1 Important Series

1. \[ \sum_{n \geq 0} a^n z^n = \frac{1}{1 - az}, \]

2. \[ \sum_{n \geq 0} \binom{m}{n} z^n = (1 + z)^m \quad (m \in \mathbb{Z}), \]

3. \[ \sum_{n \geq 0} \binom{m + n - 1}{n} z^n = \frac{1}{(1 - z)^m} \quad (m \in \mathbb{Z}), \]

4. \[ \sum_{n \geq 0} \binom{m + n}{n} z^n = \frac{1}{(1 - z)^{m+1}} \quad (m \in \mathbb{Z}), \]

5. \[ \sum_{n \geq 0} \binom{n}{m} z^n = \frac{z^m}{(1 - z)^{m+1}} \quad (m \in \mathbb{N}), \]

6. \[ \sum_{n \geq 0} \binom{p + n}{m} z^n = \frac{z^{m-p}}{(1 - z)^{m+1}}, \]

7. \[ \sum_{n \geq 0} \frac{z^n}{n!} = e^z, \]

8. \[ \sum_{n \geq 0} (-1)^{n-1} \frac{z^n}{n} = \log(1 + z), \]

9. \[ \sum_{n \geq 0} \frac{z^n}{n} = - \log(1 - z), \]

10. \[ \sum_{n \geq 0} \binom{2n}{n} z^n = \frac{1}{\sqrt{1 - 4z}} , \]

11. \[ C(z) = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} z^n = \frac{1 - \sqrt{1 - 4z}}{2z} , \]

12. \[ \sum_{n \geq 0} \binom{2n + k}{n} z^n = \frac{C(z)^k}{\sqrt{1 - 4z}}, \]

13. \[ \sum_{n \geq 0} \frac{k}{2n + k} \binom{2n + k}{n} z^n = C(z)^k. \]
2 Combinatorial Identities

1. \[
\binom{-c}{k} = (-1)^k \binom{c+k-1}{k}, \quad (-1)^k \binom{c}{k} = \binom{k-c-1}{k}
\]

2. \[
\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}
\]

3. \[
\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}
\]

4. Binomial Theorem
\[
\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n
\]

5. Vandermonde Identity
\[
\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}
\]

6. Stirling number of the second kind
\[
x^n = \sum_{k=0}^{n} S_{n,k} x(x-1) \cdots (x-k+1)
\]

7. Signless Stirling number of the first kind
\[
x(x+1) \cdots (x+n-1) = \sum_{k=0}^{n} s_{n,k} x^k
\]

8. Worpitzky's Identity
\[
x^n = \sum_{k=0}^{n} A_{n,k} \binom{x+n-k}{n} = \sum_{k=0}^{n} A_{n,k} \binom{x+k-1}{n}
\]
where \(A_{n,k}\) is the number of permutations of length \(n\) and \(k-1\) descents.

9. Mahonian Statistics of permutations
\[
\sum_{\pi \in \mathfrak{S}_n} q^{\text{inv}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\text{maj}(\pi)} = [n]_q!
\]

10. \(q\)-Binomial Theorem
\[
(1+xq)(1+xq^2) \cdots (1+xq^n) = \sum_{k=0}^{n} \binom{n}{k} q^{\binom{k+1}{2}} x^k.
\]
11. \( q \)-Vandermonde Identity

\[
\binom{r+s}{n}_q = \sum_{k=0}^{n} \binom{r}{k}_q \binom{s}{n-k}_q q^{(r-k)(n-k)}
\]

12. \( q \)-Newton Binomial Theorem

\[
\sum_{n \geq 0} \binom{m+n-1}{n}_q z^n = \frac{1}{(1-z)(1-zq) \cdots (1-zq^{m-1})}
\]

13. \[
\binom{-1/2}{n} = \left( -\frac{1}{4} \right)^n \binom{2n}{n}, \quad \binom{-3/2}{n} = \left( -\frac{1}{4} \right)^n (2n+1) \binom{2n}{n}.
\]


\[
\prod_{k \geq 1} (1 + z q^k)(1 + z^{-1} q^{-k-1})(1 - q^k) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} z^n
\]

15. Lagrange Inversion Formula.

**Version 1.** Suppose \( F(z) = zG(F(z)) \), where \( G(0) = 0 \). Then

\[
[z^n]F(z) = \frac{1}{n} [z^{n-1}]G(z)^n \quad n \geq 1.
\]

**Version 2.** Let \( F(Z) \) and \( H(z) \) be compositional inverse. Assume \( H(z) = zK(z) \) Then

\[
[z^n]F(z) = \frac{1}{n} [z^{n-1}] (K(z))^{-n} = \frac{1}{n} [z^{-1}] (H(z))^{-n}.
\]

**Version 3.** (Stronger). Let \( F(Z) \) and \( H(z) \) be compositional inverse. Assume \( H(z) = zK(z) \). Then for any \( k \geq 1 \),

\[
n[z^n]F(z)^k = k[z^{n-k}] (K(z))^{-n} = k[z^{-k}] (H(z))^{-n}.
\]

**Version 4.** (Stronger). Let \( F(Z) \) and \( H(z) \) be compositional inverse. Assume \( H(z) = zK(z) \). Then for any \( T(z) \in \mathbb{C}[[z]] \),

\[
n[z^n]T(F(z)) = [z^{n-1}] T'(z)(K(z))^{-n}.
\]

16. Principle of Inclusion-Exclusion. Let \( X \) be a set with a set of \( n \) properties \( P = \{p_1, \ldots, p_n\} \). For any \( T \subseteq P \), let

\[
N_\subseteq T = \# \{ x \in X : x \text{ possesses precisely the properties in } T \},
\]

\[
N_{\supseteq T} = \# \{ x \in X : x \text{ possesses at least the properties in } T \}.
\]
Then
\[ N_{=\emptyset} = \sum_{T \subseteq P} (-1)^{|T|} N_{\geq T} = \sum_{k=0}^{n} (-1)^k \sum_{T:|T|=k} N_{\geq T}. \]

In general,
\[ N_{=A} = \sum_{T \supseteq A} (-1)^{|T-A|} N_{\geq T}; \]

and if \( N_p \) is the number of elements \( x \) that possess exactly \( p \) properties, then
\[ N_p = \sum_{k=p}^{n} (-1)^{k-p} \binom{k}{p} \sum_{T:|T|=k} N_{\geq T}. \]

The above formulas remain true if we change \( \subseteq \) to \( \supseteq \).