An Enumeration Problem and Branching Processes

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1 Introduction

Many problems in enumerative combinatorics can be expressed in probabilistic terms by considering random combinatorial objects with uniform distribution. Often this probabilistic approach furnishes a convenient format for the discussion, and enables us to use the effective methods of asymptotic analysis. In this paper we consider an enumerative problem of particle allocations. The problem can be stated as follows. Given integers $m \geq n$. Allocate m particles in n distinct boxes. Let x_i be the number of particles in the i-th box. An allocation is said to be *good* if $x_1 + \ldots + x_i \geq i$ for all $1 \leq i \leq n$. We are interested in the number of good allocations $F(n,m)$, or equivalently, the probability $f(n,m) = F(n,m)/n^m$ that a random allocation with uniform distribution is good, and the behavior of $f(n,m)$ when m is a function of n.

This problem is related to the classical enumeration of parking functions. A *parking function of length* n is a sequence $(a_1, a_2, \ldots, a_n)$ of positive integers such that its non-decreasing rearrangement $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies $b_i \leq i$ for all $i = 1, 2, \ldots, n$. It is well known that the number of parking functions of length n is $(n+1)^{n-1}$. In fact, for $m = n$ the number of good allocations $F(n,n)$ is the same as the number of parking functions of length n, (c.f. Theorem 1). Using this fact, in section 2 we give an exact formula of $F(n,m)$ as a summation. If $m = tn$ for a fixed $t \geq 1$, then the probability $f(n,m) = F(n,m)/n^m$ approaches to a positive limit $h(t)$, which depends only on t with $h(0) = 0$ and $\lim_{t\to\infty} h(t) = 1$. Moving to asymptotics, we have $f(n,m) \sim 2(m-n)/n$ if $n^{1/2} \ll (m-n) \ll n$, (c.f. section 3).

In a general particle allocation problem, the distribution of the occupancy of the boxes can be represented as a conditional distribution of independent random variables conditioned on their

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sum being fixed, [4]. Such a representation allows one to explore the combinatorial problems by powerful techniques in probabilistic analysis on independent random variables. For the uniform random allocation, the corresponding random variables have Poisson distributions. In section 4 and 5, We reveal such a correspondence by relating our enumeration to the theory of branching processes. Branching process is a mathematical description of the growth of a population for which the individual produces offsprings according to stochastic laws. For $m = tn$ we show that the limit of the particle-allocation problem is the branching process with a Poisson offspring distribution with mean t, and the probability of good allocations goes to the probability of survival in the branching process. Moreover, this approximation extends to asymptotic level: Denote by $T^{(\epsilon)}$ the total number of the population in the branching process with the offspring distribution $Po(1+\epsilon)$. As $\epsilon \to 0^+$, the probability $T^{(\epsilon)} > N$ is asymptotically 2ϵ, provided that $N \gg \epsilon^{-2}$. This matches the asymptotic result of $f(n,m)$ in section 3.

2 Enumeration for the particle allocation

In this section we give the exact formula for the good particle allocations in terms of a summation.

Lemma 1 *The number of good allocation of n particles in n boxes is $(n+1)^{n-1}$.*

Proof. For an allocation of n particles in n boxes, let a_i be the place of the i-th particle, and let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the non-decreasing rearrangement of $(a_1, a_2, \ldots, a_n)$. Then $(a_1, a_2, \ldots, a_n)$ is a good allocation if and only if $b_i \leq i$ for $i = 1, 2, \ldots, n$. That is, $(a_1, a_2, \ldots, a_n)$ is a parking function. Hence the number of good allocations is the same as the number of parking functions of length n, which is $(n+1)^{n-1}$. □

Theorem 2 *Let $m \geq n$. The number of good allocations of m particles in n boxes is given by the following formula*

$$F(n,m) = n^m - \sum_{i=1}^{n} \binom{m}{i-1} i^{i-2} (n-i)^{m+1-i} = \sum_{j=n}^{m} \binom{m}{j} (j+1)^{j-1} (n-j-1)^{m-j}.$$

Proof. For convenience, we say an allocation of $m = n + r$ particles in n boxes is *bad* if it is not good, and denote the number of bad allocations by $B(n,m)$. We enumerate $B(n,m)$. Let BAD_i be the number of bad allocations in which i is the smallest number such that $x_1 + \ldots + x_i < i$. In any such allocation,

$$\begin{aligned} x_1 &\geq 1, \\ x_1 + x_2 &\geq 2, \\ \ldots\ldots &\geq \ldots \\ x_1 + x_2 + \ldots + x_{i-1} &\geq i-1, \\ x_1 + x_2 + \ldots + x_i &< i. \end{aligned}$$

This implies $x_1 + x_2 + \ldots + x_{i-1} = i - 1$ and $x_i = 0$. So this allocation can be decomposed into two parts: The first part is an good allocation of $i-1$ particles in the first $i-1$ boxes. The ith box is empty. And the second part is a random allocation of the other $m-i+1$ particles in the last $N-i$ boxes.

By Lemma 1,

$$BAD_i = \binom{m}{i-1} i^{i-2}(n-i)^{m-i+1},$$

and

$$B(n,m) = \sum_{i=1}^{n} BAD_i = \sum_{i=1}^{n} \binom{m}{i-1} i^{i-2}(n-i)^{m+1-i},$$

which proved the first equation of Theorem 2. The second equation follows from Abel identity

$$x^{-1}(x+y+Na)^N = \sum_{k=0}^{N} \binom{N}{k} (x+ka)^{k-1}(y+(N-k)a)^{N-k},$$

with the substitution $N = m, x = a = 1$ and $y = n - m - 1$. □

Next we consider the ratio $f(n,m) = F(n,m)/n^m$, that is, the probability that a random allocation is good. For the case $m = tn$, $(t \geq 1)$, we have the following limit result.

Theorem 3 *Fix $t > 1$ and let $m = tn$. Then the probability $f(n,tn)$ that a random allocation is good goes to a positive constant $1-s/t$, where s is the unique solution in $(0,1)$ satisfying $se^{-s} = te^{-t}$. In particular, $1 - s/t = 0$ if $t = 1$, and $1 - s/t \to 1$ if $t \to \infty$.*

Proof. It is equivalent to show that

$$\lim_{n\to\infty} b(n,tn) := \lim_{n\to\infty} \frac{B(n,tn)}{n^{tn}} = \frac{s}{t}.$$

¿From the proof of Theorem 2,

$$b(n,tn) = \sum_{i=1}^{n} \frac{BAD_i}{n^{tn}} = \sum_{i=1}^{n} \frac{(tn)_{i-1}}{n^{i-1}} \frac{i^{i-2}}{(i-1)!}(1-\frac{i}{n})^{tn+1-i}. \tag{1}$$

We approximate $(tn)_{i-1}/n^{i-1}$ by t^{i-1}, and $(1-\frac{i}{n})^{tn+1-i}$ by e^{-ti}, then the right-hand side of (1) becomes

$$\sum_{i=1}^{n} \frac{i^{i-2}}{(i-1)!} t^{i-1} e^{-ti}.$$

As $n \to \infty$, this summation goes to

$$\sum_{i=1}^{\infty} \frac{i^{i-2}}{(i-1)!} t^{i-1} e^{-ti} = \frac{1}{t} R(te^{-t})$$

where $R(x) = \sum_{i=1}^{\infty} \frac{i^{i-1}}{i!} x^i$ is the exponential generating function for labeled rooted trees, whose radius of convergence is $1/e$. It is well-known that $R(x)$ is the inverse function of $y = xe^{-x}$, (e.g., [7][Sect.5.3]). Hence

$$\frac{R(te^{-t})}{t} = \frac{s}{t},$$

where $se^{-s} = te^{-t}$ and $0 < s < 1$.

To show that s/t is the limit of $b(n, tn)$, it is sufficient to show that the error between $b(n, tn)$ and s/t goes to 0 as n goes to infinity. Fix λ such that $1 \ll \lambda \ll n^{1/2}$. Let $b_i = BAD_i/n^{tn}$. Then the error between $b(n, tn)$ and s/t is the sum of following three terms

$$\begin{aligned} I_1 &= b(n, tn) - \sum_{i=1}^{\lambda} b_i = \sum_{i>\lambda}^{n} b_i, \\ I_2 &= \sum_{i=1}^{\lambda} b_i - \sum_{i=1}^{\lambda} \frac{i^{i-2}}{(i-1)!} t^{i-1} e^{-ti}, \\ I_3 &= -\sum_{i=\lambda}^{\infty} \frac{i^{i-2}}{(i-1)!} t^{i-1} e^{-ti}. \end{aligned}$$

For I_1, since $i \le n$,

$$\frac{(tn)_{i-1}}{n^{i-1}} = t^{i-1}(1)(1 - \frac{1}{tn}) \ldots (1 - \frac{i-2}{tn}),$$

where

$$\log \prod_{j=1}^{i-2} (1 - \frac{j}{tn}) = \sum_{j=1}^{i-2} \log(1 - \frac{j}{tn}) = -\sum_{j=1}^{i-2} \left(\frac{j}{tn} + \frac{j^2}{2t^2n^2} + \ldots \right) < -\frac{i^2}{2tn}(1 + o(1)).$$

Similarly, since $t \ge 1$,

$$\begin{aligned} (1 - \frac{i}{n})^{tn+1-i} &= \exp\left(-(tn + 1 - i)(\frac{i}{n} + \frac{i^2}{2n^2} + \ldots) \right) \\ &< \exp\left(-ti - \frac{ti^2}{2n} + \frac{i^2}{2n} + \frac{i^3}{6n^2} + \frac{i^4}{12n^3} + \cdots \right) < \exp\left(-ti - \frac{ti^2}{2n} + \frac{i^2}{n} \right) \end{aligned}$$

Hence for $\lambda \le i \le n$,

$$b_i \le \frac{i^{i-2}}{(i-1)!} t^{i-1} \exp\left(-ti - \frac{i^2}{2n}(t + \frac{1}{t} - 2 + o(1))\right) < \frac{i^{i-2}}{(i-1)!} t^{i-1} e^{-ti}.$$

By Stirling's formula,

$$I_1 \le \frac{1}{t} \sum_{i=\lambda}^{\infty} \frac{(te^{1-t})^i}{i\sqrt{2\pi i}} < C \frac{(te^{1-t})^\lambda}{\sqrt{\lambda}} = o(1),$$

for some constant $C > 0$.

For I_2, from the above calculation, we note that for $1 \le i \le \lambda$,

$$\frac{(tn)_{i-1}}{n^{i-1}} = t^{i-1} \exp\left(-\frac{i^2}{2tn}(1 + o(1))\right),$$

and

$$\exp\left(-ti - \frac{ti^2}{n}(1 + o(1))\right) = (1 - \frac{i}{n})^{tn} \le (1 - \frac{i}{n})^{tn+1-i} \le \exp\left(-ti - \frac{ti^2}{n} + \frac{i^2}{n}\right).$$

Hence for $1 \le i \le \lambda$,

$$b_i \ge \frac{i^{i-2}}{(i-1)!} t^{i-1} \exp\left(-ti - \frac{i^2}{2n}(t + \frac{1}{t})\right)$$

and

$$b_i \le \frac{i^{i-2}}{(i-1)!} t^{i-1} \exp\left(-ti - \frac{i^2}{2n}(t + \frac{1}{t} - 2)\right) < \frac{i^{i-2}}{(i-1)!} t^{i-1} \exp(-ti).$$

Let $k = t + \frac{1}{t}$. Note that

$$\frac{i^2}{2n} k = i(\frac{ik}{2n}) < i\epsilon,$$

where $\epsilon = \lambda k / 2n = o(n^{-1/2})$, we have

$$|I_2| \le \sum_{i=1}^{\infty} \frac{i^{i-2}}{(i-1)!} t^{i-1} e^{-ti} - \sum_{i=1}^{\infty} \frac{i^{i-2}}{(i-1)!} t^{i-1} e^{-i(t+\epsilon)} \le \frac{|s_1 - s_2|}{t}$$

Here s_1, s_2 are real numbers in $(0,1)$ satisfying $s_1 e^{-s_1} = te^{-t}$ and $s_2 e^{-s_2} = te^{-t-\epsilon}$. Since $y = xe^{-x}$ is a continuous function, $|s_1 - s_2| < C'\epsilon$ for some constant C'. This proves $|I_2| = o(1)$.

The term I_3 can be treated similarly as I_1.

$$|I_3| = \sum_{i=\lambda}^{\infty} \frac{i^{i-2}}{(i-1)!} t^{i-1} e^{-ti} \sim \frac{1}{t} \sum_{i=\lambda}^{\infty} \frac{(te^{1-t})^i}{i\sqrt{2\pi i}} < C \frac{(te^{1-t})^\lambda}{\sqrt{\lambda}} = o(1).$$

This finishes the proof. □

3 The asymptotic formula for the good allocations

¿From Theorem 3, we see that $\lim_{n\to} f(n,tn)$ approaches 0 as t goes to 1^+. More precisely, if $t = 1+\epsilon$, then $1 - s/t = 2\epsilon + o(\epsilon)$ as $\epsilon \to 0^+$. In this section we discuss the asymptotics of $f(n,tn)$ for $t \to 1^+$.

Theorem 4 *Assume $r = r(n)$ is a function of n such that $\sqrt{n} \ll r(n) \ll n$, and $m = n + r(n)$. Then the good probability $f(n,m)$ is asymptotically $2r/n$.*

Proof. Again, we use the formula $b(n,m)$ and prove

$$b(n,m) = \sum_{i=1}^{n} \binom{m}{i-1} i^{i-2}(n-i)^{m+1-i}/n^m = 1 - \frac{2r}{n} + o(\frac{r}{n}).$$

As before, let

$$b_i = \frac{i^{i-2}}{(i-1)!}\frac{(m)_{i-1}}{n^{i-1}}(1-\frac{i}{n})^{m+1-i}.$$

Then

$$\begin{aligned}
(1-\frac{i}{n})^{m+1-i} &= \exp\left(-(m+1-i)(\frac{i}{n} + \frac{i^2}{2n^2} + \frac{i^3}{3n^3} + \dots)\right) \\
&= \exp\left(-i - \frac{i^2}{2n} - \frac{i^3}{3n^2} - \dots \right. \\
&\qquad \left. - \frac{(r+1-i)i}{n} - \frac{(r+1-i)i^2}{2n^2} - \frac{(r+1-i)i^3}{3n^3} - \dots\right) \\
&= \exp\left(-\frac{ri}{n} - \frac{ri^2}{2n^2} - \frac{ri^3}{3n^3} - \frac{ri^4}{4n^4} - \cdots \right. \\
&\qquad \left. -i + \frac{i^2}{2n} + \frac{i^3}{6n^2} + \frac{i^4}{12n^3} + \frac{i^5}{20n^4} + \cdots + O(\frac{i}{n})\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{(m)_{i-1}}{n^{i-1}} &= (1+\frac{r}{n})(1+\frac{r-1}{n})\dots(1+\frac{r-i+2}{n}) \\
&= \exp\left(\sum_{j=0}^{i-2}(\frac{r-j}{n} - \frac{(r-j)^2}{2n^2} + \frac{(r-j)^3}{3n^3} - \dots)\right) \\
&= \exp\left(\frac{(i-1)r - i^2/2}{n} - \frac{(i-1)r^2 - ri^2 + i^3/3}{2n^2} + \frac{r^3(i-1) - 3r^2\cdot i^2/2 + ri^3 - i^4/4}{3n^3} - \dots + O(\frac{i}{n})\right) \\
&= \exp\left(\frac{(i-1)r}{n} - \frac{(i-1)r^2}{2n^2} + \frac{(i-1)r^3}{3n^3} + \cdots \right. \\
&\qquad \left. - \frac{i^2}{2n} - \frac{i^3}{6n^2} - \frac{i^4}{12n^3} - \frac{i^5}{20n^4} - \cdots + \frac{ri^2}{2n^2} + \frac{ri^3}{3n^3} + \frac{ri^4}{4n^4} + \cdots + O(\frac{i}{n} + \frac{r^2i^2}{n^3})\right).
\end{aligned}$$

Thus

$$\begin{aligned}\frac{(m)_{i-1}}{n^{i-1}}(1-\frac{i}{n})^{m+1-i} &= \exp\left(-i-\frac{r}{n}+\frac{r^2}{2n^2}-\frac{ir^2}{2n^2}+O(\frac{i}{n}+\frac{r^2i^2}{n^3}+\frac{ir^3}{n^3})\right) \\ &= \exp\left(-i-\frac{r}{n}+\frac{r^2}{2n^2}-\frac{ir^2}{2n^2}+o(\frac{ir^2}{n^2})\right)\end{aligned}$$

where the last equation holds for $i \ll n$.

Taking

$$\lambda = 20\left(\frac{n^2}{r^2}\right)\ln\left(\frac{r^2}{n}\right). \tag{2}$$

Then $\lambda \ll n$. We have

$$\sum_{i=1}^{\lambda} b_i \sim \sum_{i=1}^{\lambda} \frac{i^{i-2}}{(i-1)!}\exp\left(-\frac{r}{n}+\frac{r^2}{2n^2}-i(1+\frac{r^2}{2n^2})\right)$$

Fix n and r, the sum in the right hand side goes to

$$e^{-r/n+r^2/2n^2}R(e^{-(1+\frac{r^2}{2n^2})}) = e^{-r/n+r^2/2n^2}\cdot s$$

where $s \in (0,1)$ satisfying $se^{-s} = e^{-(1+\frac{r^2}{2n^2})}$. Assume $s = 1-\delta$. It is easy to compute that $\delta = \frac{r}{n} + o(\frac{r}{n})$. Thus

$$e^{-r/n+r^2/2n^2}\cdot s = (1-\frac{r}{n}+o(\frac{r}{n}))^2 = 1-\frac{2r}{n}+o(\frac{r}{n}).$$

To prove that $f(n,r) \sim 2r/n$, it is sufficient to check that the error is $o(r/n)$, where the error is the sum of three terms J_1, J_2 and J_3.

$$\begin{aligned} J_1 &= \exp(-\frac{r}{n}+\frac{r^2}{2n^2})\sum_{i=\lambda}^{\infty}\frac{i^{i-2}}{(i-1)!}e^{-(1+\frac{r^2}{2n^2})i} \\ J_2 &= \sum_{i=1}^{\lambda}\left(\exp(-\frac{r}{n}+\frac{r^2}{2n^2})\frac{i^{i-2}}{(i-1)!}e^{-(1+\frac{r^2}{2n^2})i}-b_i\right) \\ J_3 &= -\sum_{i=\lambda}^{n} b_i \end{aligned}$$

Let $a = r^2/2n^2$.

1. Error J_1. By the Stirling formula, we have

$$\sum_{i=\lambda}^{\infty}\frac{i^{i-2}}{(i-1)!}e^{-(1+a)i} \sim \sum_{i=\lambda}^{\infty}\frac{e^{-ia}}{i\sqrt{2\pi i}} \le C\frac{e^{-a\lambda}}{\sqrt{\lambda}},$$

where the constant $C = 2/\sqrt{2\pi}$. Since $\lambda \gg n^2/r^2$, J_1 is $o(r/n)$.

2. From the computation at the beginning of the proof,

$$\sum_{i=1}^{\lambda} b_i < e^{-r/n+r^2/2n^2} \sum_{i=1}^{\lambda} \frac{i^{i-2}}{(i-1)!} e^{-(1+a(1-\epsilon))i},$$

$$\sum_{i=1}^{\lambda} g_i > e^{-r/n+r^2/2n^2} \sum_{i=1}^{\lambda} \frac{i^{i-2}}{(i-1)!} e^{-(1+a(1+\epsilon))i}$$

where $\epsilon = o(1)$. Thus

$$|J_2| < \sum_{i=1}^{\infty} \frac{i^{i-2}}{(i-1)!} e^{-(1+a(1-\epsilon))i} - \sum_{i=1}^{\infty} \frac{i^{i-2}}{(i-1)!} e^{-(1+a(1+\epsilon))i} = s_1 - s_2.$$

where $s_1 \sim 1 - \sqrt{1-\epsilon} r/n$, and $s_2 \sim 1 - \sqrt{1+\epsilon} r/n$. Therefore

$$|J_2| \le (\sqrt{1+\epsilon} - \sqrt{1-\epsilon}) \frac{r}{n} \sim \epsilon \cdot \frac{r}{n} = o(\frac{r}{n}).$$

3. We rewrite $|J_3|$ as

$$\sum_{i=\lambda}^{n} b_i = \sum_{j=1}^{n-\lambda} b_{n-j}.$$

where b_{n-j} can be expressed as

$$b_{n-j} = h_j = \frac{(m)_r}{n^r} \frac{(n)_{j+1}}{n^{j+1}} (1 - \frac{j}{n})^{n-j} \frac{n}{(n-j)^2} \frac{j^{r+j+1}}{(r+j+1)!}. \tag{3}$$

Analysis:

$$\begin{aligned} \frac{(m)_r}{n^r} &= \frac{m}{n} \frac{m-1}{n} \cdots \frac{n+1}{n} \\ &= (1+\frac{1}{n})(1+\frac{2}{n}) \ldots (1+\frac{r}{n}) \\ &= \exp\left(\sum_{i=1}^{r} (\frac{i}{n} - \frac{i^2}{2n^2} + \frac{i^3}{3n^3} - \ldots) \right) \\ &= \exp\left(\frac{r^2}{2n} - \frac{r^3}{6n^2} + \frac{r^4}{12n^3} - \frac{r^5}{20n^4} + \cdots + O(\frac{r}{n}) \right) \end{aligned} \tag{4}$$

Similarly,

$$\begin{aligned} \frac{(n)_{j+1}}{n^{j+1}} &= (1-\frac{1}{n})(1-\frac{2}{n}) \cdots (1-\frac{j}{n}) \\ &= \exp\left(-\sum_{i=1}^{j} (\frac{i}{n} + \frac{i^2}{2n^2} + \frac{i^3}{3n^3} + \ldots) \right) \\ &= \exp\left(-\frac{j^2}{2n} - \frac{j^3}{6n^2} - \frac{j^4}{12n^3} - \ldots + O(\frac{j}{n}) \right) \end{aligned} \tag{5}$$

and

$$(1-\frac{j}{n})^{n-j} = \exp\left(-j+\frac{j^2}{2n}+\frac{j^3}{6n^2}+\frac{j^4}{12n^3}+\dots\right). \qquad (6)$$

Thus

$$\frac{(n)_{j+1}}{n^{j+1}}(1-\frac{j}{n})^{n-j} \sim \exp\left(-j+O(\frac{j}{n})\right) < Ce^{-j},$$

for a constant $C>0$.

We estimate $|J_3|$ in three intervals.

Interval 1. $\mathbf{1 \le j \le 0.1r}$.

In this interval,

$$h_j < \frac{(m)_r}{n^r}\frac{2}{n}\frac{j^{r+j+1}}{(r+j+1)!} < \frac{2}{n}\exp(\frac{r^2}{2n})(\frac{ej}{r+j+1})^{r+j+1},$$

where $ej/(r+j+1)<1/2$. Thus

$$\sum_{j=1}^{0.1r} h_j < \frac{2}{n}(\frac{1}{2})^r\exp(\frac{r^2}{2n})\sum_{j=1}^{0.1r}(\frac{1}{2})^{j+1} < \frac{2}{n}(\frac{e^{r/2n}}{2})^r = o(\frac{1}{n}) = o(r/n).$$

Interval 2. $\mathbf{0.1r \le j \le 100r}$.

Let $j=tr$, where $0.1\le t\le 100$. Then

$$h_j \sim \frac{1}{n}e^{-tr+r^2/2n}\frac{1}{\sqrt{2\pi r(1+t)}}(\frac{et}{1+t})^{r(1+t)} = \frac{1}{n\sqrt{2\pi r(1+t)}}\exp(r+\frac{r^2}{2n})(1-\frac{1}{1+t})^{(1+t)r}.$$

For $0.1\le t\le 100$, we have

$$(1-\frac{1}{1+t})^{(1+t)r} < \exp(-r-\frac{r}{2(1+t)}),$$

Thus

$$\sum_{\substack{j=tr\\0.1\le t\le 100}} h_j \le \sum_{\substack{j=tr\\0.1\le t\le 100}} \frac{C}{n\sqrt{r}}\exp(\frac{r^2}{2n}-\frac{r}{2(1+t)})$$
$$\sim \frac{Cr}{n\sqrt{r}}\exp(\frac{r^2}{2n})\int_{0.1}^{100} e^{-r/2(1+t)}dt \le \frac{r}{n}\frac{100C}{\sqrt{r}}\exp(\frac{r^2}{2n}-\frac{r}{202}).$$

Since the term $r(r/2n-1/202)\to-\infty$, the last term in the above is $o(r/n)$.

Interval 3. $\mathbf{100r \le j \le n-\lambda}$.

In this interval we have

$$\frac{j^{r+j+1}}{(r+j+1)!} < \frac{e^{r+j+1}}{\sqrt{2\pi(r+j+1)}}(1+\frac{r+1}{j})^{-(r+j+1)},$$

where

$$(1+\frac{r+1}{j})^{-(r+j+1)} = \exp\left(-r-\frac{r^2}{2j}+\frac{r^3}{6j^2}-\frac{r^4}{12j^3}+\ldots+O(\frac{r}{j})\right).$$

Since $j \geq 100r$,

$$h_j < \frac{n}{(n-j)^2}\frac{C_1}{\sqrt{r+j+1}}\exp\left(\frac{r^2}{2n}-\frac{r^3}{6n^2}+\frac{r^4}{12n^3}-\cdots-\frac{r^2}{2j}+\frac{r^3}{6j^2}-\frac{r^4}{12j^3}+\cdots\right).$$

for some constant $C_1 > 0$.

For $100r \leq j < n$, $k \geq 3$,

$$\begin{aligned}\frac{r^{k+1}}{k(k+1)}\left|\frac{1}{j^k}-\frac{1}{n^k}\right| &= \frac{r^2(n-j)}{nj}\frac{r^{k-1}}{(j)^{k-1}}\frac{1}{k(k+1)}\frac{(n^{k-1}+n^{k-2}j+\cdots+j^{k-1})}{n^{k-1}} \\ &< \frac{r^2(n-j)}{nj}(\frac{1}{100})^{k-1}.\end{aligned}$$

Thus

$$\frac{r^2}{2n}-\frac{r^3}{6n^2}+\frac{r^4}{12n^3}-\cdots-\frac{r^2}{2j}+\frac{r^3}{6j^2}-\frac{r^4}{12j^3}+\cdots < \frac{r^2}{2n}-\frac{r^2}{2j}+\frac{1}{99}(\frac{r^2}{j}-\frac{r^2}{n}) < -\frac{1}{4}(\frac{r^2}{j}-\frac{r^2}{n}).$$

Therefore

$$h_j < \frac{n}{(n-j)^2}\frac{C}{\sqrt{r+j+1}}\exp(\frac{r^2}{4n}-\frac{r^2}{4j}),$$

and

$$\sum_{j=1}^{n-\lambda} h_j < \sum_{j=1}^{n-\lambda}\frac{Cn}{(n-j)^2}\frac{\exp\left(\frac{r^2}{4n}-\frac{r^2}{4j}\right)}{\sqrt{j}}.$$

Let $j = tn$, where t ranges from 0 to $1-\lambda/n$, then the above sum is asymptotically

$$\frac{C}{\sqrt{n}}\exp(-\frac{r^2}{4n})\int_0^{1-\frac{\lambda}{n}}\frac{1}{(1-t)^2}\frac{\exp(-\frac{r^2}{4nt})}{\sqrt{t}}dt. \tag{7}$$

Let $h(t) = \exp(-\frac{r^2}{4nt})/\sqrt{t}$. Substitute by $x = 1/\sqrt{t}$, we have

$$h(t) = k(x) := x \cdot \exp(-\frac{r^2}{4n}x^2).$$

$$k'(x) = (1-\frac{x^2r^2}{2n})\exp(-\frac{r^2}{4n}x^2).$$

The maximal of $k(x)$ is achieved at $x = \sqrt{2n}/r \ll 1$, that is, at $t \gg 1$. It implies that $h(t)$ is an increasing function in the interval $(0,1)$. Therefore the integral

$$\begin{aligned}&\frac{C}{\sqrt{n}}\exp(\frac{r^2}{4n})\int_0^{1-\frac{\lambda}{n}}\frac{1}{(1-t)^2}\frac{\exp(-\frac{r^2}{4nt})}{\sqrt{t}}dt \\ \leq\ &\frac{C}{\sqrt{n}}\exp(\frac{r^2}{4n})(h(t)|_{t=1-\lambda/n})\int_0^{1-\lambda/n}\frac{1}{(1-t)^2}dt \\ \sim\ &C'\frac{\sqrt{n}}{\lambda}e^{-\frac{r^2\lambda}{4n^2}},\end{aligned}$$

for some constant $C' > 0$.

Substituting by $\lambda = 20\frac{n^2}{r^2}\ln\frac{r^2}{n}$, we obtain the bound

$$\frac{C'\sqrt{n}r^2}{20n^2\ln(\frac{r^2}{n})e^{5\ln(\frac{r^2}{n})}} = \frac{Cr}{20n}(\frac{\sqrt{n}}{r})^9\frac{1}{\ln(\frac{r^2}{n})} = o(\frac{r}{n}).$$

This finishes the proof. □

4 Branching processes with Poisson offspring distribution

The particle allocations may be approximated by a branching process. A branching process is a mathematical description of the growth of a population for which each individual produces offsprings according to stochastic laws [2]. Imagine that we are in a unisexual universe and we start with a single organism. Imagine that this organism has a number of children given by a given random variable Z. These children then themselves have children, the number again being determined by Z. These grandchildren then have children, etc. As $Z = 0$ will have nonzero probability, there will be some chance that the line dies out entirely. We want to study the total number of organisms in this process, with particular eye to whether or not the process continues forever. For mathematical formulation, we use an equivalent description in terms of queueing theory, which is due to Kendall [3]. Let $Z_1, Z_2, \ldots$ be independent random variables, each with distribution Z. Define a random walk $\mathcal{Y} = Y_0, Y_1, \ldots$ by the recursion

$$\begin{aligned} Y_0 &= 1, \\ Y_i &= Y_{i-1} + Z_i - 1, \end{aligned} \tag{8}$$

and let T be the least t for which $Y_t = 0$. If no such t exists, we say that $T = +\infty$, in which case we also say the random walk $\mathcal{Y}$ escapes. The Y_i and Z_i mirror the branching process as follows. We view all organisms as living or dead. Initially there is one live organism and no dead ones. At each time unit we select one of the live organisms, it has Z_i children, and then it dies. The number Y_i of live organisms at time i is then given by the recursion. The process stops when $Y_t = 0$. T represents the total number of of organisms, including the original, in this process.

Let $c = E[Z]$. A basic result about branching processes (e.g., [2]) states that if $c \leq 1$ the process dies out with probability one. If, however, $c > 1$, then there is a nontrivial probability $\alpha \in (0, 1)$ that the process goes on forever, where α can be computed as $1 - x$, and x is the unique fixed point in $(0, 1)$ of the probability generating function $P(x) = \sum_{i=0}^{\infty} x^i \Pr[Z = i]$.

The relation between the particle allocation problem and the branching processes can be described as follows. In the particle allocation, if $m = n + r$ particles are allocated in n boxes independently and uniformly, and x_i is the number of particles in the i-th box, then x_i has the binomial distribution $bin(m, \frac{1}{n})$, which can be approximated by the Poisson distribution $Po(\frac{m}{n}) = Po(t)$ in the case $m = tn$. The condition that $x_1 + x_2 + \cdots + x_i \geq i$ for all i then corresponds to the condition that $Y_i > 0$ for all i. Thus the probability for a random allocation to be good is roughly the probability of escape of a random walk (8) with $Z = Po(t)$. Precisely, we have

Theorem 5 *If $m = tn$ for a fixed $t > 1$, then the probability of good allocation $f(n, m)$ converges to the escaping probability* $\Pr[T = +\infty]$ *of the random walk $\mathcal{Y}$ defined by* (8) *with $Z = Po(t)$.*

Proof. In Theorem 3 we showed that the limit of $f(n, tn)$ is $1 - \frac{s}{t}$. For $Z = Po(t)$ the probability generating function of Z is $P(x) = e^{-t(1-x)}$. It is easy to check that the unique fixed point of $P(x)$ in $(0, 1)$ is s/t, where $s < 1$ and $se^{-s} = te^{-t}$. The theorem then follows from the basic result of branching processes. □

The proof of the basic result of branching processes, which uses the probability generating functions, appears in many textbooks on probability theory, (e.g., [2]). However, here we want to outline another proof with $Z = Po(t)$. This proof not only yields the escaping probability, but also gives the estimation of $\Pr[T \geq N]$, which will used in the next section.

First, we need a lemma on the Chernoff bound of Poisson distributions, which can be found, e.g., in [1][Appen.A.1.15].

Lemma 6 *Let P have Poisson distribution with mean μ. For $\epsilon > 0$,*

$$\begin{aligned}\Pr[P < \mu(1-\epsilon)] &\leq e^{-\epsilon^2 \mu/2}, \\ \Pr[P > \mu(1+\epsilon)] &\leq \left[e^{\epsilon}(1+\epsilon)^{-(1+\epsilon)}\right]^{\mu}.\end{aligned}$$

Another proof of Theorem 5. We just need to show that the probability of extinction for $\mathcal{Y}$ with $Z = Po(t)$ is s/t. Set a new random walk $\mathcal{R} = R_0, R_1, \ldots,$ as

$$\begin{aligned}R_0 &= 1, \\ R_i &= \begin{cases} R_{i-1} + Z_i - 1, & \text{if } R_{i-1} > 0, \\ 0, & \text{if } R_{i-1} = 0. \end{cases}\end{aligned}$$

Note that $T < +\infty$ for $\mathcal{Y}$ if and only if $R_i = 0$ for some i.

First, find $\lambda > 0$ with $E[e^{-\lambda(Z-1)}] = 1$. It is easily computed that λ is given implicitly by the formula

$$t = \frac{\lambda}{1 - e^{-\lambda}}. \tag{9}$$

Claim 1. $E[e^{-\lambda R_n}] = e^{-\lambda}$ for all n.

We prove by induction on n. For $n = 0$ it is immediate. Assume it holds for $n-1$, i.e., $E[e^{-\lambda R_{n-1}}] = e^{-\lambda}$. We prove it for n. Note that

$$E[e^{-\lambda R_n}] = \Pr[R_{n-1} = 0] + \sum_{i>0} \Pr[R_{n-1} = i] E[e^{-\lambda(i+(Z_n-1))}]$$

since $R_n = 0$ if $R_{n-1} = 0$, and $R_n = i + (Z_n - 1)$ if $R_{n-1} = i > 0$. But

$$E[e^{-\lambda(i+(Z_n-1))}] = e^{-\lambda i} E[e^{-\lambda(Z_n-1)}] = e^{-\lambda i}$$

so

$$E[e^{-\lambda R_n}] = \Pr[R_{n-1} = 0] + \sum_{i>0} \Pr[R_{n-1} = i] e^{-\lambda i} = E[e^{-\lambda R_{n-1}}] = e^{-\lambda}$$

as desired. This proves Claim 1.

Now we have

$$e^{-\lambda} = \Pr[R_n = 0] + \sum_{i>0} \Pr[R_n = i] e^{-\lambda i}. \tag{10}$$

Claim 2.

$$\lim_{n\to\infty} \sum_{i>0} \Pr[R_n = i] e^{-\lambda i} = 0.$$

It suffices to show

$$\lim_{n\to\infty} \sum_{i>0} \Pr[Y_n = i] e^{-\lambda i} = 0$$

as this is bigger. Note $E[Y_n] = 1 + (t-1)n$. We split the sum into intervals $0 < i \le (t-1)n/2$ and $i > (t-1)n/2$. When $i > (t-1)n/2$ we have $e^{-\lambda i} \le e^{-\lambda(t-1)n/2}$, so

$$\sum_{i>(t-1)n/2} \Pr[Y_n = i] e^{-\lambda i} \le e^{-\lambda(t-1)n/2} = o(1).$$

When $0 < i \le (t-1)n/2$ we bound $e^{-\lambda i}$ by 1, so

$$\sum_{0<i\le(t-1)n/2} \Pr[Y_n = i] e^{-\lambda i} \le \sum_{0<i\le(t-1)n/2} \Pr[Y_n = i] \le \Pr[Y_n \le (t-1)n/2].$$

Here Y_n is $1 - n + Po(tn)$. By Lemma 6 $\Pr[Y_n \leq (t-1)n/2] \leq e^{-\frac{(t-1)^2}{8t}n}$, which goes to zero as $n \to \infty$.

As $n \to \infty$, $\Pr(R_n = 0) \to \Pr(T < +\infty)$. Therefore $\Pr(T < +\infty) = e^{-\lambda}$. Finally we check that it is the same solution as s/t: Let $y = te^{-\lambda}$. Then (9) implies $y = t - \lambda$. So $ye^{-y} = te^{-\lambda}e^{\lambda - t} = te^{-t}$. It verifies $y = s$. □

REMARK.

1. One generalization of the above proof is, suppose instead of starting with $Y_0 = 1$ we started with $Y_0 = a$. Then we would still have $E[e^{-\lambda R_t}] = E[e^{-\lambda R_{t-1}}]$ for all t. In this case $E[e^{-\lambda R_t}] = e^{-a\lambda}$.

2. This method applies to the random walk $\mathcal{Y}$ with any offspring distribution Z provided that $E[Z] > 1$ and $var(Z)$ exists. The first condition is necessary to guarantee that

$$E[e^{-\lambda(Z-1)}] = 1$$

has a positive solution λ. The second is a sufficient condition for $\lim_{n\to\infty} \sum_{i>0} \Pr[R_n = i]e^{-\lambda i} = 0$, (e.g., by Chebyshev's inequality). Under these assumption, we always get

$$\Pr[T < +\infty] = e^{-\lambda}.$$

5 Asymptotics of Branching processes

In the previous section, we see that the particle allocation can be approximated by a branching process with Poisson offspring distribution. It is natural to extend such an approximation to the asymptotics of $\Pr[T > N]$ for branching processes. Explicitly, we have the following theorem, which can be viewed as the counterpart of Theorem (4).

Theorem 7 *Consider the random walk* $\mathcal{Y}^{(\epsilon)} = Y_0^{(\epsilon)}, Y_1^{(\epsilon)}, \ldots$ *defined by the recursion*

$$Y_0^{(\epsilon)} = 1, \qquad Y_i^{(\epsilon)} = Y_{i-1}^{(\epsilon)} + Z_i^{(\epsilon)} - 1,$$

where $Z_1^{(\epsilon)}, Z_2^{(\epsilon)}, \ldots$ *are independent random variables each with distribution* $Z^{(\epsilon)} \sim Po(1+\epsilon)$, $(\epsilon > 0)$. *Let* $T^{(\epsilon)}$ *be the least* t *for which* $Y_i^{(\epsilon)} = 0$. *Then*

$$Pr(T^{(\epsilon)} > N) \sim 2\epsilon \qquad \text{for } N \gg \epsilon^{-2}.$$

Denote by Δ_N the probability of $T^{(\epsilon)} > N$, and by $\Delta_{N,i}$ the conditional probability $\Pr(T^{(\epsilon)} > N | Y_N^{(\epsilon)} = i)$.

Lemma 8 *Fix N, $\Delta_{N,i}$ is a non-decreasing function in i.*

Proof. We prove by induction on N. Let $p_i = \Pr[Z = i]$. For $N = 1$, $\Delta_{1,0} = 0 < 1 = \Delta_{1,1} = \Delta_{1,2} = \cdots = \Delta_{1,k} = \cdots$. The lemma is true. Assume the lemma holds for $N = n-1$. For $N = n$,

$$\begin{aligned}
\Delta_{n,i} &= \Pr[T^{(\epsilon)} > n | Y_n^{(\epsilon)} = i) \\
&= \sum_{j=1}^{\infty} \Pr[T^{(\epsilon)} > n-1 | Y_{n-1}^{(\epsilon)} = j) \Pr[Z_i = i - j + 1] \\
&= \sum_{j=1}^{i+1} \Delta_{n-1,j} \Pr[Z_i = i + 1 - j] \\
&= \sum_{j=1}^{i+1} \Delta_{n-1,j} p_{i+1-j}.
\end{aligned}$$

Similarly, we have

$$\Delta_{n,i+1} = \sum_{j=1}^{i+2} \Delta_{n-1,j} \cdot p_{i+2-j}.$$

Thus

$$\Delta_{n,i+1} - \Delta_{n,i} = \left(\sum_{j=0}^{i} p_j (\Delta_{n-1,i+2-j} - \Delta_{n-1,i+1-j}) \right) + p_{i+1}\Delta_{n-1,1} \geq 0$$

as desired. □

Proof of Theorem 7. It is clear that

$$\Delta_N = \sum_{i>0} \Pr(Y_N^{(\epsilon)} = i)\Delta_{N,i}.$$

As in the proof of Theorem 5, taking λ so that

$$E[e^{-\lambda(Z-1)}] = 1,$$

then

$$e^{-\lambda} = 1 - \Delta_N + \sum_{i>0} \Pr[Y_N^{(\epsilon)} = i]\Delta_{N,i} e^{-i\lambda}.$$

Note that $\lambda = 2\epsilon + o(\epsilon)$ as $\epsilon \to 0^+$.

We show that if $N \gg \epsilon^{-2}$, the sum $\sum_{i>0} \Pr[Y_N^{(\epsilon)} = i]\Delta_{N,i} e^{-i\lambda}$ is $o(\Delta)$. Let $K = N\epsilon^2$. We have

$$\sum_{i>\epsilon N/2} \Pr[Y_N^{(\epsilon)} = i]\Delta_{N,i} e^{-i\lambda} \leq e^{-\frac{\lambda\epsilon k}{2}} \sum_{i>\epsilon N/2} \Pr[Y_n^{(\epsilon)} = i]\Delta_{N,i} \leq e^{-K}\Delta = o(\Delta).$$

For the other half of the summation, note that $Y_N^{(\epsilon)}$ has distribution $1 - N + Po(N(1+\epsilon))$. Again applying Lemma 6, we have $\Pr[Y_N^{(\epsilon)} \leq \epsilon N/2] \leq e^{-N\epsilon^2/8} = e^{-K/8}$. Thus

$$\sum_{i \leq \epsilon N/2} \Pr[Y_N^{(\epsilon)} = i]\Delta_{N,i} e^{-i\lambda} \leq \sum_{i \leq \epsilon N/2} \Pr[Y_N^{(\epsilon)} = i]\Delta_{N,i} \leq \Delta_{N,\epsilon N/2} \Pr[Y_N^{(\epsilon)} \leq \epsilon N/2] \leq e^{-K/8}\Delta_{N,\epsilon N/2}.$$

As $\Delta_{N,i}$ increases along i,

$$\Delta_N \geq \sum_{i \geq \epsilon N/2} \Pr[Y^{(\epsilon)} = i]\Delta_{N,i} \geq \Delta_{N,\epsilon N/2} \Pr[Y_N^{(\epsilon)} \geq \epsilon N/2] \geq \Delta_{N,\epsilon N/2}(1 - e^{-K/8}) = \Delta_{N,\epsilon N/s}(1 - o(1)).$$

Combining the above inequalities,

$$\sum_{i \leq \epsilon N/2} \Pr[Y_N^{(\epsilon)} = i]\Delta_{N,i} e^{-i\lambda} \leq e^{-K/8}\Delta(1 + o(1)) = o(\Delta_N).$$

It follows

$$e^{-\lambda} = 1 - \Delta_N + o(\Delta_N),$$

and hence $\Delta_N \sim 1 - e^{-\lambda} \sim 2\epsilon$. □

Remark. Theorem 7 implies that if the random walk $\mathcal{Y}$ survives up to time N, where $N \gg \epsilon^{-2}$, then almost surely it will survive for ever. In other words,

$$Pr(T = \infty | T \geq N) \sim 1 \qquad \text{for } N \gg \epsilon^{-2}. \tag{11}$$

The above proof is a probabilistic argument. On the other hand, for the branching process with a Poisson offspring distribution, we can compute the exact value of $\Pr[T = k]$ by enumerative techniques.

Lemma 9 *For the random walk $\mathcal{Y}^{(0)}$ with the offspring distribution $Po(1)$, let $T^{(0)}$ be the least t for which $Y_i^{(0)} = 0$. Then*

$$\Pr[T^{(0)} = k] = \frac{k^{k-1}}{k!} e^{-k}.$$

Proof. Take any choice of $Z_1, Z_2 \ldots, Z_k$ that gives $T^{(0)} = k$, we have

$$\Pr[Z_1 = z_1, Z_2 = z_2, \ldots, Z_k = z_k] = \prod_{i=1}^{k} \left(\frac{e^{-1}}{z_i!}\right) = e^{-k} \prod_{i=1}^{k} \frac{1}{z_i!}.$$

Therefore

$$\Pr[T^{(0)} = k] = e^{-k} \sum_{z_1 + \ldots z_k = k-1} \left(\prod_{i=1}^{k} \frac{1}{z_i!}\right) = \frac{e^{-k}}{(k-1)!} \sum_{z_1 + \ldots z_k = k-1} \binom{k-1}{z_1, \ldots, z_k},$$

where the $z_1, z_2, \ldots, z_k$ are non-negative integers such that $z_1 + \cdots + z_i \geq i$ for $i = 1, 2, \ldots, k-1$. Note that this sum enumerates the number of trees on the set $\{1, 2, \ldots, k\}$: Given any labeled tree on $\{1, 2, \ldots, k\}$, fix 1 to be the root. Then $z_1, z_2, \ldots, z_k$ is the degree sequence of this labeled rooted tree under the breadth-first search. (For details on breadth-first search, please see [5].) A well-known theorem of Cayley states that the number of such trees on a given k-set is k^{k-2}. Thus

$$\Pr[T^{(0)} = k] = \frac{k^{k-2}e^{-k}}{(k-1)!} = \frac{k^{k-1}}{k!}e^{-k}.$$

□

Corollary 10 *For the random walk $\mathcal{Y}^{(0)}$,*

$$\Pr(T^{(0)} > N) \sim \frac{2}{\sqrt{2\pi}} N^{-1/2}.$$

Proof.

$$\Pr(T^{(0)} > N) = \sum_{k>N} \frac{k^{k-1}}{k!}e^{-k} \sim \frac{1}{\sqrt{2\pi}} \sum_{k>N} k^{-3/2} \sim \frac{2}{\sqrt{2\pi}} N^{-1/2}.$$

□

Lemma 11 *For the random walk $\mathcal{Y}^{(\epsilon)}$ with the offspring distribution $Po(1+\epsilon)$, let $T^{(\epsilon)}$ be the least t for which $Y_i^{(\epsilon)} = 0$. Then*

$$\Pr[T^{(\epsilon)} = k] = \frac{k^{k-1}}{k!} e^{-k-k\epsilon}(1+\epsilon)^{k-1}.$$

Proof. For any choice of $Z_1, Z_2, \ldots, Z_k$ that gives $T^{(\epsilon)} = k$,

$$\begin{aligned}
& \Pr[T^{(\epsilon)} | Z_1 = z_1, Z_2 = z_2, \ldots, Z_k = z_k] \\
= & \prod_{i=1}^{k} e^{-(1+\epsilon)} \frac{(1+\epsilon)^z_i}{z_i!} \\
= & e^{-(1+\epsilon)k}(1+\epsilon)^{k-1} \prod_{i=1}^{k} \frac{1}{z_i!} \\
= & e^{-k\epsilon}(1+\epsilon)^{k-1} \Pr(T_0 = k | Z_1 = z_1, Z_2 = z_2, \ldots, Z_k = z_k).
\end{aligned}$$

By Lemma 9,

$$\Pr[T^{(\epsilon)} = k] = e^{-k\epsilon}(1+\epsilon)^{k-1} \Pr[T^{(0)} = k] = \frac{k^{k-1}}{k!} e^{-k-k\epsilon}(1+\epsilon)^{k-1}.$$

□

Using Lemma 11, we can improve the result in Theorem 7.

Theorem 12 *For the random walk $\mathcal{Y}^{(\epsilon)}$ with the offspring distribution $Po(1+\epsilon)$, we have*

$$\Pr[T^{(\epsilon)} > N] \sim 2\epsilon \qquad \text{if and only if} \qquad N \gg \epsilon^{-2}.$$

Proof. The sufficient part is proved in Theorem 7. For the other direction, let $t = 1+\epsilon$. By the Stirling formula,

$$\begin{aligned}
\sum_{i=N+1}^{\infty} \Pr[T^{(\epsilon)} = i] &= \sum_{i=N+1}^{\infty} e^{-ti} t^{i-1} \frac{i^{i-1}}{i!} \\
&\sim \sum_{i=N+1}^{\infty} e^{-ti} t^{i-1} \frac{e^i}{i\sqrt{2\pi i}} \\
&= \frac{1}{t\sqrt{2\pi}} \sum_{i=N+1}^{\infty} \frac{(te^{-\epsilon})^i}{i^{3/2}}.
\end{aligned}$$

Let $\beta = te^{-\epsilon}$. Note that $\beta = (1+\epsilon)e^{-\epsilon} = 1 - \frac{\epsilon^2}{2} + o(\epsilon^2) < 1$. Then

$$\sum_{i=N+1}^{\infty} \frac{\beta^i}{i^{3/2}} < \beta^N \sum_{i=N+1}^{\infty} i^{-3/2} \sim 2\beta^N \cdot N^{-1/2},$$

and

$$\sum_{i=N+1}^{\infty} \frac{\beta^i}{i^{3/2}} > \sum_{i=N+1}^{2N} \frac{\beta^i}{i^{3/2}} > \beta^{2N} \sum_{i=N+1}^{2N} i^{-3/2} \sim 2\beta^{2N}(1 - \frac{1}{\sqrt{2}})N^{-1/2}.$$

If $N = c\epsilon^{-2}$ for some constant c, then

$$\beta^N \sim (1 - \frac{\epsilon^2}{2})^{c\epsilon^{-2}} \sim e^{-\frac{c}{2}},$$

Thus there exists constant $c_1, c_2 > 0$ such that

$$c_1\epsilon < \sum_{i=N+1}^{\infty} \frac{\beta^i}{i^{3/2}} < c_2\epsilon.$$

This proves for $\sum_{i>N} \Pr[T^{(\epsilon)} = i] = o(\epsilon)$, it is necessary that $N \gg \epsilon^{-2}$. □

Comparing Theorem 7 with Theorem 4, one notes that Theorem 5 actually suggests that $\Pr[T^{(\epsilon)} > N | Y_N^{(\epsilon)} = N\epsilon] \sim 2\epsilon$ for $N \gg \epsilon^{-2}$. We prove that this is indeed true in the branching process with Poisson offspring distributions.

Theorem 13 *In the random walk $\mathcal{Y}^{(\epsilon)}$, let Δ_N be the probability $\Pr[T^{(\epsilon)} > N]$, and $\Delta_{N,i}$ be the conditional probability $\Pr[T^{(\epsilon)} > N | Y_n^{(\epsilon)} = i]$. Then*

$$\Delta_{N,N\epsilon} = \Pr[T^{(\epsilon)} > N | Y_N^{(\epsilon)} = N\epsilon] \sim 2\epsilon \qquad \text{for } N \gg \epsilon^{-2}.$$

Proof. The key point of the proof is the observation that $\Delta_{N,i}$ depends only on N and i, but independent of the choice of ϵ. The reason is, for any choice of $Z_1, Z_2, \ldots, Z_N$ that leads to $T^{(\epsilon)} = N$, the conditional probability

$$\Pr[Z_1 = z_1, Z_2 = z_2, \ldots, Z_N = z_n | Y_N^{(\epsilon)} = i] = \frac{\prod_{j=1}^{N} e^{-t} \frac{t^{z_i}}{z_i!}}{e^{-Nt}(Nt)^{N+i-1}/(N+i-1)!} = \frac{(N+i-1)!}{N^{N+i-1} \prod z_i!},$$

which is independent of the choice of ϵ.

For the upper bound on $\Delta_{N,N\epsilon}$, from the proof of Theorem 7 we have

$$2\epsilon \sim e^{-\lambda} \quad \sim \quad \Delta_N,$$

$$\Delta_N = \sum_{i>0} \Pr[Y_n^{(\epsilon)} = i]\Delta_{N,i} \quad \geq \quad \Delta_{N,N\epsilon(1-\delta)} \Pr[Y_N^{(\epsilon)} \geq N\epsilon(1-\delta)].$$

for any small $\delta > 0$. Applying Lemma 6, we have

$$\Pr[Y_N^{(\epsilon)} \leq N\epsilon(1-\delta)] \leq e^{-\frac{\epsilon^2\delta^2 N}{2(1+\epsilon)}}.$$

Hence for any ϵ, δ, there is a small $\delta' > 0$ such that $\Delta_{N,N\epsilon(1-\delta)} \leq 2\epsilon(1+\delta')$ when $N \gg eps^{-2}$. Let $\epsilon_1 = \epsilon(1-\delta)$, then $\Delta_{N,N\epsilon_1} \leq 2\epsilon_1(1+\delta')/(1-\delta)$. Since δ, δ' can be arbitrarily small, we have $\Delta_{N,N\epsilon_1} \leq 2\epsilon_1(1+o(1))$ for $N \gg \epsilon_1^{-2}$.

The lower bound now uses the upper bound. We have already seen $2\epsilon \sim \sum_i \Delta_{N,i} \Pr[Y_N^{(\epsilon)} = i]$. We prove that for any $\delta > 0$, $\Delta_{N,N\epsilon} \geq (2-\delta)\epsilon$ for $N \gg \epsilon^{-2}$. The proof is by the method of contradiction. Assume $\Delta_{N,N\epsilon} \leq (2-\delta)\epsilon$ for some fixed δ, $0 < \delta < 2$. Take $\alpha > 1$ such that $\alpha(2-\delta) < 2$. Let $\epsilon = \alpha\epsilon_1$. Then $\Delta_{N\epsilon_1\alpha} = \Delta_{N\epsilon} \leq (2-\delta)\epsilon = (2-\delta)\alpha\epsilon_1$. In the random walk with parameter ϵ_1, we have

$$\sum_{i \leq N\epsilon_1\alpha} \Pr[Y_N^{(\epsilon_1)} = i]\Delta_{N,i} \leq \Delta_{N,N\epsilon_1\alpha} \Pr[Y_N^{(\epsilon_1)} \leq N\epsilon_1\alpha] \leq (2-\delta)\alpha\epsilon_1.$$

We continue estimating the contribution of $\Pr[Y_N^{(\epsilon_1)} = i]\Delta_{N,i}$ for larger i. For $\alpha N\epsilon_1 < i \leq 2N\epsilon_1$,

$$\sum_{i>N\alpha\epsilon_1}^{2N\epsilon_1} \Pr[Y_N = i]\Delta_{N,i} \leq \Delta_{N,2N\epsilon_1} \Pr[Y_N^{(\epsilon_1)} > N\epsilon_1\alpha] \leq 2 \cdot 2\epsilon_1 e^{-c_\alpha(1+\epsilon_1)N}.$$

where $c_\alpha = (1+\gamma)\ln(1+\gamma) - \gamma$ and $\gamma = (\alpha-1)\epsilon_1/(1+\epsilon_1)$. The last inequality in the above formula is by the Chernoff bound of Poisson distribution, (c.f. Lemma 6). In general, for $k \geq 2$,

$$\sum_{i>kN\epsilon_1}^{(k+1)N\epsilon_1} \Pr[Y_N^{(\epsilon_1)} = i]\Delta_{N,i} \leq 2(k+1)\epsilon_1 \Pr[Y_N^{(\epsilon_1)} \geq kN\epsilon_1] \leq 2(k+1)\epsilon_1 e^{-c_k(1+\epsilon_1)N},$$

where $c_k = (1+\gamma_k)\ln(1+\gamma_k) - \gamma_k$ and $\gamma_k = (k-1)\epsilon_1/(1+\epsilon_1)$.

Since

$$(1+\gamma)\ln(1+\gamma)-\gamma > \begin{cases} \gamma^2/4 & \text{if } \gamma < 1, \\ \gamma/4 & \text{if } \gamma \geq 1. \end{cases}$$

we have

$$e^{-c_k(1+\epsilon_1)N} \leq \max(e^{-\gamma_k^2(1+\epsilon_1)N/4}, e^{-\gamma_k(1+\epsilon_1)N/4}) = \max(e^{-\frac{(k-1)^2\epsilon_1^2}{4(1+\epsilon_1)}N}, e^{-\frac{(k-1)\epsilon_1}{4}N}).$$

Now it is easy to check that the integrals

$$\int_{\alpha}^{\infty}(k+1)e^{-\frac{N\epsilon_1^2}{4(1+\epsilon_1)}(k-1)^2}dk, \qquad \int_{\alpha}^{\infty}(k+1)e^{-\frac{N\epsilon_1}{4}(k-1)}dk$$

both are $o(1)$ as $N\epsilon_1^2 \to \infty$. Hence $\Delta_N = \sum_i \Pr(Y_N^{(\epsilon_1)} = i)\Delta_{N,i} \leq (2-\delta)\alpha\epsilon_1 + o(\epsilon_1)$, contradicting the fact that $\Pr[T^{(\epsilon_1)} > N] \sim 2\epsilon_1$ if $N \gg \epsilon_1^{-2}$.

Therefore $\Delta_{N,N\epsilon} \geq (2-\delta)\epsilon$ for all $\delta > 0$. Combining with $\Delta_{N,N\epsilon} \leq 2\epsilon(1+o(1))$ we proved the theorem. □

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