Two-boundary lattice paths and parking functions

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Key words and phrases. Lattice path, parking function, area and sum enumerators, Appell relations.

Mathematics Subject Classification. 05A15

Abstract

We describe an involution on a set of sequences associated with lattice paths with north or east steps constrained to lie between two arbitrary boundaries. This involution yields recursions (from which determinantal formulas can be derived) for the number and area enumerator of such paths. An analogous involution can be defined for parking functions with arbitrary lower and upper bounds. From this involution, we obtained determinantal formulas for the number and sum enumerator of such parking functions. For parking functions, there is an alternate combinatorial inclusion-exclusion approach. The recursions also yield Appell relations. In certain special cases, these Appell relations can be converted into rational or algebraic generating functions.

[4]The third author was supported in part by NSF grant #DMS-0245526.

1 Introduction

Let x be a positive integer and $(x_0, x_1, x_2, \ldots, x_{n-1})$ be a non-decreasing sequence of non-negative integers with $x_{n-1} < x$. We can associate with the sequence a lattice path in the plane going from the origin $(0, 0)$ to the point $(x - 1, n)$ with unit north and east steps as follows: for $i = 0, 1, \ldots, n - 1$, the right-most point with y-coordinate equal to i is (x_i, i). Put another way, the path associated with $(x_0, x_1, x_2, \ldots, x_{n-1})$ is the path with x_0 east steps, one north step, $x_1 - x_0$ east steps, one north step, $x_2 - x_1$ east steps, one north step, and so on.

Let $\underline{r}$ and $\underline{s}$ be non-decreasing sequences with non-negative integer terms $r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$, thought of as *left* and *right boundaries.* An *$(\underline{r}, \underline{s})$-lattice path of length n* is defined to be a non-decreasing sequence $(x_0, x_1, \ldots, x_{n-1})$ such that $r_i \le x_i$ and $x_i < s_i$. We denote by $\mathrm{Path}_n(\underline{r}; \underline{s})$ (respectively, $\mathrm{LP}_n(\underline{r}, \underline{s})$) the set (respectively, the number) of all $(\underline{r}, \underline{s})$-lattice paths of length n.

Parking functions are rearrangements of lattice paths. An *$(\underline{r}, \underline{s})$-parking function of length n* is a sequence $(x_0, x_1, \ldots, x_{n-1})$ of positive integers such that its rearrangement $(x_{(0)}, x_{(1)}, \ldots, x_{(n-1)})$ into a non-decreasing sequence satisfies the inequalities

$$r_i \le x_{(i)} \quad \text{and} \quad x_{(i)} < s_i.$$

When $\underline{r}$ equals $\underline{1}$, the sequence with all terms equal to 1, and $\underline{s}$ is the sequence $2, 3, \ldots$, $(\underline{r}, \underline{s})$-parking functions are, up to a shift in indexing, "ordinary" parking functions, as defined in [1]. We shall denote the set (respectively, the number) of all $(\underline{r}, \underline{s})$-parking functions of length n by $\mathrm{Park}_n(\underline{r}; \underline{s})$ (respectively, $\mathrm{P}_n(\underline{r}, \underline{s})$).

Parking functions and lattice paths have very similar enumeration theories. We shall make this expectation precise by showing that the basic enumeration formulas for parking functions and lattice paths are "equivalent" under the substitution of a binomial coefficient for a power, (c.f. Eqns. (2.5) and (2.8)). Analogous or equivalent probability formulas for real or integer random variables have been derived and studied, by Steck, Niederhausen, Pitman, and others [7, 8, 9, 12, 13]. These formulas partly inspired our work.

We begin by showing that a certain set constructed from lattice paths has an involution. This involution implies a recursion which yields a triangular system of linear equations. Solving this equation gives us determinantal formulas for the number and area enumerators of $(\underline{r}, \underline{s})$-lattice paths. The involution and determinantal formulas have analogs for $(\underline{r}, \underline{s})$-parking functions. For parking functions, there is an alternate inclusion-exclusion approach.

What kind of generating functions do these determinantal formulas yield? We show how Appell relations can be easily obtained from any triangular system of linear equations. Usually, these Appell relations cannot be converted to generating functions.

Our exposition is deliberately elementary and focused on combinatorial arguments which allow a uniform derivation of counting and area enumerator formulas for lattice paths and parking functions. The theory of biorthogonal polynomials developed for $(\underline{1}, \underline{s})$-parking functions in [2]) can be extended to general $(\underline{r}, \underline{s})$-parking functions. In addition, a lattice-path theory can be obtained by replacing the differential operator with a difference operator. In such an intensively cultivated area as lattice-path counting, it is difficult to reference all earlier work. We have restrict our citation to papers which are directly relevant.

2 Combinatorial decompositions and bijections

We shall use the following notation. If x is a real number, then $x_+ = \max\{0, x\}$. If y and x are integers, then $[y, x)$ is the half-open interval $\{y, y+1, y+2, \ldots, x-2, x-1\}$ if $y < x$, and the empty set otherwise. If H is a set of integers, then $\binom{H}{m}$ is the set of all m-subsets of H or, equivalently, all length-m (strictly) increasing sequences with terms in H.

Let $\underline{r}$ and $\underline{s}$ be given left and right boundaries. Let

$$\mathcal{A} = \bigcup_{i=0}^{n} \mathrm{Path}_i(\underline{r}; \underline{s}) \times \binom{[r_{n-1}, s_i)}{n-i}. \tag{2.1}$$

Put another way, $\mathcal{A}$ is the set of all pairs $(\underline{x}, i)$ where $\underline{x}$ is a sequence of non-negative integers with terms $x_0, x_1, \ldots, x_{n-1}$ such that the initial segment $x_0, x_1, \ldots, x_{i-1}$ of length i is an $(\underline{r}, \underline{s})$-lattice path of length i and the final segment $x_i, x_{i+1}, \ldots, x_{n-1}$ is an increasing sequence all of whose terms lie in $[r_{n-1}, s_i)$. Note that we are not assuming that the initial segment is an $(\underline{r}, \underline{s})$-lattice path of maximum length. Let $\mathcal{A}_0$ be the subset of $\mathcal{A}$ consisting of those pairs $(\underline{x}, i)$ with i even and $\mathcal{A}_1$ be the complement of $\mathcal{A}_0$, the subset of pairs $(\underline{x}, i)$ with i odd.

Theorem 2.1 *Let $n \geq 1$. Then there is an bijection from $\mathcal{A}$ to itself sending $\mathcal{A}_0$ to $\mathcal{A}_1$ and $\mathcal{A}_1$ to $\mathcal{A}_0$.*

If $\underline{x}$ is a finite sequence, let $m(\underline{x})$ be the maximum $\max\{x_0, x_1, \ldots, x_{n-1}\}$ of all its terms. If the sequence $\underline{x}$ occurs as the first component of a pair $(\underline{x}, i)$ in $\mathcal{A}$, then either

$$x_{i-1} > x_{n-1},\ x_{n-1} \neq m(\underline{x}),\ \text{and } x_{i-1} = m(\underline{x}), \tag{2.2}$$

or

$$x_{i-1} \leq x_{n-1} \text{ and } x_{n-1} = m(\underline{x}). \tag{2.3}$$

Let $\mathcal{A}_0'$ (respectively $\mathcal{A}_1'$) be the subset of pairs in $\mathcal{A}_0$ (respectively $\mathcal{A}_1$) satisfying Condition (2.2) and let $\mathcal{A}_0''$ and $\mathcal{A}_1''$ be their complementary subsets in $\mathcal{A}_0$ and $\mathcal{A}_1$.

Let $\sigma : \mathcal{A} \to \mathcal{A}$ be the function defined in the following way.

- If $(\underline{x}, i)$ is in $\mathcal{A}_0'$ or $\mathcal{A}_1'$, then

$$\sigma(\underline{x}, i) = ((x_0, x_1, \ldots, x_{i-2}, x_i, x_{i+1}, \ldots, x_{n-1}, x_{i-1}), i-1),$$

that is, σ decreases i by 1 (changing its parity) and moves x_{i-1} to the end of the entire sequence. Since

$$r_{n-1} \le x_i < x_{i+1} < \ldots < x_{n-1} < x_{i-1} < s_{i-1},$$

$(x_i, x_{i+1}, \ldots, x_{n-1}, x_{i-1})$ an increasing sequence with all its terms in $[r_{n-1}, s_{i-1})$.

- If $(\underline{x}, i)$ is in $\mathcal{A}_0''$ or $\mathcal{A}_1''$,

$$\sigma(\underline{x}, i) = ((x_0, x_1, \ldots, x_{i-2}, x_{i-1}, x_{n-1}, x_i, x_{i+1}, \ldots, x_{n-2}), i+1),$$

that is, σ increases i by 1 and moves x_{n-1} to the end of the initial lattice path. Since $x_{n-1} < s_i$, the initial segment $(x_0, x_1, \ldots, x_{i-1}, x_{n-1})$ is an $(\underline{r}, \underline{s})$-lattice path of length $i+1$. Moreover,

$$r_{n-1} \le x_i < x_{i+1} < \ldots < x_{n-2} < s_i \le s_{i+1},$$

$(x_i, x_{i+1}, \ldots, x_{n-2})$ an increasing sequence with all its terms in $[r_{n-1}, s_{i+1})$.

It is routine to check that σ^2 is the identity function, that is, σ is an involution, and

$$\sigma(\mathcal{A}_0') = \mathcal{A}_1'',\ \sigma(\mathcal{A}_1') = \mathcal{A}_0'',\ \sigma(\mathcal{A}_0'') = \mathcal{A}_1',\ \sigma(\mathcal{A}_1'') = \mathcal{A}_0'.$$

Thus completes the proof of the theorem.

Because involutions are bijections, $|A_1| = |A_0|$. Hence, by Definition (2.1), if $n \ge 1$,

$$\sum_{i \text{ even}} \binom{(s_i - r_{n-1})_+}{n-i} \mathrm{LP}_i(\underline{r}, \underline{s}) = \sum_{i \text{ odd}} \binom{(s_i - r_{n-1})_+}{n-i} \mathrm{LP}_i(\underline{r}; \underline{s}).$$

We conclude that

$$\sum_{i=0}^{n} (-1)^i \binom{(s_i - r_{n-1})_+}{n-i} \mathrm{LP}_i(\underline{r}, \underline{s}) = \delta_{n,0}. \tag{2.4}$$

The equations (2.4) for $n, n-1, \ldots, 0$ form an upper triangular system of linear equations with $(-1)^i \mathrm{LP}_i(\underline{r}, \underline{s})$ as the unknowns. Solving this by Cramer's rule, we conclude that

$\mathrm{LP}_n(\underline{r},\underline{s})$ equals

$$\left|\begin{array}{cccccc} \binom{(s_0-r_0)_+}{1} & \binom{(s_0-r_1)_+}{2} & \binom{(s_0-r_2)_+}{3} & \cdots & \binom{(s_0-r_{n-2})_+}{n-1} & \binom{(s_0-r_{n-1})_+}{n} \\ 1 & \binom{(s_1-r_1)_+}{1} & \binom{(s_1-r_2)_+}{2} & \cdots & \binom{(s_1-r_{n-2})_+}{n-2} & \binom{(s_1-r_{n-1})_+}{n-1} \\ 0 & 1 & \binom{(s_2-r_2)_+}{1} & \cdots & \binom{(s_2-r_{n-2})_+}{n-3} & \binom{(s_2-r_{n-1})_+}{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \binom{(s_{n-1}-r_{n-1})_+}{1} \end{array}\right|. \tag{2.5}$$

A version of this determinantal formula was obtained by Steck [12, 13] earlier. In addition, Eqn (2.4) was obtained also earlier (in a different way) by Mohanty (Eqn (2.37) in [6]).

The combinatorial involution also yields weighted enumeration formulas. Let H be a set of sequences of length n. The *area* or *sum enumerator* $\mathrm{Area}(q;H)$ *of* H is the polynomial in the variable q defined by

$$\mathrm{Area}(q;H) = \sum_{(x_0,x_1,\ldots,x_{n-1})\in H} q^{x_0+x_1+\cdots+x_{n-1}}$$

We define $\mathrm{Area}_n(q;\underline{r};\underline{s})$ to be the area enumerator of $\mathrm{Path}_n(\underline{r};\underline{s})$.

Using the known result that

$$\begin{aligned} \mathrm{Area}(q;\binom{[y,x)}{n}) &= q^{ny+\binom{n}{2}}\binom{x-y}{n}_q \\ &= q^{ny+\binom{n}{2}}\frac{(1-q^{x-y})(1-q^{x-y-1})\cdots(1-q^{x-y-n+1})}{(1-q^n)(1-q^{n-1})\cdots(1-q)} \end{aligned}$$

(where $\binom{x-y}{n}_q$ is a q-binomial coefficient) and proceeding as earlier, we obtain

$$\mathrm{Area}_n(q;\underline{r};\underline{s}) = \det\left[q^{r_j(j-i+1)+\binom{j-i+1}{2}}\binom{(s_i-r_j)_+}{j-i+1}_q\right]_{0\le i,j\le n-1}.$$

In other words, the area enumerator of $\mathrm{Path}_n(\underline{r};\underline{s})$ equals the determinant obtained from determinant (2.5) by the substitutions

$$\binom{(s_i-r_j)_+}{j-i+1} \longleftarrow q^{r_j(j-i+1)+\binom{j-i+1}{2}}\binom{(s_i-r_j)_+}{j-i+1}_q.$$

Finally, we observe that the involution yield recursions for the moments of areas of lattice paths. It is possible to get formulas for such moments. For an indication of how this can be done, see [3, 4].

There is an analog of Theorem 2.1 for parking functions. For given boundaries $\underline{r}$ and $\underline{s}$, let $\mathcal{Q}$ be the set of all pairs $(\underline{x}, J)$ where $\underline{x}$ is a sequence of non-negative integers with terms $x_0, x_1, \ldots, x_{n-1}$ and J is a subset of the index set $\{0, 1, \ldots, n-1\}$ such that the subsequence $(x_j : j \in J)$ is an $(\underline{r}, \underline{s})$-parking function of length $|J|$ and all the terms of the complementary subsequence $(x_j : j \notin J)$ lie in $[r_{n-1}, s_{|J|})$. That is,

$$\mathcal{Q} = \bigcup_{J : J \subseteq \{0,1,2,\ldots,n-1\}} \mathrm{Park}_J(\underline{r}; \underline{s}) \times \prod_{j \notin J} [r_{n-1}, s_{|J|}), \tag{2.6}$$

where $\mathrm{Park}_J(\underline{r}; \underline{s})$ is the set of all $(\underline{r}, \underline{s})$-parking functions of length i indexed by J and $\prod_{j \notin J} [r_{n-1}, s_{|J|})$ is the set of all sequences indexed by the complement $\{0, 1, 2, \ldots, n-1\} \backslash J$ with terms in $[r_{n-1}, s_{|J|})$.

If $\underline{x}$ is a sequence, let k be the largest index such that x_k equals the maximum $m(\underline{x})$. Let

$$\begin{aligned} \mathcal{Q}_0' &= \{(\underline{x}, J) : |J| \text{ is even and } k \in J\}, \\ \mathcal{Q}_0'' &= \{(\underline{x}, J) : |J| \text{ is even and } k \notin J\}. \end{aligned}$$

Define $\mathcal{Q}_1'$ and $\mathcal{Q}_1''$ similarly. Finally, define the function $\tau : \mathcal{Q} \to \mathcal{Q}$ as follows:

$$\tau(\underline{x}, J) = \begin{cases} (\underline{x}, J \setminus \{k\}) & \text{if } (\underline{x}, i) \in \mathcal{Q}_0' \cup \mathcal{Q}_1' \\ (\underline{x}, J \cup \{k\}) & \text{if } (\underline{x}, i) \in \mathcal{Q}_0'' \cup \mathcal{Q}_1'' \end{cases}.$$

It is easy to check that τ is an involution on $\mathcal{Q}$ sending $\mathcal{Q}_0'$ to $\mathcal{Q}_1''$, $\mathcal{Q}_1'$ to $\mathcal{Q}_0''$, $\mathcal{Q}_0''$ to $\mathcal{Q}_1'$, and $\mathcal{Q}_1''$ to $\mathcal{Q}_0'$.

The involution τ implies the following identity:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (s_i - r_{n-1})_+^{n-i} \mathrm{P}_i(\underline{r}, \underline{s}) = \delta_{n,0}. \tag{2.7}$$

We conclude that

$$\mathrm{P}_n(\underline{r}; \underline{s}) = n! \det \left[\frac{(s_i - r_j)_+^{j-i+1}}{(j-i+1)!} \right]_{0 \le i,j \le n-1}, \tag{2.8}$$

where, by convention, the i,j-th entry is 0 if $j - i + 1 < 0$. Once again, a version of this determinantal formula was obtained earlier by Steck [12, 13]. Using the fact that the sum enumerator of $[y, x)^m$ equals $[q^y (x-y)_{q,+}]^m$, where $(n)_{q,+}$ equals $1 + q + q^2 + \cdots + q^{n-1}$ if $n > 0$ and equals 0 otherwise, we obtain

$$\mathrm{Sum}_n(q; \underline{r}; \underline{s}) = n! \det \left[\frac{[q^{r_j} (s_i - r_j)_{q,+}]^{j-i+1}}{(j-i+1)!} \right]_{0 \le i,j \le n-1}.$$

Again, the i,j-th entry in the determinant is 0 if $j - i + 1 < 0$.

3 An inclusion-exclusion approach

Another way to prove the recursion (2.7) and its sum enumerator analog is to use inclusion-exclusion. This approach works only for parking functions.

Let $w(\underline{x}, J)$ be a weight function defined from $\mathcal{Q}$ to a ring which depends only on the sequence $\underline{x}$. If $\mathcal{R} \subseteq \mathcal{Q}$, then we define $w(\mathcal{R})$ to be the sum

$$\sum_{(\underline{x},J)\in\mathcal{R}} w(\underline{x}, J).$$

For a fixed subset J in the index set $\{0, 1, \ldots, n-1\}$, let $\mathcal{Q}(J)$ be the subset of pairs $(\underline{x}, J)$ in $\mathcal{Q}$ with second component equal to J. The sets $\mathcal{Q}(J)$ partition $\mathcal{Q}$. The other natural way of partitioning $\mathcal{Q}$ is to fix the sequence $\underline{x}$. For a fixed sequence $\underline{x}$, let $\mathcal{T}(\underline{x})$ be the collection of subsets K in $\{0, 1, 2, \ldots, n-1\}$ such that $(\underline{x}, K)$ is in $\mathcal{Q}$. For example, if the left boundary be $1, 1, 1, 1, \ldots$ and the right boundary be $2, 3, 4, 5, \ldots$ (so that we are considering the case of ordinary parking functions), then

$$\begin{aligned}
\mathcal{T}(1,3,3,3) &= \emptyset, \\
\mathcal{T}(1,2,2,3) &= \{\{0,1\},\{0,2\},\{0,1,2\},\{0,1,3\},\{0,2,3\},\{0,1,2,3\}\}, \\
\mathcal{T}(1,2,3,4) &= \{\{0,1,2\},\{0,1,2,3\}\}.
\end{aligned}$$

Theorem 3.1 *Let $n \geq 1$. Then,*

$$\sum_{J\subseteq\{0,1,\ldots,n-1\}} (-1)^{|J|} w(\mathcal{Q}(J)) = 0.$$

We begin the proof of theorem 3.1 by observing that

$$\begin{aligned}
\sum_{J\subseteq\{0,1,\ldots,n-1\}} (-1)^{|J|} w(\mathcal{Q}(J)) &= \sum_{(\underline{x},J)\in\mathcal{Q}} (-1)^{|J|} w(\underline{x}, J) \\
&= \sum_{\underline{x}} \left(\sum_{K\subseteq\mathcal{T}(\underline{x})} (-1)^{|K|} \right) w(\underline{x}, J).
\end{aligned}$$

Thus, to prove Theorem 3.1, it suffices to show that

$$\sum_{K\subseteq\mathcal{T}(\underline{x})} (-1)^{|K|} = 0. \tag{3.1}$$

Lemma 3.2 *Let $\underline{x}$ be a fixed sequence and $\mathcal{T}(\underline{x})$ be the collection of all subsets K in the index set $\{0, 1, 2, \ldots, n-1\}$ such that $(\underline{x}, K)$ is in $\mathcal{Q}$. Then $\mathcal{T}(\underline{x})$ is a filter in the Boolean algebra $2^{\{0,1,\ldots,n-1\}}$ of all subsets of $\{0, 1, \ldots, n-1\}$. In particular, $\mathcal{T}(\underline{x})$ is non-empty if and only if $\underline{x}$ is an $(\underline{r}, \underline{s})$-parking function.*

Proof. Suppose $J \in \mathcal{T}(\underline{x})$. Since $[r_{n-1}, s_{|J|}) \subseteq [r_{n-1}, s_i)$ for $i \geq |J|$ and $[r_{n-1}, s_{|J|}) \subseteq [r_{n-1}, s_k)$ for $k \geq |J|$, any subset in the complement of J can be added to J to obtain a subset in $\mathcal{T}(\underline{x})$. $\square$

Although $\mathcal{T}$ may not be an interval of the Boolean algebra, it satisfies similar regularity properties.

Lemma 3.3 *Suppose that $\mathcal{T}(\underline{x})$ is non-empty. Let $J_1, J_2, \ldots, J_r$ be the minimal subsets in $\mathcal{T}(\underline{x})$.*

1. $|J_i| < n$.
2. $|J_1| = |J_2| = \ldots = |J_r|$.
3. $J_1 \cup J_2 \cup \ldots \cup J_r \neq \{0, 1, 2, \ldots, n-1\}$.

Proof. By Lemma 3.2, $\underline{x}$ is a $(\underline{r}, \underline{s})$-parking function. If x_k is a term equal to the maximum $\max\{x_0, x_1, \ldots, x_{n-1}\}$, then $x_k \in [r_{n-1}, s_{n-1})$. Hence, $\{0, 1, \ldots, n-1\}\backslash\{k\}$ is in $\mathcal{T}(\underline{x})$ and $\{0, 1, \ldots, n-1\}$ is not minimal.

Next, suppose that $k = |J_1| < |J_2| = l$. Since $J_1 \in \mathcal{T}(\underline{x})$, at least $n-k$ terms are in $[r_{n-1}, s_k)$. Consider the $(\underline{r}, \underline{s})$-parking function $(x_j : j \in J_2)$. Since $|J_2| > k$, at least one of its term x_i in $[r_{n-1}, s_k)$ which is contained in $[r_{n-1}, s_{l-1})$. Hence, the subsequence $(x_j : J_2\backslash\{i\})$ is an $(\underline{r}, \underline{s})$-parking function, and the terms x_i and x_j, $j \notin J_2$ are all in $[r_{n-1}, s_{l-1})$. We conclude that J_2 is not minimal, a contradiction.

We may now assume that $|J_1| = |J_2| = \ldots = |J_r| = k$. Suppose that $J_1 \cup J_2 \cup \ldots \cup J_r = \{0, 1, 2, \ldots, n-1\}$. Then every term x_j in $\underline{x}$ is in some $(\underline{r}, \underline{s})$-parking function of length k; in particular, for all j, $x_j < s_{k-1}$. Consider an index i in J_1 but not in J_2. Then $i \in [r_{n-1}, s_{k-1})$. We conclude that $J_1\backslash\{i\}$ is in $\mathcal{T}(\underline{x})$, contradicting the hypothesis that J_1 is minimal. This completes the proof of the lemma. $\square$

If h is a function defined from the Boolean algebra $2^{\{0,1,\ldots,n-1\}}$ to a ring, then, by an inclusion-exclusion argument, the sum of $h(K)$ over all subsets K in a filter can be written as the alternating sum of sums of $h(K)$, where K ranges over principal filters. Specifically, let $\mathcal{F}(J_1, J_2, \ldots, J_r)$ be the filter with minimal subsets $J_1, J_2, \ldots, J_r$. Then $\mathcal{F}(J_1, J_2, \ldots, J_r) =$

$\mathcal{F}(J_1) \cup \mathcal{F}(J_2) \cup \ldots \cup \mathcal{F}(J_r)$ and $\mathcal{F}(J_1 \cup J_2 \cup \ldots \cup J_r) = \mathcal{F}(J_1) \cap \mathcal{F}(J_2) \cap \ldots \cap \mathcal{F}(J_r)$. Then, by inclusion-exclusion,

$$\sum_{K \in \mathcal{F}(J_1, J_2, \ldots, J_r)} h(K) = \sum_{\{k_1, k_2, \ldots, k_l\} \subseteq \{1,2,\ldots,r\}} (-1)^l \left(\sum_{K \in \mathcal{F}(J_{k_1} \cup J_{k_2} \cup \ldots \cup J_{k_l})} h(K) \right).$$

The proof can now be completed by setting $h(K) = (-1)^{|K|}$ and observing that for a principal filter $\mathcal{F}$, $\sum_{K \in \mathcal{F}} (-1)^{|K|}$ equals 0 except in the case when $\mathcal{F} = \{\{0, 1, 2, \ldots, n-1\}\}$.

4 Appell relations and generating functions

We begin with an elementary lemma relating solutions of systems of linear equations and Appell relations. Our assumption that the system is triangular is not the most general possible.

Lemma 4.1 *Suppose that g_n is a solution to the triangular system*

$$\begin{aligned}
b_{0,0} g_0 &= a_0 \\
b_{1,1} g_1 + b_{1,0} g_0 &= a_1 \\
b_{2,2} g_2 + b_{2,1} g_1 + b_{2,0} g_0 &= a_2 \\
&\vdots \\
b_{m,m} g_m + b_{m,m-1} g_{m-1} + b_{m,m-2} g_{m-2} + \cdots + b_{m,1} g_1 + b_{m,0} g_0 &= a_m \\
&\vdots
\end{aligned}$$

Then, as formal power series in the variable t,

$$\sum_{m=0}^{\infty} g_m t^m \Phi_m(t) = \Psi(t), \tag{4.1}$$

where

$$\Psi(t) = \sum_{k=0}^{\infty} a_k t^k$$

and

$$\Phi_m(t) = \sum_{k=0}^{\infty} b_{m+k,m} t^k.$$

To prove the lemma, multiply the mth equation by t^m and sum the columns.

How useful Appell relations are depends on how simple (or, subjectively, how familiar) the formal power series $\Phi_m(t)$ and $\Psi(t)$ are.

Our first example concerns $(\underline{0}, \underline{s})$-lattice paths, where $\underline{0}$ is the sequence of all zeroes. Then

$$\begin{aligned} \Phi_m(t) &= (-1)^m \sum_{k=0}^{\infty} \binom{s_m}{k} t^k && (4.2) \\ &= (-1)^m (1+t)^{s_m}. && (4.3) \end{aligned}$$

Hence,

$$\sum_{m=0}^{\infty} (-1)^m \mathrm{LP}_m(\underline{0}, \underline{s}) t^m (1+t)^{s_m} = 1.$$

For area enumerators, the functions $\Phi_m(t)$ is given by Eqn (4.2), with the binomial coefficient $\binom{s_m}{k}$ replaced by $q^{\binom{k}{2}} \binom{s_m}{k}_q$. However, there seems to be no closed form for $\Phi_m(t)$ in this case. Similar formulas can be derived for $(\underline{1}, s)$-parking functions (see [2]).

In the case where $\underline{s}$ is an arithmetic progression $(a + id)_{i=0}^{\infty}$,

$$\sum_{m=0}^{\infty} \mathrm{LP}_m(\underline{0}, (a+id)) [-t(1+t)^d]^m = \frac{1}{(1+t)^a}.$$

and hence,

$$\sum_{m=0}^{\infty} \mathrm{LP}_m(\underline{0}, (a+id)) u^m = \frac{1}{(1+t)^a}.$$

where t satisfies polynomial equation $t(1+t)^d + u = 0$. We conclude that the ordinary generating function of $\mathrm{LP}_m(\underline{0}, (a+id))$ is an algebraic function. The stronger result, that the generating function of $\mathrm{LP}_m(\underline{0}, \underline{s})$ is algebraic if $\underline{s}$ is periodic, was proved by de Mier and Noy [5].

Another case where the Appell relation can be simplified is when the left boundary $(a + id)_{i=0}^{\infty}$ and the right boundary $(b + id)_{i=0}^{\infty}$ are arithmetic progressions with the same difference d and $a < b$. In this case,

$$\sum_{m=0}^{\infty} \mathrm{LP}_m((a+id), (b+id)) t^m = \frac{1}{p(t)},$$

where

$$p(t) = \sum_{k=0}^{\lfloor (b-a+d)/d \rfloor} \binom{b-a+d-kd}{k} t^k. \qquad (4.4)$$

Similarly, the generating function of $\mathrm{Area}_m((a+id),(b+id))$ equals $1/p_q(t)$, where $p_q(t)$ is obtained from $p(t)$ in Eqn (4.4) by replacing the binomial coefficients by a product of q-binomial coefficients and a power of q. Since a formal power series $\sum_{m=0}^{\infty} a_m t^m$ is rational if and only if for some finite index M, the series $\sum_{m=M}^{\infty} a_m t^m$ is rational, we conclude that if $\underline{r}$ and $\underline{s}$ are arithmetic progressions with the same common difference after a finite number of terms, then the ordinary generating functions of $\mathrm{LP}_m(\underline{r},\underline{s})$ and $\mathrm{Area}_m(\underline{r},\underline{s})$ are rational functions. An analogous result holds for $(\underline{r},\underline{s})$-parking functions. Motivated by the algebraic result of de Mier and Noy, we conjecture that the generating function of $\mathrm{LP}_m(\underline{r},\underline{s})$ is algebraic if pair $(\underline{r},\underline{s})$ is "periodic", that is, there exist finite sequences $\underline{a}$ and $\underline{b}$ of the same length and positive integer d such that $\underline{r}$ is the concatenation of $\underline{a}, \underline{a}+d, \underline{a}+2d, \ldots$ and $\underline{s}$ is the concatenation of $\underline{b}, \underline{b}+d, \underline{b}+2d, \ldots$.

References

1. A.G. Konheim and B. Weiss, An occupancy discipline and applications, SIAM J. Appl. Math. 14(1966) 1266–1274.

2. J.P.S. Kung, C. Yan, Gončarov polynomials and parking functions, J. Combin. Theory Ser. A, 102(2003), 16–37.

3. J.P.S. Kung, C. Yan, Expected Sums of Moments General Parking Functions, Annals of Combinatorics, 7(2003), 481–493.

4. J.P.S. Kung, C. Yan, Exact Formula for Moments of Sums of Classical Parking Functions, Adv. Appl. Math. 31(2003) 215–241.

5. A. de Mier, M. Noy, A solution to the tennis ball problem, Theoret. Comput. Sci. 346 (2005) 254–264.

6. S.G. Mohanty, Lattice Path Counting and Applications, Academic Press, New York, 1979.

7. H. Niederhausen, Sheffer polynomials for computing exact Kolmogorov-Smirnov and Rényi type distributions. Annals of Statistics 9(1981), 923- 944.

8. H. Niederhausen, Lattice Path Enumeration and Umbral Calculus, Advances in Combinatorial Methods and Applications (editor: N. Balakrishnan), Birkhäuser Boston (1997).

9. E.J.G. Pitman, Simple proofs of Steck's determinantal expressions for probabilities in the Kolmogorov and Smirnov tests. *Bull. Austral. Math. Soc.*, 7(1972), 227–232.

10. J. Pitman, R.P. Stanley, A polytope related to empirical distributions, plane trees, parking functions, and the associahedron, *Discrete Comput. Geom.* 27(2002), no.4, 603–634.

11. R.P. Stanley, Enumerative Combinatorics, Volume 2. Cambridge University Press, 1999.

12. G.P. Steck, The Smirnov two sample tests as rank tests, Ann. Math. Statist., 40 (1969) 1449–1466.

13. G.P. Steck. Rectangle probabilities for uniform order statistics and the probability that the empirical distribution function lies between two distribution functions, Ann. Math. Statist., 42 (1971), 1–11.