-1}(E) = F \cup N$ where F is Borel measurable and N is a null set with respect to Lebesgue measure. Since Lebesgue measure is completion of Borel measure, thus there is a Borel measurable set N' such that $N \cup N'$ and $m(N) = 0$. Hence,

$$m(T^{-1}(N)) \leq m(T^{-1}(N')) \underbrace{=}_{\text{(a)}} |\det T^{-1}| m(N') = 0 \implies T^{-1}(N) \text{ is Lebesgue measurable.}$$

Thus, $T^{-1}(E)$ and $T^{-1}(N)$ are Lebesgue measurable, this implies $(f \circ T)^{-1}$ is Lebesgue measurable. Thus the above argument can be applied for any Lebesgue measurable sets. ☐

Corollary 3.126 (Corollary 2.46 in [1] p.74). *m^n is invariant under rotation.*

Proof. If T is a rotation, then $TT^* = I$, which implies $(\det T)^2 = 1 \implies |\det T| = 1$. Note that T^* is the transpose of T. $\square$

Let $\Omega \subseteq \mathbb{R}^n$ be open. Let $g : \Omega \to \mathbb{R}^n$ be C^1. Then, $g(x) := (g_1(x), g_2(x), \cdots, g_n(x))$. Let's denote

$$(D_x g =) Dg(x) := \left(\frac{\partial g_i}{\partial x_j}(x)\right)_{1 \le i \le n, 1 \le j \le n}$$

Definition 3.127 (Diffeomorphism). *The above g is called* **Diffeomorphism** *if $D_g(x)$ is invertible for all $x \in \Omega$ and g is injective, i.e., 1-1.*

If g is a diffeomorphism, then by the inverse function theorem, g^{-1} is in C^1, and that $D_x(g^{-1}) = \left(D_{g^{-1}(x)} g\right)^{-1}$ for all $x \in g(\Omega)$. Also, for any $T \in GL(n, \mathbb{R})$, $D_x T \circ g = T D_x g$; this can be obtained by using the result of elementary matrices.

Theorem 3.128 (Theorem 2.47 in [1] p.74). *Suppose $\Omega \subset \mathbb{R}^n$ and $g : \Omega \to \mathbb{R}^n$ is C^1 diffeomorphism.*

1. *If $f : g(\Omega) \to \mathbb{R}^n$ is m^n-measurable and $f \in L^1$, then $f \circ g$ is m^n-measurable. If $f \ge 0$ or $f \in L^1$, then*

$$\int_{g(\Omega)} f dm^n = \int_{\Omega} f \circ g(x) |\det(D_g(x))| dm^n.$$

2. *If $E \subset \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and $m(G(E)) = \int_E |\det D_x g| dx$.*

Before solving this, we need a notation for ∞ norm.

Definition 3.129 (∞-norm). *For any $x \in \mathbb{R}^n$ or $T = (T_{ij}) \in GL(n, \mathbb{R})$, we set*

$$\|x\| = \max_{1 \le j \le n} |x_j| \text{ and } \|T\| = \max_{1 \le i \le n} \sum_{j=1}^{n} |T_{ij}|.$$

Then, $\|Tx\| \le \|T\| \, \|x\|$ and $\{x : \|x - a\| \le h\}$ is the cube of side length $2h$ centered at a.

Proof. It suffices to show that Borel measurable functions and Borel measurable sets. If this holds for Borel measurable sets and functions, then we can extend this for the case of Lebesgue measurable function and sets using $E = B \cup N$ for $E \in \mathcal{L}^n$, $B \in \mathcal{B}^n, N \in \mathcal{L}^n$ with $m(N) = 0$.

Let $Q \subset \Omega$ be a cube say $Q = \{x : \|x - a\| \le h\}$ for some $a \in \Omega$ and $h > 0$. By the mean value theorem,

$$g_i(x) - g_i(a) = \sum_{j=1}^{n} (x_j - a_j) \left(\frac{\partial g_i}{\partial x_j}\right)(y)$$

for some y in a line segment from x to a. Thus, for any $x \in Q$,

$$\begin{aligned} \|g(x) - g(a)\| &= \max_{1 \le i \le n} \left\{ \left| \sum_{j=1}^{n} (x_j - a_j) \left(\frac{\partial g_i}{\partial x_j}\right)(y_i) \right| \right\} \le h \max_{1 \le i \le n} \left\{ \left| \sum_{j=1}^{n} \left(\frac{\partial g_i}{\partial x_j}\right)(y_i) \right| \right\} \le h \max_{1 \le i \le n} \left\{ \sum_{j=1}^{n} \left| \left(\frac{\partial g_i}{\partial x_j}\right)(y_i) \right| \right\} \\ &\le h \max_{1 \le i \le n} \{D_{y_i} g\} \le h \sup_{y \in Q} \|D_y g\| \end{aligned}$$

since $h \ge (x_j - a_j)$ for any $j \in [n]$ and each $\sum_{j=1}^n \left| \left(\frac{\partial g_i}{\partial x_j}\right)(y_i) \right|$ is less than $D_{y_i} g$ by definition of norm. Thus, $g(Q)$ is contained in a rectangle with side length $h \sup_{y \in Q} \|D_y g\|$. Therefore, $g(Q)$ is contained in $\sup_{y \in Q} \|D_y g\| \cdot I(Q)$, where I is an identity matrix in $GL(n, \mathbb{R})$. Hence,

$$m(g(Q)) \le m(\sup_{y \in Q} \|D_y g\| \cdot I(Q)) \underbrace{=}_{\text{Thm 2.44(a)}} (\sup_{h \in Q} \|D_y g\|)^n m(Q).$$

Since g was arbitrary, for any $T \in GL(n, \mathbb{R})$, we can apply the same argument of $T^{-1} \circ g$, thus we have

$$m(g(Q)) \underbrace{=}_{2.44(a)} |\det T| m(T^{-1} \circ g(Q)) \leq |\det T| \left(\sup_{y \in Q} \left\| D_y(T^{-1} \circ g) \right\| \right)^n m(Q) \leq |\det T| \left(\sup_{y \in Q} \left\| T^{-1} D_y g \right\| \right)^n m(Q).$$

Since g is C^1-Diffeomorphism, $D_y g$ and $(D_y g)^{-1}$ is continuous with respect to y. Thus for any $\epsilon > 0, \exists \delta > 0$ so that if $\|y - z\| \leq \delta$,

$$\left\| (D_z g)^{-1} D_y g \right\|^n \leq 1 + \epsilon$$

Thus, let $Q_1, \cdots Q_N$ be subdivided disjoint cubes of Q such that its side length is less than δ and whose center is $x_1, \cdots x_N$ respectively. Then,

$$m(g(Q)) \leq \sum_{j=1}^{N} m(g(Q_j)) \leq \sum_{j=1}^{N} |\det D_{x_j} g| \left(\sup_{y \in \mathbb{Q}_j} \left\| (D_{x_j} g)^{-1} D_y g \right\| \right)^n m(Q_j) \leq (1 + \epsilon) \sum_{j=1}^{N} |\det D_{x_j} g| m(Q_j)$$

And the last sum $(1 + \epsilon) \sum_{j=1}^{N} |\det D_{x_j} g| m(Q)$ is integral of $(1 + \epsilon) \sum_{j=1}^{N} |\det D_{x_j} g| 1_{Q_j}$. Therefore, it tends to go $|\det D_x g|$ as $\delta \to 0$ (Why??? I don't understand.). Thus, letting $\delta \to 0$ and $\epsilon \to 0$, we find that

$$m(g(Q)) \leq \int_Q |\det D_x g| dx.$$

Now we claim that this estimate hold with Q is replaced by any Borel set in Ω. Let $U \subset \Omega$ is open. Then by Lemma 2.43 $U = \bigcup_1^\infty Q_j$ where Q_j are cubes with disjoint interiors. Since the boundary of the cubes have Lebesgue measure 0, we hvae

$$m(g(U)) \leq \sum_{j=1}^{\infty} m(g(Q_j)) \leq \sum_{j=1}^{\infty} \int_{Q_j} |\det D_x g| dx = \int_U |\det D_x g| dx.$$

Moreover, if $E \subset \Omega$ is any Borel set of finite measure, by theorem 2.40 there is a decreasing sequence of $U_j \subset \Omega$ of finite measure such that $E \subset \bigcap_{j=1}^\infty U_j$ and $m((\bigcap_{j=1}^\infty U_j) \setminus E) = 0$. Thus, by the Dominated Convergence Theorem,

$$m(g(E)) \leq m\left(g\left(\bigcap_{j=1}^{\infty} U_j \right) \right) = \lim m\left(g\left(U_j\right)\right) \leq \lim \int_{U_j} |\det D_x g| dx \underbrace{=}_{\text{DCT}} \int_E |\det D_x g| dx.$$

Finally, since m is σ-finite, from the above equation we can draw the conclusion that

$$m(g(E)) \leq \int_E |\det D_x g| dx$$

for any borel set, using continuity from below and Monotone Convergence Theorem.

If $f = \sum_{j=1}^{N} a_j 1_{A_j}$ is a nonnegative simple function on $g(\Omega)$, we have

$$\int g(\Omega) f(x) dx = \sum_{j=1}^{N} a_j m(A_j) \leq \sum_{j=1}^{N} a_j \int_{g^{-1}(A_j)} |\det D_x g| dx = \int f \circ g(x) |det D_x g| dx.$$

Theorem 2.10 and the monotone convergence theorem implies that

$$\int_{g(\Omega)} f(x) dx \leq \int_\Omega f \circ g(x) |\det D_x g| dx$$

for any nonnegative measurable f. But the same reasoning applies with g replaced by g^{-1}, thus

$$\int_\Omega f \circ g(x) |\det D_x g| dx \leq \int_{g(\Omega)} f \circ g \circ g^{-1}(x) |\det D_{g^{-1}(x)} g| |\det D_x g^{-1}| dx = \int_{g(\Omega)} f(x) dx.$$

Thus it establishes (a) for $f \geq 0$ and the case $f \in L^1$ follows immediately. And (b) is the case when $f = 1_{g(E)}$, thus the proof is complete. $\square$

3.7 Integration in Polar Coordinates

It is omitted in the class.

4 Signed Measures and Differentiation

4.1 Signed Measures

Definition 4.1 (Signed measure). *A* **signed measure** *on* $(X, \mathcal{M})$ *is* $\gamma : \mathcal{M} \to [-\infty, +\infty]$ *satisfying*

(i) $\nu(\emptyset) = 0$

(ii) At most one of $+\infty, -\infty$ *is in the range of* γ.

(iii) If $E_1, E_2, \cdots \in \mathcal{M}$ *are disjoint, then* $\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j)$.

Remark 4.2.

$$\{\textit{measures}\} \subseteq \{\textit{ signed measures}\}.$$

Let us denote a measure as **positive measure**.

Example 4.3 (Example of signed measure). *Firstly,* μ_1, μ_2 *are positive measures, at least one of which is finite, then* $\nu := \mu_1 - \mu_2$ *is a signed measure on* $(X, \mathcal{M})$.

Next, if f *is* **extended** μ**-integrable**, *i.e., at least one of* $\int f^+ d\mu$ *or* $\int f^- d\mu$ *is finite, and* $f : X \to [-\infty, \infty]$ *is measurable with respect to a positive measure* μ, *then* $\nu(E) := \int_E f d\mu$ *is a signed measure. Note that* $f^+ = \min(f, 0), f^- = -\max(0, -f)$.

Proposition 4.4 (Proposition 3.1. in [1] p.86).

1. *If* $(E_n)_{n=1}^{\infty}$ *is an increasing sequence in* $\mathcal{M}$, *then* $\nu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \nu(E_n)$.
2. *If* $(E_n)_{n=1}^{\infty}$ *is an decreasing sequence in* $\mathcal{M}$ *and if* $\nu(E_1)$ *is finite, then* $\nu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \nu(E_n)$.

Proof. For the first one, let $F_n = E_n \setminus \bigcup_{j=1}^{n-1} E_j$. Then, F_ns are disjoint and $\bigcup_{j=1}^{n} E_j = E_n = \bigcup_{j=1}^{n} F_n$ for any $n \in \mathbb{N}$. Thus, by the property (iii) of a signed measure,

$$\lim_{n\to\infty} \nu(E_n) \underbrace{=}_{\text{(iii)}} \sum_{j=1}^{\infty} \nu(F_j) \underbrace{=}_{\text{(iii)}} \nu\left(\bigcup_{j=1}^{\infty} F_j\right) = \nu\left(\bigcup_{j=1}^{\infty} E_j\right).$$

For the second one, if $(E_n)_{n=1}^{\infty}$ is a decreasing sequence, then

$$\bigcap_{n=1}^{\infty} E_n = E_1 \setminus \left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right),$$

and $E_1 \setminus \bigcap_{n=1}^{\infty} E_n$ and $\bigcap_{n=1}^{\infty} E_n$ are disjoint. Thus,

$$\begin{aligned}
\nu\left(\bigcap_{n=1}^{\infty} E_n\right) &= \nu\left(E_1 \setminus \left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right)\right) \\
&= \nu(E_1) - \nu\left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right) && \text{from disjointness of } E_1 \setminus \bigcap_{n=1}^{\infty} E_n \text{ and } \bigcap_{n=1}^{\infty} E_n \\
&= \nu(E_1) - \nu\left(E \cap \left(\bigcup_{n=1}^{\infty} E_n^c\right)\right) && \text{from De Morgan's Law} \\
&= \nu(E_1) - \nu\left(\bigcup_{n=1}^{\infty} (E_1 \cap E_n^c)\right) && \text{from Distribution law of ZF axiom}
\end{aligned}$$

And since $(E_1 \setminus E_n)_{n=1}^{\infty}$ is an increasing sequence, we can use the part (a). Thus,

$$\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = \nu(E_1) - \nu\left(\bigcup_{n=1}^{\infty}(E_1 \cap E_n^c)\right) \underbrace{=}_{\text{part (a)}} \nu(E_1) - \lim_{n\to\infty} \nu(E_1 \setminus E_n) = \nu(E_1) - \nu(E_1) + \lim_{n\to\infty} \nu(E_n).$$

Thus, $\nu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \nu(E_n)$, as desired. □

Definition 4.5 (Positive, negative and null set)**.** *Let ν be a signed measure on $(X, \mathcal{M})$ and let $E \in \mathcal{M}$. We say E is*

- **Positive** *if* $\forall F \in \mathcal{M}, F \subseteq E \implies \nu(F) \geq 0$.
- **Negatvie** *if* $\forall F \in \mathcal{M}, F \subseteq E \implies \nu(F) \leq 0$.
- **Null set** *if* $\forall F \in \mathcal{M}, F \subseteq E \implies \nu(F) = 0$.

Lemma 4.6 (Lemma 3.2 in [1] p.86)**.**

(a) Every measurable subset of a positive set is a positive set.

(b) Every countable union of positive sets is a positive set.

Proof. For (a), let E be a positive set and F be a subset of E. Since every subset of F is also a subset of E, thus it has positive measure, therefore F is also a positive set.

For (b), suppose $E_1, \cdots \in \mathcal{M}$ are positive sets. Let $E = \bigcup_{n=1}^{\infty} E_n$. It suffices to show that E is positive. Replacing E_n by $E_n \setminus_{k=1}^{n-1} E_k$ if necessary, we may assume that $E_1, \cdots$ are disjoint without loss of generality. Then, let $F \subset E$. Then,

$$\nu(F) = \nu\left(\bigcup_{n=1}^{\infty} F \cap E_n\right) = \sum_{n=1}^{\infty} \nu(F \cap E_n) \geq 0,$$

where first inequality comes from $E = \bigcup_{n=1}^{\infty} E_n$ and the second equality comes from disjointness of $F \cap E_n$ inherited from E_ns, jand the last inequality comes from the fact that $F \cap E_n$s are positive sets since each of them is a subset of E_n respectively. Thus, E is a positive set. □

Theorem 4.7 (The Hanh Decomposition Theorem, Theorem 3.3 in [1] p.86)**.** *Let ν be a signed measure on $(X, \mathcal{M})$. Then $\exists$ a positive set $P \in \mathcal{M}$ such that $N = X \setminus P$ is negative. We call $X = P \cup N$ a* **Hanh decomposition** *for ν. And this decomposition is unique up to symmetric difference of null set; if there exists another $P', N' \subset X$ such that $P' \cup N' = X$ and $P' \cap N' = \emptyset$ and P' is positive and N' is negative, then $P \Delta P' = N \Delta N'$ is a null set.*

Proof. Without loss of generality, let $\nu : \mathcal{M} \to [-\infty, +\infty)$. Let $r = \sup\{\nu(P) : P \in \mathcal{M}, P \text{ is positive}\}$. Since $\emptyset \in \{\nu(P) : P \in \mathcal{M}, P \text{ is positive}\}$, r is well-defined, thus we have a sequence of positive sets P_n such that $\lim_{n\to\infty} \nu(P_n) = r$. Let $P = \bigcup_{n=1}^{\infty} P_n$. Then by the lemma, P is a positive set, thus

$$r \geq \nu(P) = \nu(P_n) + \nu(P \setminus P_n)$$

and since $\nu(P \setminus P_n) \geq 0$ since $P \setminus P_n$ is a subset of a positive set P,

$$r \geq \nu(P) \geq \nu(P_n).$$

By letting $n \to \infty$, we have $\nu(P) = r$. Let $N = X \setminus P$.

Claim 4.8. *If $Ein\mathcal{M}, E \subset N$, and E is positive, then E is a null set.*

Proof. Let E' be a measurable subset of E. Then,

$$\nu(P \cup E') = \nu(P) + \nu(E') = r + \nu(E')$$

and $P \cup E$ is positive. However, by choice of r, $\nu(E') \leq 0$. Since E' is positive, $\nu(E') \geq 0$. Thus, $\nu(E') = 0$. Since E' was arbitrarily chosen, E is a null set. $\square$

Claim 4.9. *If $A \in \mathcal{M}, A \subseteq N$ and $\nu(A) > 0$, then $\exists B \in \mathcal{M}, B \subseteq A$ such that $\nu(B) > \nu(A)$.*

Proof. We used $r < +\infty$ since $\nu(P) = r \in [0, +\infty)$. Suppose A is not a null set. Then by claim 1, A is not positive. Thus, $\exists C \in \mathcal{M}, C \subseteq A$ such that $\nu(C) < 0$. Let $B = A \setminus C$. Then, $\nu(B) = \nu(A) - \nu(C) > \nu(A)$. $\square$

Claim 4.10. *N is negative.*

Proof. Suppose for contradiction that N is not a negative set. Then $\exists A_0 := A \in \mathcal{M}, A \subseteq N$ such that $\nu(A) > 0$. By claim 2, $\exists A_1 \in \mathcal{M}, A_1 \subseteq A$, such that $\nu(A) > \nu(A)$. Let $n_1 \in \mathbb{N}$ be the least number such that $\exists A_1 \subset A_0$ with $\nu(A_1) > \nu(A_0) + \frac{1}{n_1}$. Likewise, let $n_2 \in \mathbb{N}$ be the least number such that $\exists A_2 \in \mathcal{M}, A_2 \subseteq A_1$ with $\nu(A_2) > \nu(A_1) + \frac{1}{n_2}$. We can continue in this manner, thus we get

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \text{ and } 0 < \nu(A_0) < \nu(A_1) < \cdots,$$

and integers $n_1, n_2, \cdots \in \mathbb{N}$ such that $\forall j \in \mathbb{N}$. Let $E = \bigcap_{n=0}^{\infty} A_n$. Then, by the Proposition 3.1.(b) in [1],

$$\nu(E) = \lim_{n \to \infty} \nu(A_n) \in (0, +\infty)$$

since $\nu(A_n) \in (0, +\infty)$ for all $n \in \mathbb{N}$, by assumption that the range of ν doesn't contain $+\infty$. However,

$$\nu(A_k) \geq \sum_{j=1}^{k} \frac{1}{n_j} + \nu(A_0).$$

By letting $k \to \infty$, the left hand side goes to $\nu(E)$, and the right hand side goes to $\sum_{j=1}^{\infty} \frac{1}{n_j} + \nu(A_0)$. Thus,

$$\sum_{j=1}^{\infty} \frac{1}{n_j} < +\infty \implies n_j \to \infty \text{ as } j \to \infty.$$

Note that $E \subset N$ and $\nu(E) > 0$. Thus, by claim 2, $\exists B \subset E$ such that $\nu(B) > \nu(E)$. Let $m \in \mathbb{N}$ such that $\nu(B) > \nu(E) + \frac{1}{m}$. Let $j \in \mathbb{N}$ such that $n_j > m$. Since $n_j \to \infty$ as $j \to \infty$, such j exists. Then,

$$B \subseteq E \subseteq A_{j-1} \text{ and } \nu(B) \geq \nu(E) + \frac{1}{m} \geq \nu(A_{j-1}) + \frac{1}{m},$$

and this contradicts the minimality of n_j, which is chosen by the least one. Thus N should be a negative set. $\square$

To show uniqueness, let P, N, P', N' are given in the problem. Note that $P \setminus P' = N' \setminus N$ is both positive and negative, thus it is a null set. By the same argument, $P' \setminus P = N \setminus N'$ is null. Hence,

$$P \Delta P' = (P' \setminus P) \cup (P \setminus P') = (N \setminus N') \cup (N' \setminus N) = N \Delta N'$$

is a union of null sets, thus it is null. $\square$

Definition 4.11 (Mutually singular)**.** *Let μ and ν be signed measure on $(X, \mathcal{M})$. We say that μ and ν are mutually singular and write*

$$\mu \perp \nu$$

, when $X = E \cup F, E \cap F = \emptyset, E, F \in \mathcal{M}$ such that E is null for μ and F is null for ν.

Theorem 4.12 (The Jordan Decomposition Theorem, Theorem 3.4 in [1] p. 87)**.** *Let ν be a signed measure on $(X, \mathcal{M})$. Then $\exists!$ positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$, and $\nu^+ \perp \nu^-$.*

Proof. Let $X = P \cup N$ be a Hanh decomposition for ν. Let $\nu^+(A) = \nu(A \cap P), \nu_-(A) = -\nu(A \cap N)$. So ν^+, ν^- are also positive measure. Clearly, $\nu = \nu^+ - \nu^-$ since $\nu(A) = \nu(A \cap P) + \nu(A \cap N) = \nu^+(A) - \nu^-(A)$, and $\nu^-(P) = \nu(P \cap N) = 0, \nu^+(N) = \nu(P \cap N) = 0$. Thus, P is null for ν^- and N is null for ν^+.

To see uniqueness, suppose $\nu = \mu^+ - \mu^-$ for some positive measures μ^+ and μ^- with $\mu^+ \perp \mu^-$. There exists $E, F \in \mathcal{M}$ such that $E \cup F = X, E \cap F = \emptyset$ and E is μ^+-null set, and F is μ^--null set. Then, E is ν-negative, and F is ν-positive, so $X = F \cup E$ is a Hanh decomposition for ν. By the Hanh decomposition theorem, $F \Delta P = E \Delta N$ is ν-nul, so $\forall A \in \mathcal{M}$,

$$A \cap (F \Delta P) = (A \cap F) \Delta (A \cap P)$$

is ν-null. Thus,

$$\mu^+(A) = \mu^+(A) - \mu^+(A \cap E) = \mu^+(A \cap F) = \mu^+(A \cap F) - \mu^-(A \cap F) = \nu(A \cap F) = \nu(A \cap P) = \nu^+(A).$$

So $\mu^+ = \nu^+$, thus $\mu^- = \nu^-$. □

Definition 4.13 (Positive Variation, Total variation)**.** *We call ν^+ the* **positive variations** *of ν and ν^- the negative variations of ν. The* **total variation** *of ν is a measure $|\nu| := \nu^+ + \nu^-$.*

Definition 4.14 (Finite, σ-finite signed measure)**.** *A signed measure ν is finite iff $|\nu|$ is a finite (positive) measure. Also, a signed measure ν is σ-finite iff $|\nu|$ is a σ-finite (positive) measure.*

Observation 4.15. *The followings are equivalent.*

(i) ν is finite

(ii) $\nu : \mathcal{M} \to (-\infty, +\infty)$

(iii) ν^+ and ν^- are finite.

Proof. Suppose (i). Then, $|\nu|(X) = \nu^+(P) + \nu^-(N) < \infty$, thus

$$-\infty < \nu^-(N) \leq \nu^+(E) - \nu^-(E) = \nu(E) = \nu^+(E) - \nu^-(E) \leq \nu^+(P) < \infty.$$

Hence, (ii) holds. Also, if (ii) holds, then $\nu^+(X) = \nu^+(P) < \infty$ and $\nu^-(X) = \nu^-(N) < \infty$, thus they are finite measure. If (iii) holds, then, $|\nu| = \nu^+ + \nu^-$ is finite positive measure, thus μ is finite. □

Definition 4.16 (L^1 of a signed measure)**.** *Let $L^1(\nu) := L^1(|\nu|)$. Hence for any $f \in L^1(\nu), \int f d\nu = \int f d\nu^+ - \int f d\nu^- < \infty$.*

Example 4.17.

Some examples, and Push forward $m \perp (\sum_n \lambda_n \delta_{t_n})$ *where m is the Lebesgue measure, and δ_{t_n} is a Dirac measure at $t_n \in \mathbb{R}$, and $\lambda_n \in \mathbb{R}$ for any $n \in \mathbb{N}$. Note that Dirac measure gives 1 and 0 only.*

Some exbmples, bnd Push forwbrd m^2 *be the Lebesgue measure on $\mathbb{R}^2$, and $D = \{(x, x) : x \in \mathbb{R}\}$. Then, $m^2(D) = 0$. Let $\mu(A) := m(\{x : (x, x) \in A\})$. Then, $\mu \perp m^2$. Note that this $\mu = i_* m$ is called* **push-forward of** *m where $i : \mathbb{R} \to D$.*

If $f : X \to Y$ is measurable and if σ is a measure on X, then $f_*\sigma$ is a measure on Y given by $f_*\sigma(E) = \sigma(f(E))$.

Proof. To see that $m^2(D)=0$, start with $D_k=\{(x,x):|x|\leq k\}$. By rotating this D, $m^2(D)=m^2(\{(x,0):|x|\leq k\})$. Now let $A_{n,k}=\{(x,y):|x|\leq k,|y|\leq \frac{1}{n}\}$. Then, $A_{n,k}$s are decreasing sequence with $m^2(A_{n,k})=\frac{2}{n}$. And $\{(x,0):|x|\leq k\}\subseteq A_{n,k}$ for any $n\in\mathbb{N}$. Thus,

$$0\leq m^2(D_k)=m^2(\{(x,0):|x|\leq 1\})\leq m^2(A_{n,k})=\frac{2k}{n}$$

for all $n\in\mathbb{N}$. Thus, by letting $n\to\infty$, $m^2(D_k)=0$. From the fact that D_ks are increasing seqeunce, using continuity from below

$$m^2(D)=m^2\left(\bigcup_{k=1}^{\infty}D_k\right)=\lim_{n\to\infty}m^2(D_n)=0.$$

Hence, since $i(A)\subseteq D$, for any $A\in\mathcal{M}$,

$$\mu(A)=m^2(i(A))\leq m^2(D)=0$$

Since μ is zero mesure, we can conclude that $\mu\perp m^2$. $\square$

Definition 4.18 (Absolute continuity)**.** *Let ν be a signed measure and μ be a positive measure on $(X,\mathcal{M})$. We say ν is* **absolutely continuous** *with respect to μ or*

$$\nu\ll\mu$$

if $\forall E\in\mathcal{M},\mu(E)=0\implies\nu(E)=0$, equivalently, if E is a μ-null set, then E is a ν-null set.

Observation 4.19 (Exercise 3.8, proved in the Homework)**.** $\nu\ll\mu\iff\nu^+\ll\mu$ *and* $\nu^-\ll\mu$

Example 4.20. *Fix μ. Choose $f^+,f^-\in L^+(\mu)$ with $\int f^+d\mu<+\infty$ or $\int f^-d\mu<+\infty$. Then, without loss of generality, up to μ-null sets, either $f^+:X\to[0,+\infty)$ or $f^-:X\to[0,+\infty)$. Let $f:=f^+-f^-$ and let $\nu(E):=\int_E fd\mu=\int_E f^+d\mu-\int_E f^-d\mu$. Then, ν is a signed measure and $\nu\ll\mu$.*

Proof. From countable additivity of integration, and by construction avoiding ∞, ν is a signed measure. Also, for any μ-null set E, $\nu(E)=\int_E fd\mu=0$, by approximating the integral using simple functions. Hence, $\nu\ll\mu$. $\square$

Observation 4.21. *If $\nu\perp\mu$ and $\nu\ll\mu$, then $\nu=0$*

Proof. From $\nu\perp\mu$, there exists $E,F\in\mathcal{M}$ such that $E\cup F=X,E\cap F=\emptyset$ and E is μ null set and F is ν null set. Also, since $\nu\ll\mu$, for any subset $B\subset E$, $\nu(B)=0$ from the fact $\mu(B)=0$. Thus for any $A\in\mathcal{M}$,

$$\nu(A)=\nu(A\cap E)+\nu(A\cap F)=0+0=0.$$

$\square$

Theorem 4.22 (Theorem 3.5 in [1] p.89)**.** *Let μ be a positive measure and let ν be a finite signed measure on $(X,\mathcal{M})$. Then,*

$$\nu\ll\mu\iff(\forall\epsilon,\exists\delta>0\text{ such that }E\in\mathcal{M},\mu(E)<\delta\implies|\nu|(E)<\epsilon)\cdots(*)$$

Proof. Without loss of generality, let $\nu=|\nu|$ be a positive measure. Then, if $(*)$ holds, let $E\in\mathcal{M}$ with $\mu(E)=0$. Then by $(*)$, $\nu(E)<\epsilon$ for all ϵ, thus $\nu(E)=0$. Hence $\nu\ll\mu$.

Conversely, to prove by contrapositive statement, assume $(*)$ fails. Then, $\exists\epsilon>0$ such that $\forall\delta>0$, $\exists E\in\mathcal{M}$ such that $\mu(E)<\delta$ but $\nu(E)\geq\epsilon$. Choose $E_n\in\mathcal{M}$ such that $\mu(E_n)<2^{-n}$ with $\nu(E_n)\geq\epsilon$. Let $G:=\limsup_{n\to\infty}E_n=\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}E_n$, and let $F_k:=\bigcup_{n=k}^{\infty}E_n$. Then $(F_n)_{n=1}^{\infty}$ is a decreasing sequence, thus

$$\mu(G)\leq\mu(F_k)\leq\sum_{n=k}^{\infty}\mu(E_n)<\sum_{k=1}^{\infty}2^{-n}=2^{-k+1}.$$

By letting $k \to \infty$, $\mu(G) = 0$. However,

$$\nu(F_k) \geq \nu(E_k) \geq \epsilon,$$

and ν is finite, this gives us to continuity from above, thus we have

$$\nu(G) = \lim_{k\to\infty} \nu(F_k) \geq \epsilon,$$

thus $\nu \not\ll \mu$.

For the case when ν is finite signed measure, this follows from the Exercise 8 that $\nu \ll \mu \iff |\nu| \ll \mu$. ☐

Corollary 4.23 (Corollary 3.6 in [1] p.89)**.** *If $f \in L^1(\mu)$, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\int_E f d\mu| < \epsilon$ whenever $\mu(E) < \delta$.*

Proof. Since f is extended μ-integrable real-valued function, $\nu(E) := \int_E f d\mu$ is a finite signed measure and $\nu \ll \mu$. Thus, by the theorem, the statement holds. If f is complex valued function, apply the theorem on Ref and Imf, and take such δs for Ref and Imf using $\frac{\epsilon}{2}$. ☐

Lemma 4.24 (Lemma 3.7 in [1] p.89)**.** *If ν and μ are finite positive measure on $(X, \mathcal{M})$, then either $\nu \perp \mu$ or $\exists \epsilon > 0, \exists E \in \mathcal{M}$ such that $\nu \geq \epsilon\mu$ on E, i.e.,*

$$\forall A \in \mathcal{M}, A \subset E \implies \nu(A) \geq \epsilon\mu(A).$$

Proof. Suppose ν and μ are positive measures. Then, given $n \in \mathbb{N}$, consider the signed measure $\nu - \frac{1}{n}\mu$ and let $X = P_n \cup N_n$ be Hanh decomposition for $\nu - \frac{1}{n}\mu$. Let $P = \bigcup_{n=1}^{\infty} P_n$ and $N = \bigcap_{n=1}^{\infty} N_n$. Then, $P \cup N = X$, $P \cap N = \emptyset$, and $\forall n \in \mathbb{N}$, from the fact that $N \subset N_n$,

$$(\nu - \frac{1}{n}\mu)(N) \leq 0 \implies 0 \leq \nu(N) \leq \frac{1}{n}\mu(N).$$

So letting $n \to \infty$, $\nu(N) = 0$ Hence N is null set for ν. If $\mu(P) = 0$, then $\mu \perp \nu$. If $\mu(P) > 0$, then since $\mu(P) \leq \sum_{n=1}^{\infty} \mu(P_n)$, there exists $n \in \mathbb{N}$ such that $\mu(P_n) > 0$. Take $E = P_n$. Suppose $A \in \mathcal{M}, A \subset E$. Since P_n is positive for $\nu - \frac{1}{n}\mu$, we have

$$(\nu - \frac{1}{n}\mu)(A) \geq 0 \implies \nu(A) \geq \frac{1}{n}\mu(A).$$

Thus, $\nu \geq \frac{1}{n}\mu$ on E. ☐

Note that Folland says he proved the Lemma 3.7 in general case. However, this proof require that (at least) ν should be finite positive measure. Thus I limited the condition for the lemma. This limited version is enough for proving Radon-Nikodym theorem.

Theorem 4.25 (The Lebesgue-Radon-Nikodym Theorem, 3.8 in [1] p.90)**.** *Let μ be a σ-finite measure and ν be a σ-finite signed measure on $(X, \mathcal{M})$. Then, $\exists!$ pair of σ-finite signed measure (λ, ρ) on $(X, \mathcal{M})$ such that*

$$\nu = \lambda + \rho \text{ and } \lambda \perp \mu \text{ and } \rho \ll \mu.$$

Moreover, $\exists$ an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$

Definition 4.26 (Extended μ-integrable function, revisited)**.** *A μ-measurable function $f : X \to \mathbb{R}$ or $f : X \to [-\infty, \infty]$ is* **extended μ-integrable** *function if letting $f^+ = \min(f, 0), f^- = -\max(0, -f)$. so that $f = f^+ - f^-$, then either $\int f^+ d\mu$ or $\int f^- d\mu$ is finite. and $f : X \to [-\infty, \infty]$ is measurable with respect to a positive measure μ, then $\nu(E) := \int_E f d\mu$ is a signed measure.*

Then, $\nu(E) := \int_E d\mu$ defines a signed measure $\nu \ll \mu$ and we write $d\nu = f d\mu$ or $f = \frac{d\nu}{d\mu}$.

Proof of the theorem. Without loss of generality, assume ν is a positive measure.

(i) ***Case I : ν is a positive finite measure.*** Let

$$\begin{aligned}\mathcal{F} &:= \{f : X \to [0,\infty] : f \text{ is measurable and } \forall E \in \mathcal{M}, \int_E f d\mu \le \nu(E)\}\\ &= \{f : X \to [0,\infty] : \exists \text{ a measure } \sigma \text{ s.t. } d\lambda = d\nu - f d\mu \text{ is a positive measure.}\}\end{aligned}$$

Thus if $\mathcal{F}$ is nonempty, then for any $f \in \mathcal{F}$ we can write $d\nu = d\lambda + d\mu$ for some positive measure λ. (Note that $0 \in \mathcal{F}$, thus $\mathcal{F}$ is nonempty.

Claim 4.27. *If $f, g \in \mathcal{F}$ then $h := \max(f,g) \in \mathcal{F}$*

Proof. Let $h = \max(f,g)$, $A := \{x \in X : f(x) \ge g(x)\}$. Then for any $E \in \mathcal{M}$,

$$\int_E h d\mu = \int_{E\cap A} h d\mu + \int_{E\setminus A} h d\mu = \int_{E\cap A} f d\mu + \int_{E\setminus A} g d\mu \underbrace{\le}_{f,g\in\mathcal{F}} \nu(E\cap A) + \nu(E \setminus A) = \nu(E).$$

Let $r = \sup\{\int f d\mu : f \in \mathcal{F}\}$. Then, $r \le \nu(X) < +\infty$, from finiteness of ν. □

Claim 4.28. *There exists $f \in \mathcal{F}$ such that $\int f d\mu = r$.*

Proof. Choose $f_n \in \mathcal{F}$ such that $r = \lim_{n\to\infty} \int f_n d\mu$. Let $g_n = \max(f_1, \cdots, f_n)$. Then, by the previous claim, $g_n \in \mathcal{F}$. And $g_1 \le g_2 \le \cdots$. Also,

$$\int g_n d\mu \ge \int f_n d\mu.$$

Thus,

$$r = \lim_{n\to\infty} f_n \le \lim_{n\to\infty} \int g_n d\mu \le r \implies \lim_{n\to\infty} \int g_n d\mu = r.$$

Let $f = \lim_{n\to\infty} g_n$ pointwisely. Then, $f : X \to [0,+\infty]$ Then, by the Monotone convergence theorem,

$$\int f d\mu = \lim_{n\to\infty} \int g_n d\mu = r.$$

To see $f \in \mathcal{F}$, note that for any $E \in \mathcal{M}$,

$$\int_E f d\mu = \lim_{n\to\infty} \int_E g_n d\mu \le \nu(E).$$

Thus $f \in \mathcal{F}$, hence done. □

The f obtained by above claim is $f : X \to [0,\infty]$. Since $f \in L^1(\mu)$, this implies $f^{-1}(\{\infty\})$ is μ-null set. Thus, by redefining f on $f^{-1}(\{\infty\})$, for example, $\forall x \in f^{-1}(\{\infty\}), f(x) := 0$, we have $f : X \to [0,\infty)$. Similarly, we define the measure ρ by $\rho(E) := \int_E f d\mu$, thus $d\rho = f d\mu$, and let $\lambda = \nu - \rho$. Note that λ is a positive measure, since we just take f such that integration with f on any measurable set is less than ν-measure of the set.

Claim 4.29. $\lambda \perp \mu$.

Proof. Suppose not. Then by the lemma 3.7, $\exists \epsilon > 0, \exists F \in \mathcal{M}$ such that

$$\mu(F) > 0 \text{ and } \lambda \ge \epsilon\mu \text{ on } F.$$

This implies

$$f + \epsilon 1_F \in \mathcal{F}$$

since $\forall E \in \mathcal{M}$,

$$\begin{aligned}\int_E (f+\epsilon 1_F)d\mu = \int_E f d\mu + \epsilon\mu(F\cap E) = \rho(E) + \epsilon\mu(F\cap E) &= \rho(E\setminus F) + (\rho+\epsilon\mu)(E\cap F)\\ &\le \rho(E\setminus F) + (\rho+\lambda)(E\cap F) = \rho(E\setminus F) + \nu(E\cap F)\\ &\le \nu(E\setminus F) + \nu(E\cap F) = \nu(E).\end{aligned}$$

where first inequality comes from $\nu \ge \epsilon\mu$ on F and the last inequality comes from $\rho+\lambda = \nu$. However,

$$\int (f+\epsilon 1_F)d\mu = \int f d\mu + \epsilon\mu(F) = r + \epsilon\mu(F) > r,$$

contradicting to the maximality of r. □

Thus $\rho \ll \mu$, $\nu \perp \mu$, and $\nu = \lambda + \rho$, and $d\rho = f d\mu$ is constructed.

(ii) ***Case II: General case.*** If μ is σ-finite measure and ν is σ-finite signed measure on $(X, \mathcal{M})$, let $E_1, E_2, \cdots \in \mathcal{M}$ be disjoint such that

$$\bigcup_{n=1}^{\infty} E_n = X \text{ and } \forall n \in \mathbb{N}, \mu(E_n) < +\infty, \text{ and } |\nu|(E_n) < +\infty.$$

Perform the construction for each pair (μ_n, ν_n) when

$$\mu_n(E_n) := \mu(A\cap E_n) \text{ and } \nu_n(A) := \nu(A\cap E_n).$$

Then, define

$$\lambda = \sum_{n=1}^{\infty} \lambda_n, \rho = \sum_{n=1}^{\infty} \rho_n, f = \sum_{n=1}^{\infty} f_n.$$

This works since E_ns are disjoint.

To see uniqueness, if $\nu = \lambda + \rho = \lambda' + \rho'$, $\lambda \perp \mu$, $\lambda' \perp \mu$, and $\rho \ll \mu$, $\rho' \ll \mu$, then $\lambda - \lambda' = \rho' - \rho$, and $\lambda - \lambda' \perp \mu$, $\rho' - \rho \ll \mu$. Thus,

$$\lambda - \lambda' = 0 = \rho' - \rho,$$

as desired.

□

Definition 4.30 (Lebesgue Decomposition)**.**

(1) For ν, μ in the theorem, such $\nu = \lambda + \rho, \lambda \perp \mu, \rho \ll \mu$ is called the **Lebesgue decomposition of ν with respect to μ.**

(2) If $\nu \ll \mu$, then $\rho = \nu$, thus $\exists$ extended μ-integrable function f such that $d\nu = f d\mu$. This f is called the **Radon-Nikodym derivative of ν with respect to μ.** *And we write*

$$f = \frac{d\nu}{d\mu} \text{ so that } d\nu = \left(\frac{d\nu}{d\mu}\right) d\mu.$$

Note that $\frac{d\nu}{d\mu}$ is unique up to redefinements of μ-null sets, as we constructed above.

Observation 4.31. *If on $(X, \mathcal{M})$, ν_1, ν_2 are σ-finite signed measure and μ is σ-finite (positive) measure, and*

$$\nu_1 \ll \mu, \nu_2 \ll \mu$$

and if either $\nu_1, \nu_2 : \mathcal{M} \to (-\infty, +\infty]$, then $\nu_1 + \nu_2$ is a σ-finite signed measure, $\nu_1 + \nu_2 \ll \mu$. And

$$(\nu_1 + \nu_2)(E) = \nu_1(E) + \nu_2(E) = \int_E \left(\frac{d\nu_1}{d\mu}\right) d\mu + \int_E \left(\frac{d\nu_2}{d\mu}\right) = \int_E \left(\frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}\right) d\mu$$

Thus, $\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}$.

Proof. Since ν_1, ν_2 are σ-finite measure, $|\nu_1|$, $|\nu_2|$ are σ-finite measure. Hence, there exists $E_1, \cdots$, and $F_1, \cdots$ such that $\bigcup_{i=1}^{\infty} E_i = X = \bigcup_{i=1}^{\infty} F_i$ and $|\nu_1|(E_i) < \infty, |\nu_2|(F_i) < \infty$. Let $G_{ij} = E_i \cap F_j$. Then, for a bijection between $\mathbb{N}^2$ and $\mathbb{N}$, we can regard $(G_{ij}) = (G_i)$, and

$$(\nu_1 + \nu_2)(G_i) \le \nu_1(E_k) + \nu_2(F_{k'}) < \infty \text{ for some } k, k' \in \mathbb{N},$$

also $\bigcup_{i=1}^{\infty} G_i = X$. Hence $(\nu_1 + \nu_2)$ is σ-finite. And if $E \in \mathcal{M}$ with $\mu(E) = 0$, then

$$(\nu_1 + \nu_2)(E) = \nu_1(E) + \nu_2(E) = 0 + 0 = 0.$$

Thus, $\nu_1 + \nu_2 \ll \mu$. The rest is obvious from calculation. □

Proposition 4.32 (Proposition 3.9 in [1] p.91)**.** *Suppose on $(X, \mathcal{M})$ μ and λ are σ-finite measure, ν is a σ-finite signed measure and $\nu \ll \mu$ and $\mu \ll \lambda$.*

(a) If $g \in L^1(\nu)$ then $g\frac{d\nu}{d\mu} \in L^1(\mu)$ and $\int g d\nu = \int g \left(\frac{d\nu}{d\mu}\right) d\mu$. Furthermore, if $g \in L^+(\mathcal{M})$ and ν is a positive measure, then $\frac{d\nu}{d\mu} \ge 0$ and $\int g d\nu = \int g \left(\frac{d\nu}{d\mu}\right) d\mu$.

(b) (Chain rule) From given condition, $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \left(\frac{d\lambda}{d\mu}\right)\left(\frac{d\mu}{d\lambda}\right) \text{ for } \lambda\text{-a.e.}$$

Proof. Using the Jordan Decomposition, $\nu = \nu^+ - \nu^-$. We may without loss of generality assume that ν is a positive measure. Suppose $g = 1_E$ for some $E \in \mathcal{M}$. Then,

$$\int 1_E d\nu = \nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int 1_E \left(\frac{d\nu}{d\mu}\right) d\mu.$$

So the given integration holds when $g = 1_E$ case. Since E was arbitrary and integration is linear functional, the integration formula holds for any simple function. Now let $g \in L^+(\mathcal{M})$. Then by theorem 2.10, there exists a sequence $(\phi_n)_{n=1}^{\infty}$ of simple functions,

$$\phi_1 \le \phi_2 \le \cdots, \phi_n \to g.$$

Thus, by the Monotone Convergence theorem,

$$\int g d\nu \underbrace{=}_{\text{MCT}} \lim_{n\to\infty} \int \phi_n d\nu = \lim_{n\to\infty} \int \phi_n \left(\frac{d\nu}{d\mu}\right) d\mu \underbrace{=}_{\text{MCT}} \int g \left(\frac{d\nu}{d\mu}\right) d\mu.$$

If ν is positive, then After redefining μ-null set of $\frac{d\nu}{d\mu}$'s domain, it is positive.

If $g \in L^1(\nu)$, then $g = g^+ - g^-$ for $g^+, g^- \in L^+(\mathcal{M}) \cap L^1(\nu)$. Thus,

$$\int g d\nu = \int g^+ d\nu - \int g^- d\nu = \int (g^+ - g^-) \frac{d\nu}{d\mu} d\mu = \int g \frac{d\nu}{d\mu} d\mu.$$

For part (b) with positive measure ν, let $E \in \mathcal{M}$. Then,

$$\nu(E) = \int_E \frac{d\nu}{d\lambda} d\lambda$$

whereas

$$\int_E \left(\frac{d\nu}{d\mu}\right)\left(\frac{d\mu}{d\lambda}\right) d\lambda = \int 1_E \cdot \left(\frac{d\nu}{d\mu}\right)\left(\frac{d\mu}{d\lambda}\right) d\lambda = \int g d\mu.$$

where the last equality comes from the fact that $1_E \left(\frac{d\nu}{d\mu}\right) \geq 0$ and $\int g \left(\frac{d\mu}{d\lambda}\right) d\lambda = \int g d\mu$ when we replace ν with μ and μ with λ in the part (a). Also,

$$\int g d\mu = \int 1_E \frac{d\lambda}{d\mu} d\mu = \int_E \frac{d\nu}{d\mu} d\mu = \nu(E),$$

where the last eqaulity comes from the definition of Radon-Nikodym derivative. Thus, by summing these calculation,

$$\int_E \frac{d\nu}{d\lambda} d\lambda = \nu(E) = \int_E \left(\frac{d\nu}{d\mu}\right)\left(\frac{d\mu}{d\lambda}\right) d\lambda,$$

which implies $\frac{d\nu}{d\lambda} = \left(\frac{d\nu}{d\mu}\right)\left(\frac{d\mu}{d\lambda}\right)$ λ-a.e. ☐

Corollary 4.33 (Corollary 3.10 in citefo, p.91)**.** *If μ and λ are σ-finite measure and $\mu \ll \lambda, \lambda \ll \mu$, then $\left(\frac{d\mu}{d\lambda}\right)\left(\frac{d\lambda}{d\mu}\right) = 1$ μ-a.e. and ν-a.e.*

Example 4.34 (Nonexample: Dirac δ-function)**.** *Let μ be Lebesgue measure and ν the point of mass at 0 on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then, $\nu \perp \mu$. Thus there is no Radon-Nikodym Derivative, even if Dirac δ-function behave like Radon-Nikodym derivative. (Actually, it is not the function.)*

Proposition 4.35 (Proposition 3.11 in [1] p.91)**.** *If $\mu_1, \cdots, \mu_n$ are positive measures on $(X, \mathcal{M})$ there is a measure μ such that $\mu_j \ll \mu$ for all j, namely, $\mu = \sum_{j=1}^n \mu_j$.*

Proof. Since each μ_i is positive, $\mu(E) = 0 \implies \mu_i(E) = 0$. for all $i \in \mathbb{N}$. ☐

4.2 Complex Measures

Definition 4.36 (Complex Measure)**.** *A complex measure on $(X, \mathcal{M})$ is $\nu : \mathcal{M} \to \mathbb{C}$ such that*

(i) $\nu(\emptyset) = 0$

(ii) If $E_1, E_2, \cdots \in \mathcal{M}$ are disjoint, then $\nu(\bigcup_j = 1)^\infty E_j = \sum_{j=1}^\infty \nu(E_j)$

where the series converges absolutely.

Observation 4.37. *If ν is a $\mathbb{C}$-measure and $\nu_r(E) = Re\nu(E), \nu_i(E) = Im\nu(E)$, then ν_r and ν_i are signed measure, and each omits $\pm\infty$. So, $\nu_r, \nu_i : \mathcal{M} \to [-k, k]$ for some $k \in \mathbb{N}$. Thus,* $\sup\{|\nu(E)| : E \in \mathcal{M}\} < \infty$.

Proof. This follows from that series converges absolutely. So a positive measure is a complex measure only if it is finite. Or if μ is positive measure and $f \in L^1(\mu)$, then $f d\mu$ is a positive measure. ☐

Definition 4.38 (Mutual singularity, absolute continuity for complex measure)**.** *If ν is a complex measure and μ is a signed measure, then $\nu \perp \mu$ means $\nu_r \perp \mu$ and $\nu_i \perp \mu$. If ν and μ are both complex measure then $\nu \perp \mu$ means $\nu \perp \mu_r$ and $\nu \perp \mu_i$. If ν is a complex measure and λ is a positive measure, then $\nu \ll \lambda$ means $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$.*

Theorem 4.39 (Complex version of the Lebesgue-Radon-Nikodym theorem, 3.12 in [1] p. 93)**.** *Let ν be a complex measure, μ be a σ-finite positive measures on $(X, \mathcal{M})$. Then, $\exists!$ complex measure λ, ρ such that*

$$\nu = \lambda + \rho, \lambda \perp \mu, \text{ and } \rho \ll \mu.$$

Also, $\exists f \in L^1(\mu)$ such that $d\rho = f d\mu$, where f is unique up to redefining of a μ-null set.

Proof. By applying the Lebesgue-Radon-Nikodym theorem on ν_r and ν_i, we have λ_i, λ_r and ρ_i, ρ_r and f_r, f_i which satisfy the desired properties. Then let $\lambda = \lambda_r + i\lambda_i, \rho = \rho_r + i\rho_i$, and $f = f_r + if_i$. Then desired property still holds. Also, uniqueness is derived from the each ν_i and ν_r. □

Proposition 4.40 (Lemma for defining Total variation of a complex measure)**.** *Let ν be a complex measure on $(X, \mathcal{M})$. Then, $\exists$ a finite measure μ such that $\nu \ll \mu$ and letting $f = \frac{d\nu}{d\mu}$ so that $d\nu = f d\mu$. Thus, by the complex version of Radon-Nikodym theorem(3.12) there exists $f = \frac{d\nu}{d\mu}$ so that $d\nu = f d\mu$.*

If also μ' is a finite measure and $d\nu = f' d\mu'$ then $|f|d\mu = |f'|d\mu'$.

Proof. Let $\mu = |\nu_r| + |\nu_i|$ Then, $\nu \ll \mu$. Let $\rho = \mu + \mu'$. Then, $\mu \ll \rho, \mu' \ll \rho$. Thus, $\mu = \left(\frac{d\mu}{d\rho}\right) d\rho, \mu' = \left(\frac{d\mu'}{d\rho}\right) d\rho$. By the chain rule,

$$f' \frac{d\mu'}{d\rho} d\rho = f' d\mu' = d\nu = f d\mu = f \frac{d\mu}{d\rho} d\rho.$$

Thus,

$$f' \frac{d\mu'}{d\rho} d\rho = \nu f \frac{d\mu}{d\rho} d\rho.$$

Then, by the uniqueness of the Radon-Nikodym derivative, we have

$$f' \frac{d\mu'}{d\rho} = f \frac{d\mu}{d\rho} \text{ for } \rho\text{-a.e.}$$

Thus,

$$|f'| \frac{d\mu'}{d\rho} = |f| \frac{d\mu}{d\rho} \text{ for } \rho\text{-a.e.} \implies |f| d\mu = |f'| d\mu'.$$

□

Definition 4.41 (Total variation for complex measure)**.** *Given a complex measure ν, its total variation is the finite positive measures $|\nu|$ defined by*

$$d|\nu| := |f| d\mu,$$

as in the previous proposition.

Observation 4.42. *If ν is a finite signed (real-valued) measure with Jordan decomposition $\nu = \nu^+ - \nu^-$, then we already defined*

$$|\nu|_{\mathbb{R}} := \nu^+ + \nu^-$$

Let's check that our new definition agrees with original one, i.e., $|\nu| = \nu^+ + \nu^-$. Note that $\nu^+ + \nu^-$ is a finite measure and $\nu \ll \nu^+ + \nu^-$ and letting $X = P \cup N$ be a Hanh decomposition for ν. So, we know

$$\nu^+(E) = \nu(E \cap P) \text{ and } \nu^-(E) = -\nu(E \cap N).$$

Thus,

$$\nu(E) = \int_E (1_P - 1_N) d(\nu^+ + \nu^-) \implies d\nu = 1_P - 1_N d|\nu|_{\mathbb{R}}.$$

By definition,

$$d|\nu| = |1_P - 1_N| d|\nu|_{\mathbb{R}} = d|\nu|_{\mathbb{R}}$$

since $|1_P - 1_N| = 1_X = 1$. So $|\nu| = |\nu|_{\mathbb{R}}$, as desired.

Definition 4.43 (L^1, L^+ for complex measure)**.** *For a complex measure ν, $L^1(\nu) := L^1(\nu_r) \cap L^1(\nu_i)$ and for $h \in L^1(\nu)$,*

$$\int h d\nu = \int h d\nu_r + i \int h d\nu_i.$$

Proposition 4.44 (Proposition 3.13 in [1] p. 94)**.** *Let ν be a complex measure on $(X, \mathcal{M})$. Then,*

(a) $\forall E \in \mathcal{M}, |\nu(E)| \leq |\nu|(E)$

(b) $\nu \ll |\nu|$ and $\frac{d\nu}{d|\nu|}$ has absolute value 1 ν-a.e.

(c) $L^1(\nu) = L^1(|\nu|)$ and $\forall f \in L^1(\nu)$,

$$|\int f d\nu| \leq \int |f| d|\nu|.$$

Proof. Let $d\nu = f d\mu, d|\nu| = |f| d\mu$ be as in definition of $|\nu|$. Then,

$$|\nu(E)| = \left| \int_E f d\mu \right| \leq \int |f| d\mu = |\nu|(E).$$

This proves (a) and show that $\nu \ll |\nu|$. For part (b), note that

$$f d\mu = d\nu = \left(\frac{d\nu}{d|\nu|} \right) d|\nu| = h|f| d\mu$$

Thus, $h|f| = f$ μ-a.e. However, $\{x : |f|(x) = 0\}$ is $|\nu|$-null set since $|\nu| = |f| d\mu$. Thus, $|h| = 1$, $|\nu|$-a.e.

For the part (c), Suppose ν be a complex measure on $(X, \mathcal{M})$. We want to show that $L^1(\nu) = L^1(|\nu|)$, and if $f \in L^1(\nu)$, then $| \int f d\nu| \leq \int |f| d|\nu|$. Before start the proof, we need a lemma.

Lemma 4.45. *$L^1(\nu) = L^1(|\nu|)$.*

Proof of the Lemma. For any simple function $f = \sum_{j=1}^n a_j 1_{A_j}$ with $A_j \in \mathcal{M}$,

$$\int f d|\nu| = \sum_{j=1}^n a_j |\nu|(A_j) = \sum_{j=1}^n a_j \nu^+(A_j) + \sum_{j=1}^n a_j \nu^-(A_j) = \int f d\nu^+ + \int f d\nu^-.$$

Thus for any $f \in L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$,

$$\int |f| d|\nu| \leq \int |f| d\nu^+ + \int |f| d\nu^- < \infty.$$

Since f was arbitrarily chosen, $L^1(\nu) \subseteq L^1(|\nu|)$. Conversely, if $f \in L^1(|\nu|)$, then

$$\int |f| d|\nu| = \int |f| d\nu^+ + \int f d\nu^- < \infty \implies \int |f| d\nu^+ < \infty \text{ and } \int f d\nu^- < \infty,$$

thus $f \in L^1(\nu^+) \cap L^1(\nu^-) = L^1(\nu)$. Hence $L^1(\nu) = L^1(|\nu|)$ □

From the definition of complex measure space and the lemma,

$$L^1(\nu) = L^1(\nu_r) \cap L^1(\nu_i) = L^1(|\nu_r|) \cap L^1(|\nu_i|).$$

Let $\mu := |\nu_r| + |\nu_i|$ Then, since each ν_r and ν_i are finite measure, so does μ. And also $\nu_r \ll \mu$ and $\nu_i \ll \mu$ by construction. Thus, by the Lebesgue-Radon-Nikodym theorem (Theorem 3.8), we have a μ-integrable function $g_r : X \to \mathbb{R}$ and $g_i : X \to \mathbb{R}$ such that

$$\nu_r = g_r d\mu \text{ and } \nu_i = g_i d\mu.$$

Thus, for any $E \in \mathcal{M}$,

$$\nu(E) = \int_E d\nu = \int_E d\nu_r + i\int_E d\nu_i = \int_E g_r + ig_i d\mu.$$

Let $g := g_r + ig_i$. Then, by the uniqueness of Radon-Nikodym derivative from theorem 3.8, $d|\nu| = |g|d\mu, d|\nu_r| = |g_r|d\mu$, and $d|\nu_i| = |g_i|d\mu$. Also, the triangle inequality gives $|g_r| \le |g|$ and $|g_i| \le |g|$, thus

$$|\nu_r|(E) = \int_E |\nu_r| = \int_E |g_r|d\mu \le \int_E |g|d\mu = |\nu|(E)$$
$$|\nu_i|(E) = \int_E |\nu_i| = \int_E |g_i|d\mu \le \int_E |g|d\mu = |\nu|(E)$$

Thus, $\nu_r \ll |\nu|, \nu_i \ll |\nu|$, therefore by the Lebesgue-Radon-Nikodym theorem, there exists $|\nu|$-integrable function h_r, h_i such that $\nu_r = h_r|\nu|, \nu_i = h_i|\nu|$. Thus, for any $E \in \mathcal{M}$,

$$\nu(E) = \int_E h_r + ih_i d|\nu| = \int_E (h_r + ih_i)|g|d\mu,$$

Let $h := h_r + ih_i$. Since $\nu(E) = \int_E g d\mu$, this implies

$$g = h|g| \text{ for } \mu\text{-a.e.}$$

And since $|\nu| \ll \mu$, this implies $g = h|g|$ $|\nu|$-a.e. Now let $E = g^{-1}(\{0\})$. Since g is $|\nu|$ measurable, E is $|\nu|$ measurable, therefore

$$|\nu|(E) = \int_E |g|d\mu = \int_E 0 d\mu = 0.$$

Thus, $g \neq 0$ $|\nu|$-a.e., this implies $|h| = 1$ $|\nu|$-a.e. Also, triangle inequality implies $|h_r| \le |h| = 1$ and $|h_i| \le |h| = 1$.

To show the inequality, let $f \in L^1(\nu) \underbrace{=}_{\text{by the Lemma}} L^1(|\nu|)$. Then,

$$\int |f|d|\nu_r| = \int |f||h_r|d|\nu| \underbrace{\le}_{\text{since } |h_r|\le 1} \int |f|d|\nu| < \infty$$
$$\int |f|d|\nu_i| = \int |f||h_i|d|\nu| \underbrace{\le}_{\text{since } |h_i|\le 1} \int |f|d|\nu| < \infty$$

Thus, $f \in L^1(|\nu_r|) \cap L^1(|\nu_i|) = L^1(\nu_r) \cap L^1(\nu_i)$. Conversely, if $f \in L^1(|\nu_r|) \cap L^1(|\nu_i|)$, then

$$\int |f|d|\nu| = \int |f|1d|\nu| = \int |f||h_r + ih_i|d|\nu| \le \int |f||h_r|d|\nu| + \int |f||h_i|d|\nu| = \int |f|d|\nu_r| + \int |f|d|\nu_i| < \infty$$

Thus, $f \in L^1(|\nu|) = L^1(\nu)$.

Also, note that for $f \in L^1(\nu) = L^1(|\nu|)$,

$$\left|\int f d\mu\right| = \left|\int f d\nu_r + if d\nu_i\right| = \left|\int f h_r d|\nu| + if h_i d|\nu|\right| = \left|\int f(h_r + ih_i)d|\nu|\right| \le \int |f|\,|h_r + ih_i|\,d|\nu| = \int |f|d|\nu|,$$

as desired. □

Proposition 4.46 (Proposition 3.14 in [1] p.94)**.** *If ν_1 and ν_2 are complex measures on $(X, \mathcal{M})$, then $|\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|$*

Proof. By the proposition 4.40, we can write $\nu_j = f_j d\mu$ for $j = 1, 2$ and some μ-integrable function f_j. Then

$$d|\nu_1 + \nu_2| = |f_1 + f_2|d\mu \le |f_1|d\mu + |f_2|d\mu = d|\nu_1| + d|\nu_2|.$$

□

4.3 Differentiation on Euclidean Space

On $\mathbb{R}$, take $d\nu = fdm$, where dm is the Lebesgue measure, with $f \in L^1(m)$. Let

$$F(t) := \begin{cases} \nu((0,t]) & t > 0 \\ 0 & t = 0 \\ -\nu([t,0]) & t < 0 \end{cases}.$$

Does this function F differentiable? If it is differentiable, we should show that

$$\lim_{r\searrow 0} \frac{F(t_0 + r) - F(t_0 - r)}{2r} = \lim_{r\searrow 0} \frac{\nu((t_0 - r, t_0 + r])}{m((t_0 - r, t_0 + r]} = \lim_{r\searrow 0} \frac{\int_{[t_0-r,t_0+r]} f(y)dm(y)}{m([t_0 - r, t_0 + r]} \overbrace{=}^{?} f(t_0).$$

To show this, fix $n \in \mathbb{N}$ and let m denote the Lebesgue measure on $\mathbb{R}^n$.

Definition 4.47 (Locally integrable)**.** *$f : \mathbb{R}^n \to \mathbb{C}$ is* **locally integrable** *if $\int_K |f(x)|dm(x) < +\infty$ for all bounded measureable set $K \subseteq \mathbb{R}^n$. It is enough to take $K = B(r, x)$ where*

$$B(r, x) := \{y \in \mathbb{R}^n : |y - x| < r\}.$$

Let L^1_{loc} denote the set of locally integrable functions f on $\mathbb{R}^n$.

Remark 4.48 (Notation)**.** *For $f \in L^1_{loc}(\mathbb{R}^n), x \in \mathbb{R}^n, r > 0$, let*

$$A_r f(x) := \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy.$$

"A" means average.

(1) $E \in \mathbb{R}^n$ is measurable, then $m(rE) = r^n m(E)$ by the theorem 2.44 in [1][p.73].

(2) $m(B(r, x)) = m(B(r, 0)) = \frac{r^n \pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$. where

$$\Gamma(1 + k) = k!, \Gamma(k + \frac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \cdots (2k - 1)}{2^k}\sqrt{\pi}.$$

Definition 4.49 (Jointly continuous)**.** *A function $g(a, b)$ is* **jointly continuous** *iff $g(a, b) \to g(x, y)$ when $a \to x$ and $b \to y$.*

Lemma 4.50 (Lemma 3.16 in [1]p.96)**.** *If $f \in L^1_{loc}(\mathbb{R}^n)$, then $A_r f(x)$ is jointly continuous in $r > 0$ and $x \in \mathbb{R}^n$*

Proof. This is equivalent to showing $(r, x) \mapsto \int_{B(r,x)} f(y)dm(y)$ is jointly continuous. Let $r_k \to r > 0$, $x_k \to x$. Then, for some large enough k, $|1_{B(r_k,x_k)} f| \leq |1_{B(r_k,x)} f|$. Thus, by the DCT,

$$\lim_{k\to\infty} \int 1_{B(r_k,x_k)} f dm \underbrace{=}_{\text{DCT}} \int 1_{B(r,x)} f dm.$$

□

Definition 4.51 (Hardy Littlewood Maximal Function)**.** *If $f \in L^1_{loc}(\mathbb{R}^n)$ then the* **Hardy-littlewood maximal function of** *f is*

$$(Hf)(x) = \sup_{r>0} A_r|f|(x).$$

Thus,

$$(Hf)^{-1}((t, \infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((t, \infty)).$$

and this set is open by the lemma 3.16 since $A_r(f)$ is continuous on any $t \in \mathbb{R}^n$.

Theorem 4.52 (The Hardy-Littlewood maximal theorem, 1930, 3.17 in [1] p.96)**.** *$\exists C > 0$ depnding on n such that $\forall f \in L^1(m), \forall \alpha > 0$,*

$$m(\{x : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int |f| dm.$$

Or since $\|f\|_1 = \int |f| dm$, we can restate it as

$$m(\{x : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1 .$$

To see this, we need a technical lemma, 3.15 in [1].

Lemma 4.53 (Lemma 3.15 in [1], p.96)**.** *Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$ and let $u = \bigcup_{B \in \mathcal{C}} B$. If $t < m(U)$, there exists a disjoint sequence $B_1, \cdots B_k \in \mathcal{C}$ such that $\sum_{j=1}^k m(B_j) > 3^{-n} t$.*

Proof. By theorem 2.40 in [1], there exists a compact set $K \subset U$ such that $c < m(K)$. Since K is compact and $\mathcal{C}$ is a cover of K it has a finite subcover, say $\mathcal{L}_1 := \{A_1, \cdots, A_q\}$. Let $B_1 \in \mathcal{L}$ be a ball with maximal radius. Let $\mathcal{L}_2 := \{A \in \mathcal{L}_1 : A \cap B_1 = \emptyset\}$. If $\mathcal{L}_\in$ is empty, then stop. Otherwise, take $B_2 \in \mathcal{L}_2$ such that B_2 has the maximal radius in $\mathcal{L}_2$, and let $L_3 := \{A \in \mathcal{L}_2 : A \cap (B_1 \cup B_2) = \emptyset\}$. Continue this process until we stop. (Since $\mathcal{L}_1$ has finitely many balls, this process must stop at some point.) Say $\mathcal{L}_{k+1} = \emptyset$. Then, $\{B_1, \cdots B_k\}$ are chosen ball which are pairwise disjoint. Then, $\forall A \in \mathcal{L}_1 \setminus \{B_1, \cdots B_k\}$, $A \cap B_p$ for some $p \in [k]$. Now let

$$p = \min_{p \in [k]} \{A \cap B_l \neq \emptyset : A \in \mathcal{L}_1 \setminus \{B_1, \cdots B_k\}, B_l \in \{B_1, \cdots, B_k\}\}.$$

Then, $A \cap (B_1 \cup \cdots \cup B_{p-1}) = \emptyset$. Thus, $A \in \mathcal{L}_p$, thus A has a radius less than B_p, hence $A \subset B_p^*$ where B_p^* is a ball concentric with B_p whose radius is three times that of B_p. Since A was arbitrary, every element in $\mathcal{L}_1 \setminus \{B_1, \cdots B_k\}$ is contained in one of B_p^* and each B_p is contained in B_p^*. Thus,

$$K \subset \bigcup_{j=1}^k B_j \implies t < m(K) < 3^n \sum_{j=1}^k m(B_j) \implies \frac{t}{3^n} < \sum_{j=1}^k m(B_j).$$

□

Proof of the Maximal theorem. Let $E_\alpha = \{x : Hf(x) > \alpha\}$. If $x \in E_\alpha$, let $r_x > 0$ such that $A_{r_x}|f|(x) > \alpha$. Thus,

$$E_\alpha \subset \bigcup_{x \in E_\alpha} B(r_x, x).$$

Therefore, if $c < m(E_\alpha)$, then by the lemma 3.15 in [1], there exists $x_1, \cdots, x_k \in E_\alpha$ such that $B_j := B(r_{x_j}, x_j)$s are disjoint and $\sum_{j=1}^k m(B_j) > 3^{-n} t$. Since each $x_j \in E_\alpha$, this implies

$$\frac{1}{m(B_j)} \int_{B_j} |f| dm > \alpha \implies \frac{1}{\alpha} \int_{B_j} |f| dm > m(B_j) \text{ for each } j.$$

Thus,

$$t 3^{-n} \leq \sum_{j=1}^k m(B_j) < \frac{1}{\alpha} \sum_{j=1}^k \int_{B_j} |f| dm = \frac{1}{\alpha} \int 1_{\bigcup_{j=1}^k B_j} |f| dm \leq \frac{1}{\alpha} \int_{\mathbb{R}^n} |f| dm = \frac{1}{\alpha} \int |f| dm.$$

Thus,

$$t < \frac{3^n}{\alpha} \int |f| dm$$

and by taking supremum on t, we can conclude that

$$m(E_\alpha) \leq \frac{3^n}{\alpha} \int |f| dm.$$

□

Actually, Stein and Stromberg (1983) gives that $C = O(n\log n)$ result. Now we will prove that $\lim_{r\searrow 0} A_r f(y) = f(x)$ for "most" x. But for particular x, $\lim_{r\searrow 0} A_r f(y)$ need not exists. For example, let $f = \sum_{n=1}^{\infty} 1_{[2^{-n^2-1}, 2^{-n^2}]}$. If $r = 2^{-n^2}$ then $A_r f(0) > \frac{1}{4}$. If $r = 2^{-n^2-1}$ then

$$A_r f(0) = \frac{2^{-n^2} - 2^{-n^2-1}}{2\cdot 2^{-n^2}} + \sum_{k=n+1}^{\infty} \frac{2^{-k^2} - 2^{-k^2-1}}{2\cdot 2^{-n^2}} \geq \frac{2^{-n^2} - 2^{-n^2-1}}{2\cdot 2^{-n^2}} = \frac{2^{-n^2-1}}{4\cdot 2^{-n^2-1}} = \frac{1}{4}.$$

However, if $r = 2^{-n^2-1}$ then

$$A_r f(0) = \frac{2^{-(n+1)^2} - 2^{-(n+1)^2-1}}{2^{-n^2-1}} + \sum_{k=n+2}^{\infty} \frac{2^{-k^2} - 2^{-k^2-1}}{2\cdot 2^{-n^2-1}} \leq \frac{2^{-(n+1)^2}}{2^{-n^2-1}} = 2^{-2n} \to 0 \text{ as } n\to\infty.$$

So $\limsup_{r\searrow 0} A_r f(0) \geq \frac{1}{4}$ but $\liminf_{r\searrow 0} A_r f(0) = 0$, therefore limit doesn't exists.

Theorem 4.54 (Differntiation Theorem Version 1. 3.18 in [1] p.97)**.** *If $f \in L^1_{loc}(\mathbb{R}^n)$ then $\lim_{r\to 0^+} A_r f(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$.*

Proof. It will suffices to show that $\forall N \in \mathbb{N}$, $\lim_{r\to 0^+} A_r f(x) = f(x)$ for a.e. $x \in B(N,0)$, However, since we are interested in local points near x with small radius r, thus when we set $r < 1$, then

$$A_r f(x) = A_r(f 1_{B(N,0)}).$$

Thus we may assume that $f \in L^1$

By the theorem 2.41, $\exists g : \mathbb{R}^n \to \mathbb{C}$ continuous and compactly supported function, such that $\|f-g\|_1 < \epsilon$.

Claim 4.55. $\lim_{r\to 0^+} A_r g(x) = g(x)$ *for any* $x \in \mathbb{R}^n$

Proof of the claim. Since g is continuous, $\forall x \in \mathbb{R}^n$ and $\forall \delta > 0, \exists r > 0$ such that $|g(y)-g(x)| < \delta$ whenever $|y-x| < r$. Thus,

$$|A_r g(x) - g(x)| = \frac{1}{m(B(r,x))}\left|\int_{B(r,x)} (g(y)-g(x))\right| < \frac{\delta m(B(r,x))}{m(B(r,x))} = \delta.$$

Thus $A_r g(x) \to g(x)$ as $r \to 0$. ☐

Now, note that

$$\begin{aligned}\lim_{r\to 0^+} A_r f(x) = f(x) \text{ for a.e. } x\in\mathbb{R}^n &\iff \limsup_{r\to 0^+} |A_r f(x) - f(x)| = 0 \text{ for a.e. } x \in \mathbb{R}^n \\ &\iff \inf_{\delta>0}\sup_{0<r<\delta} |A_r f(x) - f(x)| = 0 \text{ for a.e. } x \in \mathbb{R}^n\end{aligned}$$

Thus,

$$\begin{aligned}\inf_{\delta>0}\sup_{0<r<\delta} |A_r f(x) - f(x)| \underbrace{\leq}_{\text{Triangle ineq.}} &\limsup_{r\to 0^+} |A_r f(x) - A_r g(x)| \\ &+ \limsup_{r\to 0^+} |A_r g(x) - g(x)| \\ &+ \limsup_{r\to 0^+} |g(x) - f(x)|\end{aligned}$$

For the first term,

$$\begin{aligned}\limsup_{r\to 0^+} |A_r f(x) - A_r g(x)| &= \left|\frac{1}{m(B(r,x))}\int_{B(r,x)} (f(y)-g(y))dy\right| \leq \frac{1}{m(B(r,x))}\int_{B(r,x)} |f(y)-g(y)|dy \\ &= A_r|f-g|(x) \leq H(f-g)(x).\end{aligned}$$

And $\limsup_{r\to 0^+} |A_r g(x) - g(x)| = 0$ by the above claim. Thus,

$$\limsup_{r\to 0^+} |A_r f(x) - f(x)| \leq H(f-g)(x) + |f-g|(x).$$

To show this, let

$$\alpha > 0, E_\alpha := \{x \in \mathbb{R}^n : \limsup_{r\to 0^+} |A_r f(x) - f(x)| \geq \alpha\}$$

Then,

$$E_\alpha \subseteq F_{\frac{\alpha}{2}} \cup G_{\frac{\alpha}{2}}$$

when

$$F_\alpha := \{x \in \mathbb{R}^n : H(f-g)(x) > \alpha\}, G_\alpha := \{x \in \mathbb{R}^n : |f-g| > \alpha\}.$$

However, by the Hardy Littlewood Maximal Theorem,

$$m(F_{\frac{\alpha}{2}}) \leq \frac{C}{\frac{\alpha}{2}} \|f-g\|_1 < \frac{2C\epsilon}{\alpha} \text{ and } m(G_{\frac{\alpha}{2}}) \leq \frac{\|f-g\|_1}{\frac{\alpha}{2}} < \frac{2\epsilon}{\alpha}.$$

Thus,

$$m(E_\alpha) \leq \frac{2C+1}{\alpha}\epsilon \to 0 \text{ as } \epsilon \to 0.$$

Therefore,

$$\forall x \in \mathbb{R}^n \backslash, \bigcup_{j=1}^{\infty} E_{\frac{1}{j}}, \lim_{r\to 0^+} A_r f(x) = f(x).$$

This implies that

$$\lim_{r\to 0^+} A_r f(x) = f(x) \text{ for a.e. } x.$$

□

So, the theorem 3.18 implies that

$$\lim_{r\to 0^+} \frac{1}{m(B(r,x))} \int_{B(r,x)} (f(y) - f(x)) dm(y) = 0 \text{ for a.e.} x.$$

However, even this is true when we change $f(y) - f(x)$ to $|f(y) - f(x)|$.

Definition 4.56 (Lebesgue set of f)**.** *For $f \in L^1_{loc}(\mathbb{R}^n)$, the* **Lebesgue set of** f *is*

$$L_f := \{x \in \mathbb{R}^n : \lim_{r\to 0^+} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| dm(y) = 0\}.$$

Theorem 4.57 (Theorem 3.20 in [1] p.98)**.** *If $f \in L^1_{loc}(\mathbb{R}^m)$ Then $m(L_f)^c) = 0$*

Proof. For $z \in \mathbb{C}$, let $f_z(x) = |f(x) - z|$ and apply the theorem 3.18 in [1]. Then, $\exists E_z \subseteq \mathbb{R}^n$ such that $m(E_z) = 0$ and $\forall x \in E_z^c$,

$$lim_{r\to 0^+} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - z| dm(y) = |f(x) - z|.$$

Now let D be a countable dense subset of $\mathbb{C}$, and let $E = \bigcup_{z\in D} E_z$. Then, $m(E) = 0$, since it is countable union of null sets. Let $x \in E^c$, $\epsilon > 0$, and let $z \in D$ such that $|z - f(x)| < \epsilon$. This choice is possible since D is countable dense. Then,

$$\forall y \in \mathbb{R}^n, |f(y) - f(x)| < \epsilon + |f(y) - z|.$$

Since $x \in E_z^c$,

$$\limsup_{r \to 0^+} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| dm(y) \le \epsilon + \limsup_{r \to 0^+} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - z| dm(y)$$
$$= \epsilon + |f(x) - z| \le 2\epsilon$$

where the equality comes from the fact that $z \in E_z^c$, and the last inequality comes from our choice of z as $|z - f(x)| < \epsilon$. Thus by letting $\epsilon \to 0$, $x \in L_f^c$. □

Definition 4.58 (Shrink Nicely). *A family $(E_r)_{r>0}$ of subsets of $\mathbb{R}^n$* **shrink nicely to** *$x \in \mathbb{R}^n$ if*

(i) $\forall r > 0, E_r \subseteq B(r,x)$.

(ii) $\exists \alpha > 0$, s.t. $\forall r > 0, m(E_r) \le \alpha m(B(r,x))$.

Example 4.59 (Example of a family shrinking nicely). *Let $E_r = x + B(\frac{r}{3}, (\frac{r}{2}, 0, \cdots, 0))$ on $\mathbb{R}^n$. Then, since the farthest point of E_r from x is $(\frac{5r}{6}, 0, \cdots, 0)$, $E_r \subseteq B(r,x)$ and*

$$m(E_r) = V(\frac{r}{3}) = Cr^n 3^{-n} > 4^{-n}(Cr^n) = 4^{-n} m(B(r,x)).$$

Thus $(E_r)_{r>0}$ shrinks nicely to x.

Proposition 4.60 (The Lebesgue Differentiation Theorem, 3.21 in [1], p.98). *If $f \in L^1_{loc}(\mathbb{R}^n)$, $x \in L_f$, and if $(E_r)_{r>0}$ is a family shrinking nicely to x, then*

$$\lim_{r \to 0^+} \int_{E_r} |f(y) - f(x)| dm(y) = 0.$$

Proof. From shirinking nicely to x property

$$\frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dm(y) \le \frac{1}{\alpha m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| dm(y)$$

and $\frac{1}{\alpha m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)| dm(y) \to 0$ as $r \to 0^+$ since $x \in L_f$. □

Definition 4.61 (Regular Borel measure). *A Borel measure ν on $\mathbb{R}^n$ is* **regular** *if*

(i) $\nu(K) < \infty$ for all compact subset $K \subseteq \mathbb{R}^n$

(ii) $\forall E \in \mathcal{B}_{\mathbb{R}^n}$, $\nu(E) = \inf\{\nu(U) : U$ is open, $U \subseteq \mathbb{R}^n, E \subseteq U\}$.

If ν is signed or complex Borel measure, then ν is **regular** *if $|\nu|$ is regular.*

Proposition 4.62. *If ν is regular, then $\forall E \in \mathcal{B}(\mathbb{R}^n)$,*

$$\nu(E) = \sup\{\nu(K) : K \subseteq \mathbb{R}^n, K \text{ is compact}, K \subseteq E\}.$$

Proof. Without loss of generality, let E be bounded, such that $E \subseteq B(N,0)$. Let $F = B(N,0) \setminus E$. Let $\epsilon > 0$. Then, by the regularity, $\exists U \subseteq \mathbb{R}^n$ open, such that $F \subseteq U$, such that

$$\nu(U) < \nu(F) + \epsilon.$$

Let $K = \overline{B(N,0)} \setminus U \subseteq E$. Then,

$$\begin{aligned}\nu(E \setminus K) \le \nu(E) - \nu(K) &= \nu(E) - (\nu(B(N-\epsilon,0)) - \nu(U)) \\ &< \nu(E) - \nu(B(N-\epsilon,0)) + \nu(F) + \epsilon \\ &= \nu(B(N,0)) - \nu(B(N-\epsilon,0)) + \epsilon \\ &= \epsilon C(\epsilon)\end{aligned}$$

where $C(\epsilon)$ is just polynomial of ϵ. Thus, letting $\epsilon \to 0$, $\nu(E \setminus K) \to 0$. Hence the desired conclusion holds. □

Theorem 4.63 (Theorem 2.40, Dr. Dykema's proof)**.** *Lebesgue measure m on $\mathbb{R}^n$ is regular.*

Proof. The case $n = 1$ is done by theorem 1.18. on [1][p.36]. For $n > 1$, condition (i) is clear. For (ii), let $E \in \mathcal{B}_{\mathbb{R}^n}$. If $m(E) = +\infty$, then it is okay, since $\mathbb{R}^n$ covers E. If $m(E) < +\infty$, then from definition of product measure, for any fixed $\epsilon > 0$, $\exists k \in \mathbb{N}$ with $B_1, \cdots, B_k \subseteq \mathbb{R}^n$ such that

$$\forall j \in [k], B_j := A_{j,1} \times A_{j,2} \times \cdots \times A_{j,n}, \text{ where } A_{j,i} \in \mathcal{B}_{\mathbb{R}} \text{ and } E \subseteq \bigcup_{j=1}^{k} B_j \text{ and } m(E) + \epsilon \geq \sum_{j=1}^{k} m(B_j).$$

By the regularity of Lebesgue measure on $\mathbb{R}$, for any $\delta > 0$, $\exists$ open $V_{i,j} \subseteq \mathbb{R}$ such that

$$A_{j,i} \subseteq V_{j,i} \text{ and } m(V_{j,i}) - \delta \leq m(A_{j,i}).$$

Let $V_j := \prod_{i=1}^{n} V_{j,i}$. Then, $V_j \supseteq B_j$ and V_j is open. By choosing δ smaller, we can have

$$m(V_j) - \frac{\epsilon}{k} \leq m(B_j).$$

Then, let $U = \bigcup_{j=1}^{k} U_j$. Then $E \subseteq U$, and $m(U) - \epsilon \leq \sum_{j=1}^{k} m(B_j)$. $\square$

Proposition 4.64 (Lemma for theorem 3.22 in [1] p.99)**.** *Let $f \in L^+(\mathbb{R}^n)$. Let $d\nu = fdm$. Then, ν is regular iff $f \in L^1_{loc}(\mathbb{R}^n)$.*

Proof. If ν is regular, then $\nu(\overline{B(R,0)}) < +\infty$, so $f1_{\overline{B(R,0)}} \in L^1(\mathbb{R}^n)$. Thus, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Conversely, if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then, for ν and $K \subseteq \mathbb{R}^n$ where K is compact set, there exists $N \in \mathbb{N}$ such that $K \subseteq B(N,0)$, thus $\nu(K) \leq \nu(B(N,0)) = \int f 1_{B(N,0)} dm < \infty$, thus condition (i) of regularity holds.

To show condition (ii), suppose $E \in \mathcal{B}_{\mathbb{R}^n}$ is bounded. (If it is not bounded, then take $U = \mathbb{R}^n$.) Let $\epsilon > 0$, we want to find $U \subseteq \mathbb{R}^n$ which is open and $E \subseteq U$, and $\nu(E) + \epsilon \geq \nu(U)$. Suppose $E \subseteq B(R,0)$. Note that $f1_{B(R,0)} \in L^1(\mathbb{R}^n)$. Then by applying Corollary 3.6 in [1][p.89] to $f1_{B(R,0)}$; for any $\epsilon > 0$ there exists $\delta > 0$ such that $\forall F \in \mathcal{B}_{\mathbb{R}^n}$ with $m(F) < \delta$,

$$\int_F f 1_{B(R,0)} dm < \epsilon.$$

By regularity of Lebesgue measure, $\exists U \subseteq \mathbb{R}^n$ which is open and $E \subseteq U$ and $m(U \setminus E) < \epsilon$. Thus,

$$\int_{U \setminus E} f 1_{B(R,0)} dm < \epsilon \iff \int_{U \setminus E \cap B(R,0)} f dm < \epsilon \iff \nu(U \cap B(R,0) \setminus E) < \epsilon.$$

Since $U \cap B(R,0)$ is open set, we are done. $\square$

Lemma 4.65. *Let λ, μ be positive Borel measure on $\mathbb{R}^n$. Then, $\mu + \lambda$ is regular if and only if μ and λ are regular.*

Proof. For condition (i), if $\mu + \lambda$ is regular, then any $K \subseteq \mathbb{R}^n$ with compactness, $\mu(K) \leq (\mu + \lambda)(K) < \infty$ and $\lambda(K) \leq (\mu+\lambda)(K) < \infty$, thus (i) holds for μ and λ. Conversely, if μ and λ is regular, then $(\mu+\lambda)(K) = \mu(K) + \lambda(K) < +\infty$.

For condition (ii), suppose $\mu + \lambda$ is regular. Then, for any $E \in \mathcal{B}_{\mathbb{R}^n}$ $\exists U \subseteq \mathbb{R}^n$ which is open, $E \subseteq U$ and

$$(\mu + \lambda)(U) < (\mu + \lambda)(E) + \epsilon \implies 0 \leq \mu(U) - \mu(E) + \lambda(U) - \lambda(E) < \epsilon$$

From $U \supset E$, we know $0 < \mu(U) - \mu(E)$ and $0 < \lambda(U) - \lambda(E)$. Thus,

$$0 \leq \mu(U) - \mu(E) < \epsilon \text{ and } 0 \leq \lambda(U) - \lambda(E) < \epsilon$$

Since ϵ was arbitrary, we have desired result. Conversely, if μ and λ are regular, then for any $\epsilon > 0$, $\exists U_1, U_2$ open sets containing E and

$$\mu(U_1) < \mu(E) + \epsilon \text{ and } \lambda(U_2) < \lambda(E) + \epsilon.$$

Then, $U = U_1 \cap U_2$ is also open, since it is finite intersection, and $E \subset U$. Thus,

$\mu(U) < \mu(E)+\epsilon$ and $\lambda(U) < \lambda(E)+\epsilon \implies 0 < \mu(U)+\lambda(U)-\mu(E)-\lambda(E) < \epsilon \implies 0 < (\mu+\lambda)(U)-(\mu+\lambda)(E) < \epsilon.$

Thus, by letting $\epsilon \to 0$, we have desired result. $\square$

Theorem 4.66 (Theorem 3.22 in [1], p.99)**.** *Let ν be a signed or complex Borel measure on $\mathbb{R}^n$ that is regular. Let*

$$d\nu = d\lambda + fdm$$

(where $d\rho = fdm$) be the Lebesgue-Radon-Nikodym decomposition. (So $\lambda \perp m$). Then for m-a.e. $x \in \mathbb{R}^n$,

$$\lim_{r \searrow 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every family $(E_r)_{r>0}$ that shrink nicely to x.

Proof. Note that $\nu - \lambda = \rho = fdm$ is regular by the proposition 4.64, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. By the Lebesgue Differentiation theorem,

$$\frac{\rho(E)}{m(E)} = \frac{1}{m(E_r)} \int_{E_r} fdm \to 0 \text{ as } r \to 0 \text{ for a.e.} x \text{ and } (E_r)_{r>0} \text{ shrinking nicely to } x.$$

So it remains to show that for almost every x,

$$\frac{\lambda(E_r)}{m(E_r)} \to 0 \text{ as } r \searrow 0 \text{ for every } (E_r)_{r>0} \text{ shrinking nicely to } x.$$

Since shrinking nicely to x implies that $E_r \subseteq B(r,x)$ and that $\exists \alpha > 0$ such that $\forall r, m(E_r) > \alpha m(B(r,x))$,

$$\left|\frac{\lambda(E_r)}{m(E_r)}\right| \leq \frac{|\lambda|(E_r)}{m(E_r)} \leq \frac{|\lambda|(B(r,x))}{\alpha m(B(r,x))}$$

where first equality comes from $|\lambda(E_r)| \leq |\lambda|(E_r)$ and the last equality comes from $E_r \subseteq B(r,x)$ and $m(E_r) > \alpha m(B(r,x))$. Thus, it suffices to show that for a.e. x,

$$\lim_{r \to 0} \frac{|\lambda|(B(r,x))}{\alpha m(B(r,x))} = 0. \tag{17}$$

Note that $|\lambda| \perp m$. Thus, $\exists A \in \mathcal{B}_{\mathbb{R}^n}$ such that $|\lambda|(A) = 0 = m(A^c)$. Let

$$F_k := \{x \in A : \limsup_{r \searrow 0} \frac{|\lambda|(B(r,x))}{\alpha m(B(r,x))} > \frac{1}{k}\}.$$

It will suffices to show that $m(F_k) = 0$. Since then (17) will holds for all $x \in A \setminus \bigcup_{k=1}^{\infty} F_k$ and

$$m\left(\left(A \setminus \bigcup_{k=1}^{\infty} F_k\right)^c\right) = m\left(A^c \cup \left(\bigcup_{k=1}^{\infty} F_k\right)\right) \leq m(A^c) = 0.$$

Let $\epsilon > 0$. We have $|\lambda|(F_k) \leq |\lambda|(A) = 0$ By the regularity of $|\lambda|$ which is from definition of regularity of signed measure, $\exists U$, open in $\mathbb{R}^n$ such that $F_k \subseteq U$ and $|\lambda|(U) < \epsilon$. Given $x \in F_k, \exists r_x > 0$ such that

$$\frac{|\lambda|(B(r_x,x)}{m(B(r_x,x))} > \frac{1}{k} \text{ and } B(r_x,x) \subset U$$

. Let $V := \bigcup_{x \in F_k} B(r_x, x)$. Then $F_k \subset V \subset U$. Then by the covering lemma 3.15 in [1][p.95], $\exists x_1, \cdots, x_n \in F_k$ such that $B(r_{x_1}, x_1), \cdots, B(r_{x_n}, x_n)$ are disjoint and

$$\sum_{j=1}^{n} m(B(r_{x_j}, x_j)) \geq 3^{-n} m(V).$$

Then,

$$3^{-n} m(V) \leq \sum_{j=1}^{n} m(B(r_{x_j}, x_j)) \leq k \sum_{j=1}^{n} |\lambda|(B(r_x, x_j)) \leq k|\lambda|(V) \leq k|\lambda|(U) \leq k\epsilon.$$

So,

$$m(F_k) \leq m(V) \leq 3^n k\epsilon.$$

By $\epsilon \to 0$, we has $m(F_k) = 0$ as desired. $\square$

4.4 Functions of Bounded Variation

Suppose $G : \mathbb{R} \to \mathbb{R}$ is increasing right continuous function. Then, the ***Lebesgue-Stieltjes measure*** μ_G is determined by

$$\mu_G([a, b]) = G(b) - G(a).$$

Then, μ_G is regular, as proved in theorem 1.18.

Theorem 4.67 (Theorem 3.23 in [1]p.101)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing function. let $G(x) = F(x+) := \lim_{t \to x^+} F(t) = \inf_{t > x} F(t)$. Then,*

(a) The set of points at which F is discontinuous is at most countable.

(b) F and G are differentiable almost everywhere, and $F' = G'$ m-a.e.

This may be related Qualifier exam... But not sure.

Proof. For (a), let

$$E_N := \{x \in (-N, N : F(x+) \neq F(x-)\},$$

where $F(x-) := \lim_{t \to x-} F(t) = \sup_{t < x} F(t)$. Since

$$\sum_{x \in E_N} (F(x+) - F(x-)) \leq F(N) - F(-N) < \infty,$$

This implies E_N must be countable set; otherwise the sum should be infinity, contradiction. Hence, $\bigcup_{N=1}^{\infty} E_N = \{x \in \mathbb{R} : F(x+) \neq F(x-)\}$ is also countable.

For part (b), note that G is increasing and right continuous, and $G(x) = F(x)$ except $x \in E := \{x \in \mathbb{R} : F(x+) \neq F(x-)\}$. Moreover,

$$G(x+h) - G(x) = \begin{cases} \mu_G((x, x+h]) & \text{if } h > 0 \\ -\mu_G((x+h, x]) & \text{if } h < 0 \end{cases},$$

where μ_G is a Lebesgue Stieltjes measure with respect to G. By theorem 1.18, μ_G is regular. So, from Radon-Nikodym decomposition $d\mu_G = d\lambda + f dm$, $f \in L^1_{\text{loc}}(\mathbb{R})$. Also note that $((x, x+h])_{h>0}$ shrinks nicely to x, since $(x, x+h] \subseteq B(x, h)$ and $m(x, x+h] = h > \frac{2}{3}h = \frac{2}{3}m(B(h, x))$. Thus, by the theorem 3.22, for m-a.e. $x \in \mathbb{R}$,

$$\frac{G(x+h) - G(x)}{h} = \frac{\mu_G((x, x+h])}{m((x, x+h])} \to f(x) \text{ as } h \searrow 0.$$

Also, $((x-h,x])_{h>0}$ shrinks nicely to x, by the similar argument. Thus, by the theorem 3.22, for m-a.e. $x \in \mathbb{R}$,

$$\frac{G(x)-G(x-h)}{h} = \frac{\mu_G((x-h,x])}{m((x-h,x])} \to f(x) \text{ as } h \searrow 0.$$

Thus, for m-a.e. $x \in \mathbb{R}$, G and F are differentiable. And from part (a), $F(x) \neq G(x)$ is countable. Now it suffices to show that if $x \in \mathbb{R}$ and $F(x) = G(x)$ and $G'(x)$ exists, then $F'(x) = G'(x)$.

For some fixed$x \in \mathbb{R}$, suppose $F(x) = G(x)$ and $G'(x)$ exists, then $F'(x) = G'(x)$.Let $0 < \alpha < 1$. For $h > 0$,

$$G(x+\alpha h) = F((x+\alpha h)+) \leq F(x+h) \leq F((x+h)+) = G(x+h)$$

where the first inequality comes from F is an increasing function. Thus,

$$G(x+\alpha h) \leq F(x+h) \leq G(x+h).$$

Then, we can modify this inequality as below;

$$\alpha \cdot \frac{G(x+\alpha h)-G(x)}{\alpha h} = \frac{G(x+\alpha h)-G(x)}{h} \leq \frac{F(x+h)-F(x)}{h} \leq \frac{G(x+h)-G(x)}{h}$$

By letting $h \to 0$, we have

$$\alpha G'(x) \leq \liminf_{h\searrow 0} \frac{F(x+h)-F(x)}{h} \leq \limsup_{h\searrow 0} \frac{F(x+h)-F(x)}{h} \leq G'(x).$$

By letting $\alpha \to 1$, we can conclude that $\lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h}$ exists and equal to $G'(x)$, as desired.

Using the almost same argument, we have

$$G(x-\alpha h) = F((x-\alpha h)+) \geq F(x-h) \geq F((x-h)+) = G(x-h)$$

where the first inequality comes from F is an increasing function. Thus,

$$G(x-\alpha h) \geq F(x-h) \geq G(x-h).$$

Then, we can modify this inequality as below;

$$\alpha \cdot \frac{G(x-\alpha h)-G(x)}{\alpha - h} = \frac{G(x-\alpha h)-G(x)}{-h} \leq \frac{F(x-h)-F(x)}{-h} \leq \frac{G(x-h)-G(x)}{-h}$$

By letting $h \to 0$, we have

$$\alpha G'(x) \leq \liminf_{h\searrow 0} \frac{F(x-h)-F(x)}{-h} \leq \limsup_{h\searrow 0} \frac{F(x-h)-F(x)}{-h} \leq G'(x).$$

By letting $\alpha \to 1$, we can conclude that $\lim_{h\to 0^+} \frac{F(x-h)-F(x)}{-h}$ exists and equal to $G'(x)$, as desired. Thus we can conclude that $F'(x)$ exists and $F'(x) = G'(x)$. □

Remark 4.68 (Notation)**.** *Let ν be a σ-finite (positive, signed, or complex) Borel measure ν on $\mathbb{R}^n$. Write $\nu = \lambda + \rho$ when $\rho \ll m$ and $\lambda \perp m$ using the Radon-Nikodym theorem. Write*

$$\nu_{a.c.} = \rho$$

for the absolutely continuous part of ν. An **atom of** *ν is a singleton $\{x\}$ such that $\nu(\{x\}) \neq 0$. The* **atomic part of** *ν is*

$$\nu_{atomic} \sum_{\{x:\{x\} \text{ is an atom}\}} \nu(\{x\})\delta_x.$$

Note that

$$|\nu_{atomic}| \leq |\lambda|, \text{ and } \nu_{s.c.} := \lambda - \nu_{atomic}.$$

Then, $\nu_{s.c.} \perp m$, and we call $\nu_{s.c.}$ the **singular part of** *ν. Thus we have*

$$\nu = \nu_{a.c.} + \nu_{s.c.} + \nu_{atomic}.$$

Example 4.69 (Revisit Cantor set)**.** *Let $C \in [0,1]$ be the Cantor set. Then in the chapter 1 we know that*

$$C := \{x \in [0,1] : \exists (a_i)_{i=1}^{\infty} \in \{0,2\}^{\mathbb{N}} \text{ such that } x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}\}.$$

Also, we can represetn $[0,1]$ as

$$[0,1] = \{t \in [0,1] : \exists (b_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}} \text{ such that } t = \sum_{j=1}^{\infty} \frac{b_j}{2^j}\}.$$

Thus, the map

$$f : C \to [0,1] \text{ by } \sum_{j=1}^{\infty} \frac{a_j}{3^j} \mapsto \sum_{j=1}^{\infty} \frac{a_j/2}{2^j}$$

is well-defined, increasing, and onto map. (But not injective since $f(\frac{7}{9}) = \frac{3}{4} = f(\frac{8}{9})$.) We can extend f to $G : [0,1] \to [0,1]$ by letting G be constant on each of disjoint interval I appearing in $[0,1] \setminus C$. Such G is called **Cantor-Lebesgue Function** *and G is increasing and onto. Thus, G is continuous, since increasing onto function is continuous. Hence,*

$$\mu_G(C^c) = 0,$$

Since C^c are union of intervals where each interval is preimage of constant of G. So, $\mu_G \perp m$. However, since G is continuous, it doesn't have any jump discontinuous; so there are no atoms. Thus, μ_G is singular continuous.

Also note that $G'(x) = 0$ for m-a.e. i.e., except a points in C.

Definition 4.70 (Total Variation)**.** *Let $F : \mathbb{R} \to \mathbb{C}$. The* **total variation** *of F on $[a,b]$ is*

$$TV_F([a,b]) := \sup\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : a = x_0 < x_1 < x_2 < \cdots < x_n = b\} \in [0,+\infty].$$

Also,

$$T_F(b) := TV_F((-\infty,b]) = \sup\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| x_0 < x_1 < \cdots < x_n = b\},$$

no lower bound on the partition. Clearly, T_F is increasing function.

Observation 4.71. *If $a < b$, then*

$$T_F(b) = T_F(a) + TV_F([a,b])$$

So if $T_F(a) < \infty$, then $TV_F([a,b]) = T_F(b) - T_F(a)$.

Proof. If $T_F(a) > \infty$, then $T_F(b) \geq T_F(a)$, thus the equality holds. If $T_F(a) < \infty$, then note that for any partition P_1 of $(-\infty,a]$ and P_2 of $[a,b]$, $P_1 \cup P_2$ is a partition of $(-\infty,b]$. Hence,

$$T_F(b) \geq \sum_{x_i \in P_1 \cup P_2, i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{x_i \in P_1, i=1}^{n} |F(x_i) - F(x_{i-1})| + \sum_{x_i \in P_2, i=1}^{n} |F(x_i) - F(x_{i-1})|.$$

Since P_1, P_2 are arbitrarily chosen, so

$$T_F(b) \geq T_F(a) + TV_F([a,b]).$$

Conversely, for any partition P of $(-\infty, b]$, $a \cup P$ gives partition P_1 for $(-\infty, a]$ and P_2 for $[a, b]$. Thus,

$$\begin{aligned}
\sum_{x_i \in P, i=1}^{n} |F(x_i) - F(x_{i-1})| &\leq \sum_{x_i \in a \cup P, i=1}^{n} |F(x_i) - F(x_{i-1})| \\
&= \sum_{x_i \in P_1, i=1}^{n} |F(x_i) - F(x_{i-1})| + \sum_{x_i \in P_2, i=1}^{n} |F(x_i) - F(x_{i-1})| \\
&\leq T_F(a) + TV_F([a, b]).
\end{aligned}$$

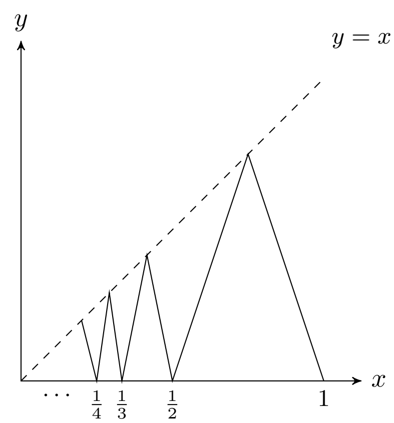
Thus

$$T_{F(b)} \leq T_F(a) + TV_F([a, b]).$$

Hence, $T_{F(b)} = T_F(a) + TV_F([a, b])$. $\square$

Example 4.72. $F(x) : [0, 1] \to [0, 1]$ *is as below. It is clearly bounded continuous function. However,*

$$TV_F([0, 1]) \geq |F(1) - F(\frac{3}{4})| + |F(\frac{3}{4}) - F(\frac{1}{2})| + \cdots \geq \sum_{j=2}^{n} \frac{1}{j} = \infty.$$



Definition 4.73 (Boudned variation)**.** *Let* $F : \mathbb{R} \to \mathbb{C}$ *be* **bounded variation** *on* $\mathbb{R}$ *if* $T_F(+\infty) := \lim_{x \to +\infty} T_F(X) < +\infty$. *Let*

$$BV := \{F : \mathbb{R} \to \mathbb{C} : F \text{ has bounded variation}\}.$$

Let $F : [a, b] \to \mathbb{C}$ *has bounded variation on the interval* $[a, b]$ *if* $TV_F([a, b]) < \infty$. *Let*

$$BV([a, b]) := \{F : \mathbb{R} \to \mathbb{C} : F \text{ has bounded variation on } [a, b]\}.$$

Observation 4.74.

(i) BV *and* $BV([a, b])$ *are vector spaces and* $BV([a, b])$ *can be identified with*

$$\{F \in BV : F \text{ is constant function on } (-\infty, a] \text{ and } [-b, +\infty).\}$$

Also, BV *can be identified with* $\{F|_{[a,b]} \in BV\} \subseteq BV([a, b])$. *Thus, property holds for* BV *also holds for* $BV([a, b])$.

(ii) $F \in BV \implies F$ *is bounded.*

(iii) If F is monotone and bounded, then $F \in BV$.

(iv) If $F : [a,b] \to \mathbb{C}$ is continuous and f is differentiable on (a,b) and if F' is bounded, then by MVT, F has bounded variation.

(v) If $F \in BV$ then $F(\pm) := \lim_{x\to\pm\infty} F(x)$ exists

(vi) $T_{\lambda F} = |\lambda| T_F$ and $T_{F+G} \leq T_F + T_G$

Proof. For part (a), BV is closed under addition and scalar multiplication comse from triangle inequality and just pulling out scalar from the definition of the sum. Also, additive, multiplicative, and distributive axioms hold.

For part (b), if F is not bounded at x, then we can take a partition P such that $a = -M < x = b$ then $\sum_P |F(x_i) - F(x_{i-1})| = F(x) - F(-M) = \infty$. Thus, $T_F(+\infty) = \infty$, contradiction.

For part (c), if F is monotone and bounded, than without loss of generality, assume F is monotone nondecreasing. Then, any partition P starting with a and ending with b gives

$$\sum_P |F(x_i) - F(x_{i-1})| = \sum_P F(x_i) - F(x_{i-1}) = F(b) - F(a).$$

Thus, $T_F(\infty) = \lim_{n\to\infty} F(n) - F(-n)$. Since F is bounded, $\lim_{n\to\infty} F(n) < +\infty, lim_{n\to\infty} F(-n) < +\infty$, thus their subtraction is also bounded, as desired.

For part (d), let $P : a = x_0 < x_1 < \cdots < x_n = b$ be arbitrary partition. Then, by Mean Value Theorem,

$$|F(x_i) - F(x_{i-1})| = |(x_i - x_{i-1})F'(c_i)| \text{ for some } c_i \in (x_{i-1}, x_i).$$

Thus, from the fact that $F' < M$ for some $M \in \mathbb{R}$,

$$\sum_{j=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} |(x_i - x_{i-1})F'_{c_i}| \leq M \sum_{i=1}^{n} (x_i - x_{i-1}) = M(b-a) < +\infty.$$

Hence, F has bounded variation.

For part (e), if $\liminf_{x\to+\infty} F(x) < \limsup_{x\to+\infty} F(x)$, then F should have infinite oscillations between these value, this may lead to get a partition giving $T_F(\infty) = +\infty$, contradiction. Similarly, if $\liminf_{x\to-\infty} F(x) < \limsup_{x\to-\infty} F(x)$, it has also infinitely many oscillations, so $F \notin BV$.

For part (f), for any partition,

$$\sum_{i=1}^{n} |\lambda F(x_i) - \lambda F(x_{i-1})| = |\lambda| \sum_{i=1}^{n} |F(x_i) - \lambda F(x_{i-1})|$$

and

$$\sum_{i=1}^{n} |G(F_{x_i}) + F(x_i) - \lambda G(x_{i-1}) + F(x_{i-1})| \leq \sum_{i=1}^{n} |F(x_i) - \lambda F(x_{i-1})| + \sum_{i=1}^{n} |G(x_i) - \lambda G(x_{i-1})|.$$

□

Lemma 4.75 (Lemma 3.26 in [1] p.102). *If $F \in BV$ is a real-valued function, then $T_F + F, T_F - F$ are increasing.*

Proof. Let $x < y, \epsilon > 0$. Choose $x_0 < x_1 < \cdots < x_n$ such that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq T_F(x) - \epsilon.$$

Then,

$$T_F(x) - \epsilon + |F(y) - F(x)| \leq \left(\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|\right) + |F(y) - F(x)| \leq T_F(y).$$

So, $|F(y) - F(x)| \leq T_F(y) - T_F(x) + \epsilon$. By letting $\epsilon \to 0$, we have for $x < y$,

$$\begin{aligned} F(y) - F(x) \leq |F(y) - F(x)| \leq T_F(y) - T_F(x) &\implies (T_F - F)(x) \leq (T_F - F)(y) \\ F(x) - F(y) \leq |F(y) - F(x)| \leq T_F(y) - T_F(x) &\implies (T_F + F)(x) \leq (T_F + F)(y), \end{aligned}$$

as desired. □

Theorem 4.76 (Theorem 3.27 part (1) in [1] p.103)**.**

(a) $F \in BV \iff ReF, ImF \in BV$

(b) Given $F : \mathbb{R} \to \mathbb{R}$, $F \in BV \iff \exists$ *bounded increasing function* $f, g : \mathbb{R} \to \mathbb{R}$ *such that* $F = f - g$.

Proof. For part (a), for any partition, note that

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} |ReF(x_i) - ReF(x_{i-1})| + \sum_{i=1}^{n} |ImF(x_i) - ImF(x_{i-1})|.$$

Thus if $ReF, ImF \in BV$, then $F \in BV$. Conversely, from $Rez \leq |z|$ and $Imz \leq |z|$, we know

$$\sum_{i=1}^{n} |ReF(x_i) - ReF(x_{i-1})| \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| \text{ and } \sum_{i=1}^{n} |ImF(x_i) - ImF(x_{i-1})| \leq \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|.$$

This implies $F \in BV \implies ReF, ImF \in BV$.

For part (b), take $f = \frac{1}{2}(T_F + F)$, $g = \frac{1}{2}(T_F - F)$. Then by the lemma, F has two such increasing function. Also, they are bounded since

$$\begin{aligned} F(y) - F(x) \leq |F(y) - F(x)| \leq T_F(y) - T_F(x) \leq T_F(\infty) - T_F(-\infty) < \infty \\ F(x) - F(y) \leq |F(y) - F(x)| \leq T_F(y) - T_F(x) \leq T_F(\infty) - T_F(-\infty) < \infty. \end{aligned}$$

□

Definition 4.77 (Jordan Decomposition of F, positive and negative variation)**.** *If* $F : \mathbb{R} \to \mathbb{R}$ *and* $F \in BV$*, the we call a representation* $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$ *be* **Jordan Decomposition of** F*. We denote*

$$F^{\pm} = \frac{1}{2}(T_F \pm F),$$

as **positive and negative** *variation of* F*. In this case, we know that*

$$\frac{1}{2}(T_F \pm F)(x) = \sup\{\sum_{j=1}^{n} (F(x_j) - F(x_{j-1}))^{\pm} : x_0 < \cdots < x_n = x\} \pm F(-\infty)$$

where

$$x^+ = \max(x, 0) = \frac{1}{2}(|x| + x) \text{ and } x^- = \max(-x, 0) = \frac{1}{2}(|x| - x)$$

Theorem 4.78 (Theorem 3.27 part (2) in [1] p.103)**.** *Let* $F \in BV$*.*

(c) $\forall x \in \mathbb{R}$, $F(x+) := \lim_{t \to x+} F(t)$ *and* $F(x-) := \lim_{t \to x-} F(t)$ *exists. Also,* $F(\pm\infty) = \lim_{t \to \pm\infty} F(t)$ *exists.*

(c) The set of points at which F is discontinuous is countable.

(c) Let $G(x) = F(x+)$. Then, $\forall$ a.e. $x \in \mathbb{R}$, $G'(x), F'(x)$ exists and $G'(x) = F'(x)$.

Proof. For part (c), by Observation 4.74 (v), $F(\pm\infty)$ exists. Also, we know that increasing bounded real-valued function g has $\lim_{x_n \to \pm x \pm} g(x_n)$, by taking $g|_{(a,x)}$ or $g|_{(x,b)}$ for some $a < x < b$ and apply the Monotone convergence theorem. Thus, for any real-valued bounded variation function, it has such limits. For complex valued function, real part and imaginary part has the limit, thus the function itself also has a limit.

For part (d) and (e) follows from the increasing function with theorem 3.23 in [1][p.101], so their subtraction also has the same property. $\square$

Lemma 4.79 (Lemma 3.28 in [1] p.104)**.** *Let $F \in BV$.*

(a) $T_F(-\infty) := \lim_{x \to -\infty} T_F(x) = 0$

(b) If F is right continuous, then so is T_F.

Proof. For part (a), $0 \leq T_F(-\infty)$ is clear by definition and

$$|T_F| \leq T_{ReF} + T_{ImF} \leq T^+_{ReF} + T^-_{ReF} + T^+_{ImF} + T^-_{ImF}.$$

So it suffices to show that the RHS goes to zero as $x \to -\infty$. Without loss of generality, assume that F is the monotone increasing bounded function. Then,

$$T_F(x) = \sup_{x_0 < x} F(x) - F(x_0) = F(x) - F(-\infty).$$

So

$$\lim_{x \to -\infty} T_F(x) = F(-\infty) - F(-\infty) = 0.$$

For part (b), Suppose F is right continuous. Let $T = T_F$. Fix $x \in \mathbb{R}$, let $\alpha = T(x+) - T(x)$. We want to show $\alpha = 0$ Let $\epsilon > 0, \delta > 0$ such that

$$0 < h < \delta \implies |F(x+h) - F(x)| < \epsilon \text{ and } T(x+h) - T(x+) < \epsilon.$$

which can be derived from the right continuity of F and definition of $T(X+)$ and the property that T is increasing function (thus by Monotone convergence theorem). Let $x = x_0 < x_1 < \cdots x_n = x + h$ such that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \geq \frac{1}{2}(T(x+h) - T(x)) \geq \frac{1}{2}\alpha,$$

using the definition of T as a supremum of such sequences. Then, $|F(x_1) - F(x_0)| < \epsilon$ since $x_1 < h$, thus

$$\sum_{j=2}^{n} |F(x_j) - F(x_{j-1})| \geq \frac{1}{2}\alpha - \epsilon.$$

Hence,

$$\frac{1}{2}\alpha - \epsilon \geq \sum_{j=2}^{n} |F(x_j) - F(x_{j-1})| \leq T(x+h) - T(x_1) \leq T(x+h) - T(x+) < \epsilon$$

where first inequality comes from the above inequality, the second inequality comes from the definition of T, and the third inequality comes from the property that T is increasing, and the last inequality comes from our choice of h. Hence,

$$\frac{1}{2}\alpha < 2\epsilon \implies \alpha < 4\epsilon.$$

By letting $\epsilon \to 0$, $\alpha = 0$. $\square$

Definition 4.80 (Normal Bounded variation). *Define* **Normal bounded variation** *as a set*

$$NBV = \{F \in BV : F \text{ is right continuous and } F(-\infty) = 0\}.$$

Observation 4.81.

(1) If $F \in BV$, let $G(x) = F(x+) - F(-\infty)$. Then $G \in NBV$ and $G(X) = F(x) - F(-\infty)$ except some countably many values of x.

(2) $G \in NBV$ then $(ReG)^{\pm}, (ImG)^{\pm} \in NBV$.

Theorem 4.82 (Theorem 3.29 in [1] p.104). *Let μ be a complex Borel measure on $\mathbb{R}$ and let*

$$F(x) := \mu((-\infty, X]). \tag{18}$$

Then, $F \in NBV$. Conversely, if $F \in NBV$, then $\exists!$ complex Borel measure μ_F on $\mathbb{R}$ such that

$$F(x) = \mu_F((-\infty, X]) \text{ for all } x \in \mathbb{R}. \tag{19}$$

Moreover,

$$|\mu_F| = \mu_{T_F}. \tag{20}$$

Proof. We can rewrite $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$ where μ_is are finite positive measure using Hanh Decomposition. Thus, $F_j(x) := \mu_j((-\infty, x]) = \begin{cases} \mu_j((0,x]) + \mu((-\infty,0]) & \text{if } x > 0 \\ 0 + \mu((-\infty,0]) & \text{if } x = 0 \\ -\mu_j((x,0]) + \mu((-\infty,0]) & \text{if } x < 0 \end{cases}$ is increasing and right continuous by theorem 1.16., and $F_j(-\infty) = 0$. And $F_j(+\infty) = \mu_j(\mathbb{R}) < \infty$. Therefore, by the theorem 3.27 (b) in [1][p. 103] $F_1 - F_2, F_3 - F_4 \in BV$ and by 3.27 (a), $F \in BV$. And from $F(-\infty) = (F_1 - F_2)(-\infty) + i(F_3 - F_4)(-\infty) = 0$, $F \in NBV$.

Conversely, if $F \in NBV$, then $ReF^{\pm}, ImF^{\pm} \in NBV$ by theorem 3.27 (a). And each of them is yields bounded, increasing and right continuous by the theorem 3.27 (b) and observations 4.81. Thus, by theorem 1.16, they gives finite Lebesgue Stieljes measures, $\mu_{ReF^{\pm}}, \mu_{ImF^{\pm}}$. So,

$$\mu_F = \mu_{ReF^+} - \mu_{ReF^-} + i(\mu_{ImF^+} - \mu_{ReF^-})$$

Thus, (19) holds.

To see $|\mu_F| = \mu_{T_F}$, we need several claim, as outlined in the Exercise 28. Let $G(x) = |\mu_F|((-\infty, x])$.

Claim 4.83. *$T_F(x) \leq G(x)$.*

Proof. For any $x \in \mathbb{R}$,

$$\begin{aligned}
T_F(x) &= \sup\{\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| : n \in \mathbb{N}, x_0 < x_1 < \cdots < x_n = x\} = \\
&= \sup\{\sum_{k=1}^{n} |\mu((-\infty, x_k]) - \mu((-\infty, x_{k-1}])| : n \in \mathbb{N}, x_0 < x_1 < \cdots < x_n = x\} \\
&= \sup\{\sum_{k=1}^{n} |\mu((x_{k-1}, x_k]) - \mu((-\infty, x_{k-1}])| : n \in \mathbb{N}, x_0 < x_1 < \cdots < x_n = x\} \\
&\leq \sup\{\sum_{k=1}^{n} |\mu(E_k) - \mu((-\infty, x_{k-1}])| : n \in \mathbb{N}, (E_k)s \text{ are disjoint sequence such that } \bigcup_{k=1}^{n} = (-\infty, x]\} \\
&= |\mu_F|((-\infty, x]) = G(x).
\end{aligned}$$

where the last equality comes from the exercise 21, $\mu_1 = |\mu|$ case. □

Claim 4.84. *$|\mu_F(E)| \le \mu_{T_F}(E)$ for all $E \in \mathcal{B}_{\mathbb{R}}$.*

Proof. For any interval $(a, b] \subseteq \mathbb{R}$,

$$T_F(b)-T_F(a) = \sup\{\sum_{k=1}^{n} |F(x_k)-F(x_{k-1})| : n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b\} \implies |F(b)-F(a)| \le T_F(b)-T_F(a).$$

Thus,

$$|\mu_F((a, b])| = |\mu_F((-\infty, b]) - \mu_F((-\infty, a])| = |F(b) - F(a)| \le T_F(b) - T_F(a) = \mu_{T_F}((a, b]).$$

Also, clearly,

$$|\mu_F(\emptyset)| = 0 = \mu_{T_F}(\emptyset).$$

And,

$$|\mu_F((-\infty, x])| = \lim_{a\to-\infty} \mu_F((a, x]) \le \lim_{a\to-\infty} \mu_{T_F}((a, x]) = \mu_{T_F}((-\infty, x]).$$

Also,

$$|\mu_F((x, \infty))| = \lim_{b\to\infty} \mu_F((x, b]) \le \lim_{b\to\infty} \mu_{T_F}((x, b]) = \mu((x, \infty)).$$

Now let $\mathcal{A}$ be an algebra generated by all half (left) open intervals. Then by the above inequalities,

$$\mathcal{A} \subseteq \mathcal{C} := \{E \in \mathcal{B}_{\mathbb{R}} : |\mu_F(E)| \le \mu_{T_F}(E)\}.$$

Also, for any increasing sequence $(E_k)_{k=1}^{\infty} \subseteq \mathcal{C}$,

$$|\mu_F(\bigcup_{k=1}^{\infty} E_k)| = |\lim_{k\to\infty} \mu_F(E_k)| = \lim_{k\to\infty} |\mu_F(E_k)| \le \lim_{k\to\infty} \mu_{T_F}(E_k) = \mu_{T_F}(\bigcup_{k=1}^{\infty} E_k).$$

Thus, $\bigcup_{k=1}^{\infty} E_k) \in \mathcal{C}$. Moreover, for any decreasing sequence $(E_k)_{k=1}^{\infty} \subseteq \mathcal{C}$,

$$|\mu_F(\bigcap_{k=1}^{\infty} E_k)| = |\lim_{k\to\infty} \mu_F(E_k)| = \lim_{k\to\infty} |\mu_F(E_k)| \le \lim_{k\to\infty} \mu_{T_F}(E_k) = \mu_{T_F}(\bigcap_{k=1}^{\infty} E_k).$$

Thus, $\bigcap_{k=1}^{\infty} E_k) \in \mathcal{C}$. Hence, $\mathcal{C}$ is closed under countable increasing union and countable decreasing intersection. Thus, $\mathcal{C}$ contains monotone class generated by $\mathcal{A}$, which is $\mathcal{B}_{\mathbb{R}}$, by the monotone class lemma. □

Claim 4.85. *If $E \in \mathcal{B}_{\mathbb{R}}, |\mu_F(E)| \le \mu_{T_F}(E)$. Hence $G \le T_F$.*

Proof. By the Exercise 21,

$$\begin{aligned} |\mu_F(E)| &= \sup\{\sum_{k=1}^{\infty} |\mu_F(E_k)| : (E_k)_{k=1}^{\infty} \text{ is a sequence of disjoint Borel sets such that } E = \bigcup_{k=1}^{\infty} E_k\} \\ &\le \sup\{\sum_{k=1}^{\infty} |\mu_{T_F}(E_k)| : (E_k)_{k=1}^{\infty} \text{ is a sequence of disjoint Borel sets such that } E = \bigcup_{k=1}^{\infty} E_k\} = \mu_{T_F}(E). \end{aligned}$$

where the inequality comes from above claims. Thus, for any $x \in \mathbb{R}$,

$$G(x) = |\mu_F|((-\infty, x]) \le \mu_{T_F}((-\infty, x]) = T_F(x) \le G(x) \implies G(x) = T_F(x).$$

Let $\mathcal{M} := \{E \in \mathcal{B}_{\mathbb{R}} : |\mu_F|(E) = \mu_{T_F}(E)\}$. Then, $(-\infty, x] \in \mathcal{M}$ for any $x \in \mathbb{R}$. Thus, $\mathcal{M}$ contains $\mathcal{B}_{\mathbb{R}}$, since $((-\infty, x])_{x\in\mathbb{R}}$ generates $\mathcal{B}_{\mathbb{R}}$. □

□

Proposition 4.86 (Proposition 3.30 in [1] p. 105)**.** *Let $F \in NBV$. Thus, $F'(x)$ exists for m-a.e. x by the theorem 3.27 (e). Then, $F' \in L^1(m)$. Let $G(x) = \int_{(-\infty,x]} F' dm$. Then $\mu_F = \mu_G + \lambda$, where λ is complex Borel measure such that $\lambda \perp m$ and $\mu_G \ll m$.*

Consequently, $\mu_F \perp m$ iff $F' = 0$ m-a.e. and $\mu_F \ll m$ iff $F(x) = \int_{(-\infty,x]} F'(x) dm$ for all $x \in \mathbb{R}$.

Proof. From theorem 3.29 in [1], μ_F exists. And by the theorem 1.18, μ_F is regular. And Lebesgue-Radon-Nikodym theorem and Theorem 3.22 in [1][Lebesgue Differentiation Theorem] says that $\mu_F = \rho + \lambda$ where $\rho \ll m$, $\lambda \perp m$ and $d\rho = f dm$ for some $f \in L^1(m)$ and for m-a.e. $x \in \mathbb{R}$,

$$f(x) = \lim_{r \to 0^+} \frac{\mu_F(E_r)}{m(E_r)} \text{ for any } (E_r)_{r>0} \text{ shrinking nicely to } x.$$

Let $E_r = (x, x+r]$ or $E_r = (x-r, x]$. Then,

$$f(x) = \lim_{r \to 0^+} \frac{\mu_F((x, x+r])}{r} = \lim_{r \to 0^+} \frac{F(x+r) - F(x)}{r}$$

and

$$f(x) = \lim_{r \to 0^-} \frac{\mu_F((x-r, x])}{r} = \lim_{r \to 0^-} \frac{F(x) - F(x-r)}{r}$$

So, $f = F'$ a.e. Hence,

$$\mu_G((a,b]) = G(b) - G(a) = \int_{(a,b]} F' dm = \int_{(a,b]} f dm = \int_{(a,b]} d\rho = \rho((a,b]).$$

The consequent facts are followed from the above equation. □

Definition 4.87 (Absolute continuity for the function)**.** *$F : \mathbb{R} \to \mathbb{C}$ is* **absolutely continuous** *if $\forall \epsilon > 0, \exists \delta > 0$ such that whenever $n \in \mathbb{N}$, and $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$ and $\sum_{j=1}^n (b_j - a_j) < \delta$, then $\sum_{j=1}^n |F(b_j) - F(a_j)| < \epsilon$. If $F : [a.b] \to \mathbb{C}$, then we say F is* **absolutely continuous in** *$[a,b]$ when the same condition holds whenever $a_1, \cdots, a_n, b_1, \cdots, b_n \in [a,b]$ as well. Then, define as below;*

$AC := \{f : \mathbb{R} \to \mathbb{C} : f$ is absolutely continuous.$\}$ and $AC[a,b] := \{f : [a,b] \to \mathbb{C} : f$ is absolutely continuous on $[a,b]\}$.

Lemma 4.88 (Lemma 3.34 in [1] p. 106)**.** *$AC[a,b] \subseteq BV[a,b]$.*

Proof. Let $F \in AC[a,b]$, $\epsilon = 1$. Choose δ as in definition of AC. Let $m \in \mathbb{N}$ such that $\frac{b-a}{m} < \delta$. To show $F \in BV[a,b]$, we must show a uniform bound on $\sum_{j=1}^n |F(x_j) - F(x_{j-1})|$ over all partitions $a = x_0 < x_1 < \cdots < x_n = b$. For any given partition P, we may take a refinements of our partition so that it contains all $a + \frac{b-a}{m} \cdot k$ for $k \in [m-1]$. Then, there exists an integer $0 = p(0) < p(1) < \cdots < p(m) <= n$ such that $x_{p(k)} = a + \frac{b-a}{m} \cdot k$. Then,

$$\sum_{j=1}^n |F(x_j) - F(x_{j-1})| = \sum_{k=1}^n \sum_{j=p(k-1)+1}^{p(k)} |F(x_j) - F(x_{j-1})|$$

and for each k,

$$\sum_{j=p(k-1)+1}^{p(k)} |F(x_j) - F(x_{j-1})| < \epsilon \text{ since } \sum_{j=p(k-1)+1}^{p(k)} |x_j - x_{j-1}| = \frac{b-a}{m} < \delta \text{ and absolute continuity of } f.$$

Thus,

$$\sum_{j=1}^n |F(x_j) - F(x_{j-1})| = \sum_{k=1}^n \sum_{j=p(k-1)+1}^{p(k)} |F(x_j) - F(x_{j-1})| \leq m\epsilon = m < +\infty.$$

Since the given partition was arbitrary, we can conclude that

$$TV_F([a,b]) \leq m < +\infty,$$

as desired. $\square$

Observation 4.89.

1. *$AC \not\subseteq BV$ in general, when we think $F(x) = x$. Then $F \in AC$ is clear but it is not in BV since $T_F(+\infty) = +\infty$.*

2. *AC is a vector space, since it is closed under addition and scalar multiplication, and AC is a subspace of a function space.*

3. *If F is absolutely continuous then it is uniformly continuous, by taking $N = 1$.*

4. *If F is everywhere differentiable and F' is bounded then $|F(b_j) - f(a_j)| \leq (\max|F'|)(b_j - a_j|$ by the Mean Value theorem, thus it is absolutely continuous by taking $\delta < \frac{\epsilon}{\max|F'|}$.*

Proposition 4.90. *Let $F \in BV \cap AC$. Then, $T_F \in AC$.*

Proof. Note that $T_F(b) - T_F(a) = TV_F([a,b])$. To show T_F is absolutely continuous, for any given ϵ, we must show $\exists \delta > 0$ such that whenever $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$ and $\sum_{j=1}^n (b_j - a_j) < \delta$ we have $\sum_{j=1}^n TV_F([a_j, b_j]) < \epsilon$.

Since $F \in AC$, $\exists \delta > 0$ such that

$$\sum_{j=1}^n (b_j - a_j) < \delta \implies \sum_{j=1}^n |F(b_j) - F(a_j)| < \frac{\epsilon}{2}.$$

For each j, choose a partition

$$a_j = x_0^{(j)} < x_1^j < \cdots x_{k(j)} = b_j$$

such that

$$TV_F((a_j, b_j]) - \frac{\epsilon}{2n} \leq \sum_{i=1}^{k(j)} |F(x_i^{(j)}) - F(x_{i-1}^{(j)})|.$$

Thus,

$$\sum_{j=1}^n TV_F((a_j, b_j]) \leq \frac{\epsilon}{2} + \sum_{j=1}^n \sum_{i=1}^{k(j)} |F(x_i^{(j)}) - F(x_{i-1}^{(j)})|.$$

Since $\sum_{j=1}^n \sum_{i=1}^{k(j)} (x_i^j - x_{i-1}^j) = \sum_{j=1}^n (b_j - a_j) < \delta$ by choice of δ,

$$\frac{\epsilon}{2} \sum_{j=1}^n \sum_{i=1}^{k(j)} |F(x_i^{(j)}) - F(x_{i-1}^{(j)})| \leq \frac{\epsilon}{2},$$

by absolute continuity of F. Thus,

$$\sum_{j=1}^n TV_F((a_j, b_j]) \leq \frac{\epsilon}{2} \sum_{j=1}^n \sum_{i=1}^{k(j)} |F(x_i^{(j)}) - F(x_{i-1}^{(j)})| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

as desired. $\square$

Corollary 4.91. *If $F \in BV$, then $F \in AC \iff (ReF)^{\pm}, (ImF)^{\pm} \in AC$.*

Proof. From left to right is clear. If $F \in AC \cap BV$, then by the triangle inequality, we can make suitable ϵ. $\square$

Proposition 4.92 (Proposition 3.32 in [1] p.105). *Let $F \in NBV$. Then, $F \in AC$ iff $\mu_F \ll m$.*

Proof. Suppose $\mu_F \ll m$. Then, $|\mu_F| \ll m$ by the Exercise 8. By theorem 3.5, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $E \in \mathcal{B}_{\mathbb{R}}, m(E) < \delta \implies |\mu_F|(E) < \epsilon$. If $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n$ and if $\sum_{j=1}^n (b_j - a_j) < \delta$, then

$$m\left(\bigcup_{j=1}^n (a_j, b_j]\right) < \delta,$$

so

$$\epsilon > |\mu_F|\left(\bigcup_{j=1}^n (a_j, b_j]\right) = \sum_{j=1}^n |\mu_F|((a_j, b_j]) \ge \sum_{j=1}^n |\mu_F|((a_j, b_j]) = \sum_{j=1}^n |F(b_j) - F(a_j)|.$$

So $F \in AC$.

Conversely, suppose $F \in AC$, Since $|\mu_F| = \mu_{T_F}$ by the theorem 3.29 in [1][p.104], $T_F \in AC$. Thus, instead of using F, we replace F with T_F, using its increasing property. In other words, without loss of generality, we can assume that F is increasing function.

Suppose $E \in \mathcal{B}_{\mathbb{R}}$ such that $m(E) = 0$. We want to show that $\mu_F(E) = 0$. Let $\epsilon > 0$. Since $F \in AC, \exists \delta > 0$ such that $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n$ and if $\sum_{j=1}^n (b_j - a_j) < \delta \implies \sum_{j=1}^n F(b_j) - F(a_j) < \epsilon$. Now there exists an open $U \subseteq \mathbb{R}$ such that $E \subseteq U$ and $m(U) < \delta$, from the regularity of m. We can write $U = \bigcup_{j=1}^\infty (a_j, b_j)$ for disjoint open intervals. Then,

$$\mu_F(E) \le \mu_F(U) = \lim_{N \to \infty} \sum_{j=1}^N \mu_F((a_j, b_j)) \le \lim_{N \to \infty} \sum_{j=1}^N \mu_F((a_j, b_j]) = \lim_{N \to \infty} \sum_{j=1}^N F(b_j - F(a_j) < \epsilon$$

since $\sum_{j=1}^N (b_j - a_j) < \delta$. So, $\mu_F(E) = 0$. If we use T_F instead of F, this implies $|\mu_F|(E) = 0 \implies \mu_F(E) = 0$. $\square$

Corollary 4.93 (Corollary 3.33 in [1]p.105). *$F \in NBV \cap AC \iff \exists f \in L^1(m)$ such that $F(x) = \int_{(-\infty,x]} f dm$ and that $F' = f$ m-a.e.*

Proof. If $F \in NBV \cap AC$, then by the proposition 3.32, $\mu_F \ll m$, thus by the proposition 3.30, $\exists f \in L^1(m)$ such that $F(x) = \int_{(-\infty,x]} f dm$ and that $F' = f$ m-a.e.

Conversely, if $\exists f \in L^1(m)$ such that $F(x) = \int_{(-\infty,x]} f dm$ and that $F' = f$ m-a.e., then $F \in NBV$ since $F(+\infty) < \infty$, and $\mu_F \ll m$. Thus by the proposition 3.32, $F \in AC$. $\square$

Theorem 4.94 (The Fundamental Theorem of Calculus, Theorem 3.35 in [1]p.105). *Let $a, b \in \mathbb{R}, a < b$. Let $F : [a, b] \to \mathbb{C}$. Then, the followings are equivalent.*

(i) $F \in AC[a, b]$.

(ii) $\exists f \in L^1([a, b])$ *such that* $\forall x \in (a, b]), F(x) = F(a) + \int_{(a,x]} f dm$.

(iii) F *is differentiable m-a.e. on* $[a, b]$, $F' \in L^1([a, b])$, *and* $\forall x \in [a, b], F(x) = F(a) + \int_{(a,x]} F dm$.

Proof. Suppose (i). Then by the lemma 3.34 in [1][p.106], $F \in BV([a, b])$. Now extend F to $\hat{F} : \mathbb{R} \to \mathbb{C}$ such that

$$\hat{F}(x) = \begin{cases} F(a) - F(a) & \text{if } x \le a \\ F(x) - F(a) & \text{if } x \in [a, b] \\ F(b) - F(a) & \text{if } x \ge b. \end{cases}$$

Then, $\hat{F} \in NBV \cap AC$. Then, (iii) follows from the Corollary 3.33. And (iii) implies (ii).

To show that (ii) implies (i), set $f(t) = 0$ for $t \notin [a, b]$ and applying corollary 3.33. $\square$

Definition 4.95 (Notation for Lebesgue-Stieltjes integrals)**.** *If $F \in NBV$, we denote $\int g d\mu_F$ as $\int g dF$ or $\int g(x) dF(x)$. Such integrals are called* **Lebesgue-Stieltjes integrals.** *Note that*

$$\int_A G dF = \int_A G d\mu_F = \int_A G F' dm.$$

If $G = \sum_{k=1}^n G(\frac{k}{n}) 1_{(\frac{k-1}{n}, \frac{k}{n}]}$ with $A = [0,1]$, then

$$\int_{[0,1]} G dF = \sum_{k=1}^{\infty} G(\frac{k}{n})(F(\frac{k}{n}) - F(\frac{k-1}{n})).$$

If $\mu_F \ll m$, then $\exists f \in L^1, d\mu_F = f dm$. Thus,

$$\int_{(a,b]} d\mu_F = F(b) - F(a) = \int_{(a,b]} f dm.$$

By the Fundamental Theorem of Calculus, $F' = f$. So,

$$\int_{(a,b]} F dG + \int_{(a,b]} G dF = \int F G' dm + \int G F' dm.$$

Theorem 4.96 (Integration by parts, Theorem 3.36 in [1]p.107)**.** *Suppose $F, G \in NBV$ and G is continuous. Then, $\int_{(a,b]} F dG + \int_{(a,b]} G dF = F(b)G(b) - F(a)G(a)$.*

Proof. Without loss of generality, F, G is increasing. (Otherwise use the decomposition $(ReF)^{\pm}, (ImF)^{\pm}, (ReG)^{\pm}, (ImG)^{\pm}$. Then, μ_F, μ_G are positive measure. Consider a set $A := \{(x,y) : a < x \le y \le b\} \subseteq (a,b]^2$. Then

$$\begin{aligned}\mu_F \times \mu_G(A) &\underbrace{=}_{\text{Fubini's Thm}} \int\int 1_A(x,y) d\mu_F(x) d\mu_G(y) \\ &= \int_{(a,b]} \int_{(a,y]} d\mu_F(x) d\mu_G(y) = \int_{(a,b]} (F(y) - F(a)) d\mu_G(y) = \int_{(a,b]} F dG - F(b)(G(b) - G(a)).\end{aligned}$$

On the other hand,

$$\mu_F \times \mu_G(A) \underbrace{=}_{\text{Fubini's Thm}} \int\int 1_A(x,y) d\mu_G(y) d\mu_F(x) = \int_{(a,b]} \int_{(x,b]} d\mu_G(y) d\mu_F(x)$$

Note that

$$\lim_{t \to x-} \mu_G((t,b]) = G(b) - G(x-) = G(b) - G(x)$$

since G is continuous. Thus,

$$\mu_F \times \mu_G(A) = \int_{(a,b]} \int_{(x,b]} d\mu_G(y) d\mu_F(x) = \int_{(a,b]} G(b) - G(x) d\mu_F(x) = G(b)(F(b) - F(a)) - \int_{(a,b]} G dF.$$

Hence,

$$\int_{(a,b]} F dG + \int_{(a,b]} G dF = G(b)(F(b) - F(a)) + F(a)(G(b) - G(a)) = G(b)F(b) - F(a)F(b).$$

$\square$

5 You should explain standard result part and missing part of the note.+Cardinal Arithmeitc Theorem + Thm 2.28 (b) =Exercise 2.23

References

[1] G. Folland, *Real Analysis: Modern Techniques and their application.* Wiley, 2nd edition 2007