I was trying to realize an item of my research plan, namely, I tried to find a direct proof of the index theorem for deformation quantization. The situation here seems a little bit strange: we have a developed machinery for deformation quantization, we know that the index may be expressed in terms of characteristic classes, but it is still insufficient to write down an explicit formula. I was sure that there must be a simple way based on the explicit formulas for deformation quantization. In the simplest situation such a direct approach actually exists (see Appendix) and the problem is to generalize it for any symplectic manifold.

To this end several steps were made.

- A convenient formula for the trace density in deformation quantization on symplectic manifold was obtained. The result was written as a preprint MPI 01-75:

- Another preprint MPI 01-103 deals with the canonical form of the Darboux chart in the presence of symplectic connection:

The formula in [1] allows one, in principle, to compute traces for arbitrary quantum observables (with compact support), not only for constants as in the index theorem. The explicit expression was obtained only for few terms of the expansion by means of first two Chern forms, the first Pontryagin form, and an exact term, which disappears in the index formula after integration.

In [2] I introduced the notion of normal Darboux coordinates and investigated the existence and uniqueness of such coordinates. To this end a special generating function (the Cayley generating function) was introduced which gives a representation of a symplectomorphism by the so-called nonlinear
Cayley transform. It turns out, however, that normal Darboux coordinates exist on the level of formal coordinate neighborhood. A counterexample shows that they do not exist even in the sense of $C^1$ smoothness. Nevertheless, the formal neighborhood is sufficient for the purposes of deformation quantization.

Unfortunately, I didn’t succeed in finding a direct proof of the index formula in full generality.

These results, as well as some my previous results, were reported at various seminars, workshops and conferences.

1. “Deformation Quantization: pro and contra”


7. “Deformation Quantization of Symplectic Manifolds and the Index Theorem”.

Appendix

0.1 A toy example

We start explaining the notion of the trace density and the index for the simplest example. After that we pass over to more and more complicated cases till the index formula for deformation quantization is derived in full generality.

Consider a symplectic manifold $(M, \omega)$ with a symplectic connection $\partial$ whose curvature vanishes. Let $E$ be a complex line bundle over $M$ with a connection $\partial^E$ of constant curvature, that is all covariant derivatives of the curvature tensor vanish: $\partial_k R_{ij} \equiv 0$. Under these very restrictive hypotheses the problem of quantization including trace density computation admits an explicit solution.

The machinery of deformation quantization begins with the construction of the Abelian connection

$$D = \partial + [s, \cdot] = \partial + \left[ \frac{i}{\hbar} \omega_{ij} y^i dx^j + r, \cdot \right]$$

(0.1)

where $\partial = 1 \otimes \partial^E + \partial^M \otimes 1$ is a connection on the Weyl algebras bundle generated by the symplectic connection $\partial^M$ and the connection $\partial^E$ on the bundle $E$. The Weyl curvature of $D$ is defined as

$$\Omega_W = R^E + \partial s + s \circ s$$

and we require this to be equal to $-i/\hbar \omega$, which gives an equation

$$\partial s + s \circ s = -\frac{i}{\hbar} \omega - R$$

(0.2)

(we drop the superscript $E$ in the notation of the curvature). This equation has a unique solution under the following normalization conditions:

$$s = \frac{i}{\hbar} \omega_{ij} y^i dx^j + r, \ deg r \geq 1,$$

$$\delta^{-1} s = 0$$

where $\delta^{-1}$ is defined on components

$$s_{pq} = s_{i_1 \ldots i_q j_1 \ldots j_q} y^{i_1} \ldots y^{i_q} dx^{j_1} \wedge \ldots \wedge dx^{j_q}$$
\[ \delta^{-1} s_{pq} = \frac{1}{p+q} \sum_{k=1}^{q} (-1)^{k+1} s_{i_1 \ldots i_k j_1 \ldots j_q} y^{i_1} \ldots y^{i_k} dx^{j_1} \wedge \ldots \wedge dx^{j_q}, \]

that is we replace \( dx^{j_k} \) by \( y^{j_k} \) and take the alternating sum of these expressions.

To solve (0.2), we make an Ansatz

\[ s = \frac{i}{\hbar} y F_{ij} dx^j = \frac{i}{h} y' F dx = -\frac{i}{\hbar} dx^l F y \]

where \( F = (F_{ij}) \) is an antisymmetric matrix whose entries \( F_{ij} \) are functions in curvature matrix, \( dx \) and \( y \) are columns with the entries

\[ dx^1, \ldots, dx^{2n}, y^1, \ldots, y^{2n}, \ 2n = \dim M. \]

Since the curvature is constant, \( \partial s \equiv 0 \) and (0.2) reduces to

\[ -\left( \frac{i}{\hbar} \right)^2 dx^l F y \circ y' F dx = -\frac{i}{2\hbar} dx^l (\omega - ihR) dx. \]

Now, since

\[ y \circ y = -\frac{i\hbar}{2} \omega^{-1}, \]

the left-hand side takes the form

\[ -\frac{i}{2\hbar} dx^l F \omega^{-1} F dx, \]

so that

\[ F \omega^{-1} F = \omega - ihR \]

or

\[ (\omega^{-1} F)^2 = 1 - ih\omega^{-1} R. \]

Thus,

\[ F = \omega \sqrt{1 - ih\omega^{-1} R} = \sqrt{1 - ihR\omega^{-1} \omega}, \]

giving a solution of (0.2)

\[ s = \frac{i}{\hbar} y' \omega \sqrt{1 - ih\omega^{-1} R} dx = -\frac{i}{\hbar} dx^l \sqrt{1 - ihR\omega^{-1} \omega} y. \quad (0.3) \]
Remark 0.1. The same formula is true for any vector bundle $E$, not necessarily a line bundle, provided the coefficients $R_{ij}$ of the curvature form are commuting matrices. In particular, this is the case if $R_{ij}$ are covariantly constant: $\partial_k R_{ij} = 0$. Indeed, using the Ricci identity

$$(\partial_k \partial_l - \partial_l \partial_k) R_{ij} = [R_{kl}, R_{ij}],$$

we see that all the commutators vanish since the left-hand side is identically zero. Thus, the solution (0.3) is valid for any vector bundle $E$ with the covariantly constant curvature. The only difference is that the matrices should be treated as block matrices whose blocks in their turn are commuting square matrices.

The next step is to give an explicit description of quantum observables, that is the solution of the equation

$$Da := \partial a + [s, a] = 0.$$  

For line bundles $\partial a = da$, where $d$ means the de Rham differential with respect to $x$. Using (0.3), we come to a partial differential equation

$$\frac{\partial a}{\partial x^i} - (\sqrt{1 - ihR\omega}^{-1})^i_k \frac{\partial a}{\partial y^k} = 0$$

which can be easily solved. Its general solution has the form

$$a(x, y) = Q(b(x)) = b(x + (1 - ih\omega)^{-1/2}y). \tag{0.4}$$

Thus, for any section $b(x) \in C^\infty(M, E)$, which is the classical observable, quantization yields a quantum observable $Q(b(x))$ defined by (0.4). This gives a complete description of the algebra of quantum observables and the quantization map $Q$.

The next step is the calculation of the trace density. To this end we consider classical observables $b(x)$ supported in a neighborhood $O$ of a point $x_0 \in M$. The algebra $W_D|_O$ is isomorphic to the standard algebra $W_{D_0}|_O$ where

$$D_0 = d - \delta = d + \left[ \frac{i}{\hbar} \omega_{ij} dy^i dx^j, \cdot \right].$$

First we construct a homotopy $D_t$ of Abelian connections with the constant Weyl curvature $\Omega_W = -i/\hbar \omega$ linking $D$ and $D_0$ in a neighborhood $O$. 


Choose a local frame of the bundle $E$ obtained by the parallel transport of a fixed frame at the point $x_0$ along the rays (geodesics) $[x_0, x]$. In such a frame $R_{ij}(x) \equiv \text{const}$ and connection coefficients $\Gamma_i(x)$ satisfy the condition $\Gamma_j(x)x^j \equiv 0$ which defines them uniquely:

$$\Gamma_j(x) = \frac{1}{2} R_{ij} x^i.$$

For $t \in [0, 1]$ consider a family of connections on the bundle $E$ defined in the neighborhood $O$

$$\partial_t = d + t\Gamma_j dx^j$$

with the curvature

$$R(t) = t d\Gamma_j dx^j = \frac{1}{2} t R_{ij} dx^i \wedge dx^j.$$

It is important that the matrices $\omega^{-1} R(t) = t \omega^{-1} R$ commute for different $t \in [0, 1]$ and are covariantly constant with respect to the connection $\partial_t$. Using (0.3) with $R$ replaced by $R(t) = tR$, we obtain the corresponding family of Abelian connections $D_t$. Next we define a section $H(t) \in W_{D_t}$ (the Hamiltonian) as a solution of the equation

$$D_t H = \dot{\Gamma} + \dot{s} \quad (0.5)$$

with the following normalization condition $H|_{y=0} = 0$. We use the Ansatz

$$H = x' F y = -y' F x \quad (0.6)$$

where $F$ is an antisymmetric matrix depending on the curvature only. Thus, $\partial_t F = 0$ and

$$\partial_t H = dx' F y = -y' F dx.$$

Further,

$$[s, H] = -x' F \sqrt{1 - iht\omega^{-1} R} dx,$$

so that (0.5) becomes

$$-y' F dx - x' F \sqrt{1 - iht\omega^{-1} R} dx = \frac{1}{2} x' R dx$$

$$+ \frac{i}{h} y' \omega \frac{d}{dt} (\sqrt{1 - iht\omega^{-1} R}) dx.$$
Comparing the first term on the left with the last term on the right, we find

\[ F = -\frac{1}{2} R(1 - iht\omega^{-1}R)^{-1/2} \]

and one easily checks that this matrix satisfies the whole equation. So, finally,

\[ H = \frac{1}{2} y' R(1 - iht\omega^{-1}R)^{-1/2} x \]  

(0.7)

### 0.2 Calculation of the trace density

Let us introduce a notion

\[ \text{str} a(y) = a(0) \]  

(0.1)

for an element of the Weyl algebra

\[ a(y) = \sum a_{k_1 \ldots i_p} h^{k_1} y^{i_1} y^{i_2} \ldots y^{i_p} \in W. \]

If we consider coefficients in \( \text{Hom}(E, E) \), then (0.1) should be modified to

\[ \text{str} a(y) = \text{tr} a(0). \]

The Weyl algebra may be treated as a superalgebra \( W = W^+ \oplus W^- \) where subspaces \( W^\pm \) consist of elements of even (respectively odd) degree. The functional (0.1) is a supertrace on this algebra, as one can easily check. Indeed, the supertrace property means

\[ \text{str} a(y) \circ b(y) = \text{str} b(-y) \circ a(y), \]

or

\[ a(y) \circ b(y)|_{y=0} = b(-y) \circ a(y)|_{y=0}, \]

the latter equality follows directly from the explicit formula for \( \circ \)-product.

The same notation may be introduced fiberwise for the sections \( a(x, y) \in C^\infty(M, W) \) of the Weyl algebras bundle.

Let us recall the variation formula for the trace density \( c(x, h) \):

\[ \int_M \text{tr} \, \dot{c} b \omega^n = \int_M \text{str} c[H, Q(b)] \omega^n \]  

(0.2)

(for the line bundle \( E \) the tr sign on the left may be dropped). Everything in this formula is considered as \( t \)-dependent \( t \in [0, 1] \), \( H \) is the corresponding
Hamiltonian. Let us apply this formula for our toy example taking $Q(b)$ and $H$ in the form (0.4) and (0.6). Thus,

$$[H, Q(b)] = x'(1 - ihtR\omega^{-1})^{-1/2}Ry \circ b(x + (1 - ihtR\omega^{-1})^{-1/2}y)$$

$$= \frac{i\hbar}{2} x'(1 - ihtR\omega^{-1})^{-1/2}R\omega^{-1}(1 - ihtR\omega^{-1})^{-1/2}$$

$$\frac{\partial b}{\partial x}(x + (1 - ihtR\omega^{-1})^{-1/2}y)$$

$$= -\frac{1}{2} x'(1 - ihtR\omega^{-1})^{-1}(1 - ihtR\omega^{-1}) \frac{\partial b}{\partial x}(x + (1 - ihtR\omega^{-1})^{-1/2}y),$$

and the right-hand side of (0.2) becomes

$$\int_M -\frac{1}{2} c x'(1 - ihtR\omega^{-1})^{-1}(1 - ihtR\omega^{-1}) \frac{\partial b}{\partial x}(x) \omega^n$$

$$= \int_M \frac{1}{2} c x'(1 - ihtR\omega^{-1})^{-1}(1 - ihtR\omega^{-1}) b(x) \omega^n.$$

Using that $b(x)$ is an arbitrary function, we come to a differential equation for the trace density

$$\dot{c} = \frac{1}{2} c x'(1 - ihtR\omega^{-1})^{-1}(1 - ihtR\omega^{-1})$$

whose solution under the initial condition $c(0) = 1$ is

$$c(t) = \sqrt{\det(1 - ihtR\omega^{-1})}.$$  

At $t = 1$ we obtain an explicit formula for the trace density

$$c = \sqrt{\det(1 - iht\omega^{-1})}.  \tag{0.3}$$

This result admits a very natural and promising interpretation. It means that the trace on $W_D$ may be defined as follows. Let

$$a(x, y) = Q(b) = b(x + (1 - ihtR\omega^{-1})^{-1/2}y).$$

Then (0.3) means that $\text{Tr} a$ for $a$ supported in a neighborhood $x \in O$ may be written as

$$\text{Tr} a(x, y) = \frac{1}{(2\pi\hbar)^n n!} \int_{R^{2n}} b(x + (1 - ihtR\omega^{-1})^{-1/2}y)(\frac{1}{2} \omega_{ij} dy^i \wedge dy^j)^n.  \tag{0.4}$$
Indeed, changing variables $y = (1 - i\hbar R\omega^{-1})^{1/2}z$, we obtain on the right

$$
\frac{1}{(2\pi \hbar)^n n!} \int_{\mathbb{R}^{2n}} \det(1 - i\hbar R\omega^{-1})^{1/2}b(z) \left(\frac{1}{2} \omega_{ij} dz^i \wedge dz^j\right)^n
$$

$$
= \frac{1}{(2\pi \hbar)^n n!} \int_M cb(x)\omega^n.
$$

which is equal to the left-hand side by the definition of the trace density. Of course, the integration over $y \in \mathbb{R}^n$ is possible because of the simple dependence (0.4) in $y$ of flat sections. There is a hope, however, that the formula

$$
\text{Tr} a = \frac{1}{(2\pi \hbar)^n n!} \int_{\mathbb{R}^{2n}} a(x, y) \left(\frac{1}{2} \omega_{ij} dy^i \wedge dy^j\right)^n
$$

still makes sense after a proper definition of the integration over $y \in \mathbb{R}^{2n}$.

Formula (0.3) has another important consequence. Namely, it leads to the index formula for deformation quantization in our toy example. Indeed, if $M$ is a compact manifold, then

$$
\text{Tr} 1 = \frac{1}{(2\pi \hbar)^n n!} \int_M c\omega^n
$$

$$
= \frac{1}{(2\pi \hbar)^n n!} \int_M \left(\frac{1}{2} dx'\sqrt{1 - i\hbar R\omega^{-1}\omega R}\right)^n
$$

$$
= \frac{1}{(2\pi \hbar)^n n!} \int_M \left(\frac{1}{2} dx'(\omega - i\hbar R)dx\right)^n
$$

$$
= \int_M \exp \left(\frac{\omega}{2\pi \hbar} - \frac{iR}{2\pi}\right),
$$

which is precisely the index formula in our example.