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Bivariate affine Gončarov polynomials

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ABSTRACT

Bivariate Gončarov polynomials are a basis of the solutions of the bivariate Gončarov Interpolation Problem in numerical analysis. A sequence of bivariate Gončarov polynomials is determined by a set of nodes $Z = \{(x_{i,j}, y_{i,j}) \in \mathbb{R}^2\}$ and is an affine sequence if Z is an affine transformation of the lattice grid $\mathbb{N}^2$, i.e., $(x_{i,j}, y_{i,j})^T = A(i,j)^T + (c_1, c_2)^T$ for some 2×2 matrix A and constants c_1, c_2. In this paper we prove that a sequence of bivariate Gončarov polynomials is of binomial type if and only if it is an affine sequence with $c_1 = c_2 = 0$. Such polynomials form a higher-dimensional analog of the Abel polynomial $A_n(x; a) = x(x - an)^{n-1}$. We present explicit formulas for a general sequence of bivariate affine Gončarov polynomials and its exponential generating function, and use the algebraic properties of Gončarov polynomials to give some new two-dimensional generalizations of Abel identities.

1. Introduction

Gončarov Interpolation is a problem in numerical analysis which asks for a polynomial of degree n whose ith derivative at point a_i equals a prefixed value b_i, for $i = 0, 1, \ldots, n$. The sequence of Gončarov polynomials forms a basis of the solutions of Gončarov Interpolation. Explicitly, given a sequence of real numbers $a_0, a_1, \ldots, a_n$, the Gončarov polynomial $g_n(x; a_0, \ldots, a_{n-1})$ is defined by the biorthogonality relation:

$$\varepsilon(a_i)D^i g_n(x; a_0, a_1, \ldots, a_{n-1}) = n!\delta_{i,n}$$

where D is the differential operator, and $\varepsilon(a)$ is evaluation at a. A special case of this is *Abel interpolation*, where the point $a_i = ai$ for a fixed constant a. In this case, the Gončarov polynomial $g_n(x; 0, a, 2a, \ldots, (n-1)a)$ is called the *Abel polynomial* and denoted by $A_n(x; a)$, which has the explicit formula $A_n(x; a) = x(x - an)^{n-1}$. Abel polynomials are named after the Norwegian mathematician Niels Henrik Abel, and are closely related to counting of labeled trees and parking functions, (e.g. [1,12,22]).

A basic property for a sequence of Abel polynomials is that it is of binomial type, where a sequence $p_0(x), p_1(x), \ldots$ of polynomials is of binomial type if and only if

$$p_m(x+y) = \sum_{i=0}^{m} \binom{m}{i} p_i(x)p_{m-i}(y).$$

The theory of polynomials of binomial type plays a fundamental role in *umbral calculus*, or *finite operator calculus*, an area of algebraic combinatorics pioneered by Rota and Mullin [13] and further developed by Rota and many others, for example, in

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[2,3,5,14,19–21], to list a few. Umbral calculus studies combinatorial problems by means of linear functionals on spaces of polynomials, and exhibits fascinating relationships between the formal power series methods of combinatorics, the calculus of finite differences, and the theory of special polynomials and identities. Polynomials of binomial type also occur naturally in the enumeration of binomial posets, and are related to graph colorings and acyclic orientations [23–25].

For multivariate polynomials, there is an analogous notion of polynomial sequences of binomial type, for which the explicit definition is given in Section 2. The basic theory of polynomial sequences of binomial type in several variables and their roles in multivariate umbral calculus was developed by Parrish [15]. Garsia and Joni [6,9] showed that this theory offers a natural setting for the study of compositional inverses of formal power series, and used higher dimensional versions of the Lagrange inversion theorem to derive certain new Steffensen-type formulas. Multivariate umbral calculus and its applications to linear recurrences are studied by Niederhausen [14].

The multivariate Gončarov polynomials were investigated by Khare, Lorentz and Yan in [11] as a basis of solutions for the multivariate Gončarov Interpolation. In the current paper we are concerned about the bivariate case unless otherwise stated. Bivariate Gončarov polynomials are related to pairs of integer sequences whose order statistics are bounded by certain weights along lattice paths in $\mathbb{N}^2$. To state the definition, let Z be a set of nodes in $\mathbb{R}^2$, that is, $Z = \left\{z_{i,j} = (x_{i,j}, y_{i,j}) \in \mathbb{R}^2 \mid 0 \le i,\ 0 \le j\right\}$, and let

$$\Pi^2_{m,n} = \left\{ \sum_{i=0}^{m} \sum_{j=0}^{n} c_{i,j} x^i y^j \mid c_{i,j} \in R \right\}$$

be the space of bivariate polynomials of coordinate degree (m, n). The bivariate Gončarov polynomial $g_{m,n}((x, y); Z)$ is the unique polynomial in $\Pi^2_{m,n}$ satisfying

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} g_{m,n}(z_{i,j}) = m!n!\, \delta_{i,m}\delta_{j,n} \tag{1}$$

for all $0 \le i \le m$ and $0 \le j \le n$. Clearly $g_{m,n}((x, y); Z)$ depends only on the nodes $Z_{m,n} = \{z_{i,j} \in Z \mid 0 \le i \le m,\ 0 \le j \le n\}$. Many algebraic and combinatorial properties of $g_{m,n}((x, y); Z)$ are given in [11]. In particular, $g_{m,n}((x, y); Z)$ can be characterized by the following linear recursion

$$x^m y^n = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} x_{i,j}^{m-i} y_{i,j}^{n-j} g_{i,j}((x, y); Z), \tag{2}$$

and the Appell relation

$$e^{sx+ty} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g_{m,n}((x, y); Z) \frac{s^m e^{x_{m,n}s}}{m!} \frac{t^n e^{y_{m,n}t}}{n!}. \tag{3}$$

Our first objective is to characterize bivariate Gončarov polynomials of binomial type, which would give a 2-dimensional generalization of the classical Abel polynomials. In Section 2 we present a necessary and sufficient condition: a sequence of bivariate Gončarov polynomials is of binomial type if and only if the set of nodes Z is a linear transformation of the standard lattice grid $\mathbb{N}^2$. We call such Gončarov polynomials *linear* Gončarov polynomials. In general, if Z is an affine transformation of $\mathbb{N}^2$, the corresponding Gončarov polynomials are called *affine Gončarov polynomials*. In Section 3 we calculate the compositional inverses of the functions appearing in the Appell relations of linear Gončarov polynomials. This allows us to transform the Appell relation into an exponential generating function for a sequence of linear/affine Gončarov polynomials. Section 4 contains explicit formulas for affine Gončarov polynomials. In the bivariate case the formula is complete. We also present some examples in the trivariate case. In the last section, we use the algebraic equations of Gončarov polynomials to derive various combinatorial identities in double summations, which offer a new family of 2-dimensional generalizations of Riordan's Abel identities [18].

We remark that in the literature, there are various generalizations of Abel polynomials, starting from Hurwitz's multinomial extensions [7] in 1902. The Abel–Hurwitz identities are further investigated by many researchers, including Riordan [18], Françon [4], Pitman [16,17], Kelmans and Postnikov [10], and are related to random mappings, subsets, forests, and forest volumes. There were also precedents of Abel polynomials deriving from finite differences. See [8] for a historical overview. Our generalization as linear Gončarov polynomials is a totally different approach which provides a new perspective to the subject. Our results are complementary to the existing research on Abel polynomials and reveal connections between combinatorics, polynomial systems, and interpolation theory.

2. Gončarov polynomials of binomial type

We adopt Parrish's definition of multivariate polynomials of binomial type [15].

Definition 1. Let $p_{m,n}(x, y) \in \Pi_{m,n}$ be a sequence of bivariate polynomials satisfying

(a) the degree of $p_{m,n}$ is $m + n$.

(b) $p_{0,0}(x, y) = 1$ and $\{p_{0,0}(x, y), p_{0,1}(x, y), p_{1,0}(x, y)\}$ span the set of linear polynomials

Then the sequence $\{p_{m,n}(x, y) \mid 0 \le m, 0 \le n\}$ is of binomial type if it satisfies

$$p_{m,n}(x + a, y + b) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} p_{i,j}(x, y) p_{m-i,n-j}(a, b)$$

for all $m, n \in N$ and $a, b \in R$.

The main result of this section is a characterization of sequences of Gončarov polynomials that are of binomial type.

Definition 2. We say that a sequence of Gončarov polynomials determined by the set of node Z is *affine* if Z is an affine transformation of the lattice grid $B = \{(i, j) \mid i, j \in \mathbb{N}\}$. In other words, there exists a 2×2 matrix A and vector $C = (c_1, c_2)^T$ such that

$$\begin{pmatrix} x_{i,j} \\ y_{i,j} \end{pmatrix} = A \begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

In this case, we will simply say that $Z = AB + C$. In particular when $C = \vec{0}$, we say that the corresponding sequence of Gončarov polynomials $g_{m,n}((x, y); AB)$ is linear.

Theorem 2.1. *A sequence of Gončarov polynomials $g_{m,n}((x, y); Z)$ is of binomial type if and only if the sequence is linear. In particular, if $A = (a_{i,j})_{2\times 2}$, the Appell relation becomes*

$$e^{sx+ty} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g_{m,n}((x, y); Z) \frac{s^m e^{(a_{1,1}m+a_{1,2}n)s}}{m!} \frac{t^n e^{(a_{2,1}m+a_{2,2}n)t}}{n!}. \tag{4}$$

The proof of Theorem 2.1 uses a result of Parrish, who characterized sequences of multinomial polynomials of binomial type by their exponential generating functions.

Definition 3. Two formal power series $f_1(x, y)$ and $f_2(x, y)$ are said to be an *admissible pair* if and only if

1. $f_1(0, 0) = 0$ and $f_2(0, 0) = 0$,
2. The Jacobian

$$J[f_1, f_2](x, y) = \det \begin{pmatrix} \frac{\partial f_1(x, y)}{\partial x} & \frac{\partial f_1(x, y)}{\partial y} \\ \frac{\partial f_2(x, y)}{\partial x} & \frac{\partial f_2(x, y)}{\partial y} \end{pmatrix}$$

does not vanish at $(x, y) = (0, 0)$.

Theorem 2.2 (*Parrish [15]*). *A sequence of bivariate polynomials $\{p_{m,n}(x, y) \mid 0 \le m, 0 \le n\}$ is of binomial type if and only if there exists an admissible pair of formal power series $f_1(x, y)$ and $f_2(x, y)$ such that*

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n}(x, y) \frac{s^m}{m!} \frac{t^n}{n!} = e^{xf_1(s,t)+yf_2(s,t)}.$$

In fact, Parrish showed that the analogous formula holds in any dimension [15, Theorem 1.1].

We shall also use the following well-known result which can be found in [6] or [14]. This result implies that the set of all admissible pairs of formal power series form a group under composition. This group is called the *umbral group* and was studied extensively in [15].

Proposition 2.3. *If $f_1(s, t)$ and $f_2(s, t)$ is an admissible pair of formal power series in two variables, then there exists a unique admissible pair of formal power series $h_1(s, t)$, $h_2(s, t)$, the compositional inverse, such that*

$$f_1(h_1(s, t), h_2(s, t)) = s, \qquad f_2(h_1(s, t), h_2(s, t)) = t.$$

If (h_1, h_2) is inverse to (f_1, f_2), then (f_1, f_2) is also inverse to (h_1, h_2).

Proof of Theorem 2.1. We will use Theorem 2.2 and the Appell relation characterization (3) for Gončarov polynomials.

Necessity: Let now the sequence of Gončarov polynomials $g_{m,n}((x,y);Z)$ be of binomial type. This is equivalent to the existence of an admissible pair of formal power series $f_1(x,y)$ and $f_2(x,y)$ such that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{m,n}((x,y);Z)\frac{s^m}{m!}\frac{t^n}{n!} = e^{xf_1(s,t)+yf_2(s,t)}. \tag{5}$$

Using the compositional inverse $h_1(s,t)$ and $h_2(s,t)$ of the pair $(f_1(s,t), f_2(s,t))$, we can write this expansion as

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{m,n}((x,y);Z)\frac{h_1(s,t)^m}{m!}\frac{h_2(s,t)^n}{n!} = e^{xs+yt}. \tag{6}$$

Comparing this with the Appell relation and using the uniqueness of the coefficients, we have for all m and n,

$$h_1(s,t)^m h_2(s,t)^n = s^m e^{x_{m,n}s} t^n e^{y_{m,n}t}.$$

Taking the same equation with $m+1$ replacing m and dividing, we have

$$h_1(s,t) = s e^{(x_{m+1,n}-x_{m,n})s} e^{(y_{m+1,n}-y_{m,n})t},$$

which holds for all m and n. It follows that $x_{m+1,n}-x_{m,n}$ is a constant independent of m and n and so is $y_{m+1,n}-y_{m,n}$. Similarly, by incrementing n by 1, we obtain that $x_{m,n+1}-x_{m,n}$ and $y_{m,n+1}-y_{m,n}$ are constants independent of m and n.

Let c_1, c_2, c_3 and c_4 be these constants. By repeating the above first by decreasing m to zero and then decreasing n to zero, we obtain that

$$x_{m,n} = mc_1 + nc_3 + x_{0,0}, \qquad y_{m,n} = mc_2 + nc_4 + y_{0,0}.$$

Comparing the $(m,n) = (0,0)$ terms of the two representations, we get

$$s^0 e^{x_{0,0}s} t^0 e^{y_{0,0}t} = h_1(s,t)^0 h_2(s,t)^0,$$

or equivalently,

$$e^{x_{0,0}s} e^{y_{0,0}t} = 1.$$

This can only hold for all s and t if $x_{0,0} = 0$ and $y_{0,0} = 0$. So $(x_{m,n}, y_{m,n})^T = A(m,n)^T$ with

$$A = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}.$$

Sufficiency: Conversely, if a sequence of Gončarov polynomials $g_{m,n}((x,y);Z)$ is linear with $Z = AB$ for a 2×2-matrix A, then $x_{m,n} = a_{1,1}m + a_{1,2}n$ and $y_{m,n} = a_{2,1}m + a_{2,2}n$, where the $a_{i,j}$ are the entries of the matrix A.

The Appell relation (3) becomes

$$e^{sx+ty} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{m,n}((x,y);Z)\frac{s^m e^{(a_{1,1}m+a_{1,2}n)s}}{m!}\frac{t^n e^{(a_{2,1}m+a_{2,2}n)t}}{n!}. \tag{7}$$

Note that

$$s^m e^{(a_{1,1}m+a_{1,2}n)s} t^n e^{(a_{2,1}m+a_{2,2}n)t} = (se^{a_{1,1}s+a_{2,1}t})^m (te^{a_{1,2}s+a_{2,2}t})^n.$$

Taking

$$h_1(s,t) = se^{a_{1,1}s+a_{2,1}t}, \qquad h_2(s,t) = te^{a_{1,2}s+a_{2,2}t}, \tag{8}$$

the sequence of polynomials satisfies Eq. (6).

We check that the pair (h_1,h_2) of (8) is admissible. Clearly $h_1(0,0) = h_2(0,0) = 0$ and since $\frac{\partial h_1}{\partial s}(0,0) = 1$, $\frac{\partial h_1}{\partial t}(0,0) = 0$, $\frac{\partial h_2}{\partial s}(0,0) = 0$ and $\frac{\partial h_1}{\partial t}(0,0) = 1$, the Jacobian is 1. Thus by Theorem 2.2 (h_1,h_2) has a compositional inverse (f_1,f_2) which is also admissible. Substituting $s = f_1(s,t)$ and $t = f_2(s,t)$ into (6) we get Eq. (5), and hence this sequence of Gončarov polynomials is of binomial type. ∎

Remark. The proof of Theorem 2.1 can be easily extended to any dimension. In particular, a sequence of univariate Gončarov polynomials is of binomial type if and only if it is a sequence of Abel polynomials, i.e., $a_n = an$ for a constant a, and $g_n(x; a_0, \ldots, a_{n-1}) = A_n(x;a)$.

3. Exponential generating functions

In this section we give an explicit formula for the exponential generating function of a sequence of bivariate affine Gončarov polynomials. That is, we will find functions f_1 and f_2 for which

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{m,n}((x,y);Z)\frac{s^m}{m!}\frac{t^n}{n!} = e^{xf_1(s,t)+yf_2(s,t)} \tag{9}$$

when $Z = AB + C$. We will deal with the linear case first.

Theorem 3.1. *Let $A = (a_{i,j})$ be a 2×2 matrix and $Z = AB$. Then we have*

$$f_1(s,t) = \sum_{m=1}^{\infty}(-a_{1,1})^{m-1}\frac{m^{m-1}}{m!}s^m + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-a_{2,1}m)\frac{(-x_{m,n})^{m-1}}{m!}\frac{(-y_{m,n})^{n-1}}{n!}s^m t^n$$

and

$$f_2(s,t) = \sum_{n=1}^{\infty}(-a_{2,2})^{n-1}\frac{n^{n-1}}{n!}t^n + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-a_{1,2}n)\frac{(-x_{m,n})^{m-1}}{m!}\frac{(-y_{m,n})^{n-1}}{n!}s^m t^n,$$

where $x_{m,n} = a_{1,1}m + a_{1,2}n$ and $y_{m,n} = a_{2,1}m + a_{2,2}n$.

Let h_1 and h_2 be the admissible pair of formal power series defined by

$$h_1(s,t) = se^{a_{1,1}s+a_{2,1}t}, \qquad h_2(s,t) = te^{a_{1,2}s+a_{2,2}t}. \tag{10}$$

Then the formal power series $(f_1(s,t), f_2(s,t))$ in Eq. (9) are the compositional inverse for $(h_1(s,t), h_2(s,t))$. To compute them we use the Lagrange–Good inversion formula in the following version, which was proved by Joni [9, Theorem 3.2]. An explicit statement and applications to linear recursions are presented by Niederhausen [14, Theorem 1.3.2].

Theorem 3.2 (*Lagrange–Good inversion formula*)**.** *Let $(h_1(s,t), h_2(s,t))$ be a multi-series with compositional inverse $(f_1(s,t), f_2(s,t))$. If we can write*

$$h_1(s,t) = s/\phi_1(s,t) \quad \text{and} \quad h_2(s,t) = t/\phi_2(s,t),$$

where $\phi_1(s,t)$ and $\phi_2(s,t)$ are formal power series with $\phi_1(0,0)\neq 0$ and $\phi_2(0,0)\neq 0$, then

$$[s^m t^n]\left(f_1(s,t)^k f_2(s,t)^\ell\right) = [s^{m-k}t^{n-\ell}]\left(\phi_1(s,t)^{m+1}\phi_2(s,t)^{n+1}|J[h_1,h_2](s,t)|\right) \tag{11}$$

for all $m, n, k, \ell \in \mathbb{N}$. In (11) $J[h_1,h_2]$ is the Jacobian of h_1 and h_2, and $[s^m t^n]f$ is the coefficient of the term $x^m y^n$ in the expansion of f.

Proof of Theorem 3.1. With the functions (h_1, h_2) given in (10), we set

$$\phi_1(s,t) = e^{-(a_{1,1}s+a_{2,1}t)}, \qquad \phi_2(s,t) = e^{-(a_{1,2}s+a_{2,2}t)}.$$

Clearly $\phi_1(s,t)$ and $\phi_2(s,t)$ are formal power series with $\phi_1(0,0) = \phi_2(0,0) = 1$, and such that

$$h_1(s,t) = \frac{s}{\phi_1(s,t)}, \qquad h_2(s,t) = \frac{t}{\phi_2(s,t)}.$$

Applying Theorem 3.2 with $k = 1$ and $\ell = 0$, we get that the compositional inverse (f_1, f_2) of the pair (h_1, h_2) satisfies

$$[s^m t^n](f_1(s,t)) = [s^{m-1}t^n](\phi_1(s,t)^{m+1}\phi_2(s,t)^{n+1}J[h_1,h_2](s,t))$$

for all $m, n \in \mathbb{N}$. In the following we compute $f_1(s,t)$ in detail. The computation of $f_2(s,t)$ is similar with Eq. (11) at $k = 0$ and $\ell = 1$.

For our choice of h_1 and h_2,

$$J[h_1,h_2](s,t) = e^{a_{1,1}s+a_{2,1}t}e^{a_{1,2}s+a_{2,2}t}\left(1 + a_{1,1}s + a_{2,2}t + \det(A)st\right).$$

So

$$\phi_1(s,t)^{m+1}\phi_2(s,t)^{n+1}J[h_1,h_2](s,t) = e^{-x_{m,n}s-y_{m,n}t}\left(1 + a_{1,1}s + a_{2,2}t + \det(A)st\right).$$

Now use that $[s^i t^j](u(s)v(t)) = [s^i](u(s))[t^j](v(t))$ and linearity to obtain

$$\begin{aligned}[s^m t^n]f_1(s,t) = {} & \det(A)[s^{m-1}](se^{-x_{m,n}s})[t^n](te^{-y_{m,n}t}) + a_{1,1}[s^{m-1}](se^{-x_{m,n}s})[t^n](e^{-y_{m,n}t}) \\ & + a_{2,2}[s^{m-1}](e^{-x_{m,n}s})[t^n](te^{-y_{m,n}t}) + [s^{m-1}](e^{-x_{m,n}s})[t^n](e^{-y_{m,n}t}).\end{aligned}$$

In evaluating the above formula, we consider the following four cases.

Case 1: $m = 0, n \geq 0$. Since the functions $\phi_1(s,t)$, $\phi_2(s,t)$ and $J[h_1,h_2](s,t)$ are formal power series in s,t, there is no term with s^{-1}. Hence $[s^0t^n](f_1(s,t)) = 0$ for all $n \geq 0$.

Case 2: $n = 0, m \geq 1$. Using $[t^0](tg(t)) = 0$ for any formal power series g and $x_{m,0} = a_{1,1}m$, we get

$$[s^mt^0](f_1(s,t)) = a_{1,1}[s^{m-1}](se^{-x_{m,0}s}) + [s^{m-1}](e^{-x_{m,0}s}) = (-a_{1,1})^{m-1}\frac{m^{m-1}}{m!}.$$

Case 3: $m = 1, n \geq 1$. Using $[s^0](sg(s)) = 0$ for any formal power series g, we get

$$[s^1t^n]f_1(s,t) = a_{2,2}[t^n](te^{-y_{1,n}t}) + [t^n](e^{-y_{1,n}t}) = -a_{2,1}\frac{(-y_{1,n})^{n-1}}{n!}.$$

Case 4: $m \geq 2$ and $n \geq 1$. Now we have

$$\begin{aligned}
[s^mt^n]f_1(s,t) &= \det(A)[s^{m-2}](e^{-x_{m,n}s})[t^{n-1}](e^{-y_{m,n}t}) + a_{1,1}[s^{m-2}](e^{-x_{m,n}s})[t^n](e^{-y_{m,n}t})\\
&\quad + a_{2,2}[s^{m-1}](e^{-x_{m,n}s})[t^{n-1}](e^{-y_{m,n}t}) + [s^{m-1}](e^{-x_{m,n}s})[t^n](e^{-y_{m,n}t})\\
&= \det(A)\frac{(-x_{m,n})^{m-2}}{(m-2)!}\frac{(-y_{m,n})^n}{(n-1)!} + a_{1,1}\frac{(-x_{m,n})^{m-2}}{(m-2)!}\frac{(-y_{m,n})^n}{n!}\\
&\quad + a_{2,2}\frac{(-x_{m,n})^{m-1}}{(m-1)!}\frac{(-y_{m,n})^n}{(n-1)!} + \frac{(-x_{m,n})^{m-1}}{(m-1)!}\frac{(-y_{m,n})^n}{n!}\\
&= \frac{(-x_{m,n})^{m-2}}{(m-1)!}\frac{(-y_{m,n})^{n-1}}{n!}\\
&\quad \cdot\{(m-1)n\det(A) - (m-1)a_{1,1}y_{m,n} - na_{2,2}x_{m,n} +_{m,n} y_{m,n}\}.
\end{aligned}$$

The quantity in brackets { } simplifies to $a_{2,1}x_{m,n}$, so that

$$[s^mt^n]f_1(s,t) = (-a_{2,1})\frac{(-x_{m,n})^{m-1}}{(m-1)!}\frac{(-y_{m,n})^{n-1}}{n!}$$

for $m \geq 2$ and $n \geq 1$.

Putting everything together, we have

$$\begin{aligned}
f_1(s,t) &= \sum_{m=1}^{\infty}(-a_{1,1})^{m-1}\frac{m^{m-1}}{m!}s^m + \sum_{n=1}^{\infty}(-a_{2,1})\frac{(-y_{1,n})^{n-1}}{n!}st^n\\
&\quad + \sum_{m=2}^{\infty}\sum_{n=1}^{\infty}(-a_{2,1}m)\frac{(-x_{m,n})^{m-1}}{m!}\frac{(-y_{m,n})^{n-1}}{n!}s^mt^n
\end{aligned}$$

where $x_{m,n} = a_{1,1}m + a_{1,2}n$ and $y_{m,n} = a_{2,1}m + a_{2,2}n$. Note that the second sum is just the $m = 1$ term of the following double sum, so

$$f_1(s,t) = \sum_{m=1}^{\infty}(-a_{1,1})^{m-1}\frac{m^{m-1}}{m!}s^m + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-a_{2,1}m)\frac{(-x_{m,n})^{m-1}}{m!}\frac{(-y_{m,n})^{n-1}}{n!}s^mt^n.$$

The formula of $f_2(s,t)$ is obtained similarly. ■

Note that if the grid of nodes is a rectangular one, i.e., if A is a diagonal matrix with $a_{2,1} = a_{1,2} = 0$, then $f_1(s,t) = f_1(s,0)$ and $f_2(s,t) = f_2(0,t)$ are univariate inverses of $h_1(s,0) = se^{a_{1,1}s}$, respectively $h_2(0,t) = te^{a_{2,2}t}$. In that case $f_1(s,t) = (1/a_{1,1})W(a_{1,1}s)$ and $f_2(s,t) = (1/a_{2,2})W(a_{2,2}t)$, where $W(s)$ is the well-known Lambert W-function

$$W(s) = \sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}s^n.$$

Using the shift invariant formula of Gončarov polynomials in [11, Theorem 3.8],

$$g_{m,n}((x+c_1, y+c_2); Z+(c_1,c_2)) = g_{m,n}((x,y);Z),$$

we obtain the exponential generating function for a sequence of affine Gončarov polynomials.

Corollary 3.3. *When $Z = AB+C$ where $C = (c_1,c_2)^T$, the exponential generating function for the affine Gončarov polynomials is*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}g_{m,n}((x,y);Z)\frac{s^m}{m!}\frac{t^n}{n!} = e^{(x-c_1)f_1(s,t)+(y-c_2)f_2(s,t)},$$

where $f_1(s,t)$ and $f_2(s,t)$ are given in Theorem 3.1.

4. Closed formula for affine Gončarov polynomials

Recall that a univariate Gončarov polynomial is affine if and only if the nodes form an arithmetic progression. In this case $g_n(x; a, a+b, a+2b, \ldots, a+(n-1)b) = (x-a)(x-a-nb)^{n-1}$ is the shifted Abel polynomials $A_n(x-a; b)$. In this section we present the explicit formula for the affine Gončarov polynomials in two variables, and some special cases in three variables. Again we start with the linear case. The formula was first conjectured based on explicit calculation with small indices, and then verified in the following theorem.

Theorem 4.1. *Let $m, n \in \mathbb{N}$, $A = (a_{i,j})$ be a 2×2 matrix and $Z = AB$ be a set of interpolation nodes. Then the linear bivariate Gončarov polynomials with the set of nodes Z are given by*

$$g_{m,n}((x, y); Z) = (x - x_{m,n})^{m-1}(y - y_{m,n})^{n-1}\left[(x - x_{0,n})(y - y_{m,0}) - x_{0,n}y_{m,0}\right], \tag{12}$$

where $x_{i,j} = a_{1,1}i + a_{1,2}j$, $y_{i,j} = a_{2,1}i + a_{2,2}j$ for all $i, j \in \mathbb{N}$.

Proof. Denote by $G_{m,n}(x, y)$ the right-hand side of (12). We use the defining interpolation equations (1) for Gončarov polynomials and check that for $0 \le i \le m$ and $0 \le j \le n$

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} G_{m,n}(x_{i,j}, y_{i,j}) = m!n!\delta_{i,m}\delta_{j,n}. \tag{13}$$

It is easy to compute that

$$\frac{\partial^i}{\partial x^i}(x - x_{m,n})^{m-1}(x - x_{0,n}) = \begin{cases} m! & \text{if } i = m, \\ (m-1)_{i-1}(x - x_{m,n})^{m-1-i}(mx - (m-i)x_{0,n} - ix_{m,n}) & \text{if } i < m, \end{cases}$$

where for a nonnegative integer k the symbol $(x)_k$ is the lower factorial $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$. For $x = x_{i,j} = a_{1,1}i + a_{1,2}j$,

$$\frac{\partial^i}{\partial x^i}(x - x_{m,n})^{m-1}(x - x_{0,n})\Big|_{x=x_{i,j}} = \begin{cases} m! & \text{if } i = m, \\ (m-1)_{i-1}(x_{i,j} - x_{m,n})^{m-1-i} \cdot a_{1,2}m(j-n) & \text{if } i < m. \end{cases} \tag{14}$$

Similarly, for $y = y_{i,j} = a_{2,1}i + a_{2,2}j$,

$$\frac{\partial^j}{\partial j^i}(y - y_{m,n})^{n-1}(y - y_{m,0})\Big|_{y=y_{i,j}} = \begin{cases} n! & \text{if } j = n, \\ (n-1)_{j-1}(y_{i,j} - y_{m,n})^{n-1-j} \cdot a_{2,1}n(i-m) & \text{if } j < n. \end{cases} \tag{15}$$

Using (14) and (15) we see immediately that

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} G_{m,n}(x, y) = m!n! \quad \forall (x, y)$$

and

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} G_{m,n}(x, y) = 0$$

if $i = m$ and $j < n$, or $i < m$ and $j = n$. Finally when $0 \le i < m$ and $0 \le j < n$, we have

$$\begin{aligned} \frac{\partial^{i+j}}{\partial x^i \partial y^j} G_{m,n}(x, y)\Big|_{(x,y)=(x_{i,j},y_{i,j})} &= (m-1)_{i-1}(n-1)_{j-1}(x_{i,j} - x_{m,n})^{m-1-i}(y_{i,j} - y_{m,n})^{n-1-j} \\ &\quad \{a_{1,2}a_{2,1}mn(m-i)(n-j) - (m-i)(n-j)x_{0,n}y_{m,0}\}. \end{aligned}$$

The term in the { } vanishes since $x_{0,n} = a_{1,2}n$ and $y_{m,0} = a_{2,1}m$. ■

Remark. When A is a diagonal matrix, i.e., $a_{1,2} = a_{2,1} = 0$, $g_{m,n}((x, y); AB)$ is a product of two univariate Abel polynomials in the variables x and y. Precisely,

$$g_{m,n}((x, y), AB) = A_m(x; a_{1,1})A_n(y; a_{2,2}).$$

When $a_{1,2} = 0$ but $a_{2,1} \neq 0$, $g_{m,n}((x, y); AB)$ can also be factored into a product of an Abel polynomial in x and a shifted Abel polynomial in y, as

$$g_{m,n}((x, y), AB) = A_m(x; a_{1,1})A_n(y - a_{2,1}m; a_{2,2}).$$

A similar equation holds for the case $a_{1,2} \neq 0$ but $a_{2,1} = 0$. In general, linear Gončarov polynomials cannot be factored as a product of univariate polynomials of x and y. Hence (12) offers a new generalization of the classical Abel polynomials.

The shift invariant formula leads to the closed formula for affine Gončarov polynomials in two variables.

Corollary 4.2. *Let $Z = AB + C$ where A is a 2×2 matrix and $C = (c_1, c_2)^T$, Then the affine bivariate Gončarov polynomials with the node-set Z are given by*

$$g_{m,n}((x,y);Z) = (x - x_{m,n})^{m-1}(y - y_{m,n})^{n-1}\left\{(x - x_{0,n})(y - y_{m,0}) - (x_{0,n} - x_{0,0})(y_{m,0} - y_{0,0})\right\}$$

where $x_{i,j} = a_{1,1}i + a_{1,2}j + c_1$, $y_{i,j} = a_{2,1}i + a_{2,2}j + c_2$ for all $i, j \in \mathbb{N}$.

In Section 2 we have shown that a sequence of Gončarov polynomials is of binomial type if and only if it is linear. This assertion is true in any dimension. However, we do not have a complete description for linear Gončarov polynomials in general dimensions, except for some special cases with three variables, which are given below. To simplify notations, we will write $x_{i,j,k}$ as x_{ijk}, and so on.

Theorem 4.3. *Let m, n, p be nonnegative integers, $A = (a_{i,k})$ be a 3×3-matrix and the set of nodes Z be given by $Z = \{(x_{ijk}, y_{ijk}, z_{ijk})^T = A(i,j,k)^T \mid 0 \le i, 0 \le j, 0 \le k\}$.*

If one of the conditions

1. $a_{1,3} = a_{2,1} = a_{3,2} = 0$
2. $a_{1,2} = a_{2,3} = a_{3,1} = 0$
3. $a_{1,2} = a_{1,3} = 0$ *and* $a_{2,3}a_{3,2} = 0$
4. $a_{2,1} = a_{2,3} = 0$ *and* $a_{1,3}a_{3,1} = 0$
5. $a_{3,1} = a_{3,2} = 0$ *and* $a_{1,2}a_{2,1} = 0$

holds, then

$$\begin{aligned} g_{m,n,p}((x,y,z);Z) = {} & (x - x_{mnp})^{m-1}(y - y_{mnp})^{n-1}(z - z_{mnp})^{p-1} \\ & \{(x - x_{0np})(y - y_{m0p})(z - z_{mn0}) + x_{0np}y_{m0p}z_{mn0}\} \end{aligned} \tag{16}$$

is the trivariate linear Gončarov polynomial with the node-set Z.

Theorem 4.3 is proved by using the interpolation definition for Gončarov polynomials [11] and verifying that the function $g_{m,n,p}((x,y,z);Z)$ defined by (16) satisfies the equations

$$\frac{\partial^{i+j+k}}{\partial x^i \partial y^j \partial z^k} g_{m,n,p}((x_{ijk}, y_{ijk}, z_{ijk})) = m!n!p!\delta_{i,m}\delta_{j,n}\delta_{k,p}. \tag{17}$$

for $0 \le i \le m$, $0 \le j \le n$ and $0 \le k \le p$.

We remark that the possible choices of the matrix A allow placing the nodes of interpolation on a plane or a line, (i.e., A is degenerate). If A is diagonal, $g_{m,n,p}((x,y,z);AB)$ is the product of three univariate Abel polynomials in the variables x, y and z. In the cases 3, 4, and 5 above, $g_{m,n,p}((x,y,z);AB)$ can be factored into three univariate polynomials, one is an Abel polynomial and two are shifted Abel polynomials.

Again using the shift invariance of Gončarov polynomials, we have

Corollary 4.4. *Let $Z = AB + C$ where A is a 3×3 matrix and $C = (c_1, c_2, c_3)^T$. If one of the five conditions in Theorem 4.3 holds, then the affine bivariate Gončarov polynomials for interpolation at the nodes Z are given by*

$$\begin{aligned} g_{m,n,p}((x,y,z);Z) = {} & (x - x_{mnp})^{m-1}(y - y_{mnp})^{n-1}(z - z_{mnp})^{p-1} \\ & \{(x - x_{0np})(y - y_{m0p})(z - z_{mn0}) + (x_{0np} - x_{000})(y_{m0p} - y_{000})(z_{mn0} - z_{000})\}, \end{aligned}$$

where $(x_{ijk}, y_{ijk}, z_{ijk})^T = A(i,j,k)^T + (c_1, c_2, c_3)^T$ for all $i, j, k \in \mathbb{N}$.

5. Two-dimensional Abel identities

In his monograph *Combinatorial Identities*, John Riordan studied a class of sums defined by

$$A_n(x,y;p,q) = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{k-p}(y+n-k)^{n-k+q},$$

and gave explicit formulas for many $A_n(x,y;p,q)$'s with small values of (p,q). Riordan's results were summarized in [18, Section 1.5] and named *Abel identities*, since the instance with $p = -1$, $q = 0$ is Abel's celebrated generalization of the binomial theorem:

$$x^{-1}(x+y+na)^n = \sum_{k=0}^{n} \binom{n}{k} (x+ka)^{k-1}(y+(n-k)a)^{n-k}.$$

In this section we use the bivariate affine Gončarov polynomials to derive combinatorial identities in double summations, which can be viewed as a two-dimensional generalization of Abel identities. Explicitly, we consider the double summation

$$\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}(s+j)^{m-i+\alpha}(t+i)^{n-j+\beta}(x+(n-j))^{i+\gamma}(y+(m-i))^{j+\delta}. \tag{18}$$

We will present explicit formulas for the case $\alpha=\beta=\gamma=\delta=0$ and the case that exactly one of $\alpha,\beta,\gamma,\delta$ is -1 and others are 0. These formulas reduce the double summations of (18) into single summations, which can be computed out at certain specializations.

First we list some basic properties of the sum $\mathcal{A}_{m,n}$.

5.1. Basic properties of $\mathcal{A}_{m,n}$

Initial values.

$$\mathcal{A}_{0,0}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} = s^\alpha t^\beta x^\gamma y^\delta,$$

$$\mathcal{A}_{1,0}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} = s^\alpha t^\beta x^\gamma y^\delta\left(s\left(1+\frac{1}{y}\right)^\delta + x\left(1+\frac{1}{t}\right)^\beta\right),$$

and

$$\mathcal{A}_{0,1}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} = s^\alpha t^\beta x^\gamma y^\delta\left(t\left(1+\frac{1}{x}\right)^\gamma + y\left(1+\frac{1}{s}\right)^\alpha\right).$$

Symmetry. Replacing i with $m-i$ and j with $n-j$ yields that

$$\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} = \mathcal{A}_{m,n}\begin{pmatrix} x & y & s & t \\ \gamma & \delta & \alpha & \beta \end{pmatrix}$$

Recurrence. Using Pascal's identity

$$\binom{m}{i} = \binom{m-1}{i} + \binom{m-1}{i-1},$$

we have

$$\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} = \mathcal{A}_{m-1,n}\begin{pmatrix} s & t & x & y+1 \\ \alpha+1 & \beta & \gamma & \delta \end{pmatrix} + \mathcal{A}_{m-1,n}\begin{pmatrix} s & t+1 & x & y \\ \alpha & \beta & \gamma+1 & \delta \end{pmatrix}. \tag{19}$$

Similarly,

$$\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} = \mathcal{A}_{m,n-1}\begin{pmatrix} s & t & x+1 & y \\ \alpha & \beta+1 & \gamma & \delta \end{pmatrix} + \mathcal{A}_{m,n-1}\begin{pmatrix} s+1 & t & x & y \\ \alpha & \beta & \gamma & \delta+1 \end{pmatrix}. \tag{20}$$

On the other hand,

$$\begin{aligned}
\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} &= \sum_{(i,j)=(0,0)}^{(m,n)}\binom{m}{i}\binom{n}{j}(s+j)(s+j)^{m-i-1+\alpha}(t+i)^{n-j+\beta}(x+n-j)^{i+\gamma}(y+m-i)^{j+\delta} \\
&= s\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha-1 & \beta & \gamma & \delta \end{pmatrix} \\
&\quad + n\sum_{(i,j)=(0,0)}^{(m,n-1)}\binom{m}{i}\binom{n-1}{j-1}(s+j)^{m-i-1+\alpha}(t+i)^{n-j+\beta}(x+n-j)^{i+\gamma}(y+m-i)^{j+\delta} \\
&= s\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha-1 & \beta & \gamma & \delta \end{pmatrix} + n\mathcal{A}_{m,n-1}\begin{pmatrix} s+1 & t & x & y \\ \alpha-1 & \beta & \gamma & \delta+1 \end{pmatrix}.
\end{aligned} \tag{21}$$

Analogously, expanding other factors leads to

$$\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} = x\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma-1 & \delta \end{pmatrix} + n\mathcal{A}_{m,n-1}\begin{pmatrix} s & t & x+1 & y \\ \alpha & \beta+1 & \gamma-1 & \delta \end{pmatrix} \tag{22}$$

$$\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} = t\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta-1 & \gamma & \delta \end{pmatrix} + m\mathcal{A}_{m-1,n}\begin{pmatrix} s & t+1 & x & y \\ \alpha & \beta-1 & \gamma+1 & \delta \end{pmatrix} \tag{23}$$

$$\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} = y\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta-1 \end{pmatrix} + m\mathcal{A}_{m,n-1}\begin{pmatrix} s & t & x & y+1 \\ \alpha+1 & \beta & \gamma & \delta-1 \end{pmatrix}. \tag{24}$$

Remark. Sometimes one is interested in the following summation with parameters b and c:

$$\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix}; b, c\Big) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}(s+bj)^{m-i+\alpha}(t+ci)^{n-j+\beta}(x+(n-j)b)^{i+\gamma}(y+(m-i)c)^{j+\delta}.$$

Such a summation can be obtained by the equation

$$\mathcal{A}_{m,n}\begin{pmatrix} sb & tc & xb & yc \\ \alpha & \beta & \gamma & \delta \end{pmatrix}; b, c\Big) = b^{m+\alpha+\gamma}c^{n+\beta+\delta}\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix}.$$

5.2. Formula for the case $\alpha = \beta = \gamma = \delta = 0$

We use the linear recursion (2) for bivariate Gončarov polynomials. Let $Z = \{(x_{i,j}, y_{i,j} : i \geq 0, j \geq 0\}$ be given by

$$\begin{pmatrix} x_{i,j} \\ y_{i,j} \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}.$$

Using Corollary 4.2 and replacing x with $x+s$, y with $y+t$, we derive from (2) the identity

$$(x+s)^m(y+t)^n = \sum_{(i,j)=(0,0)}^{(m,n)}\binom{m}{i}\binom{n}{j}(s+j)^{m-i}(t+i)^{n-j}\cdot(x-j)^{i-1}(y-i)^{j-1}((x-j)(y-i)-ij). \tag{25}$$

This is the key identity in our derivation.

Let

$$\begin{aligned} f_{m,n}(x,y) &= \mathcal{A}_{m,n}\begin{pmatrix} s & t & x-n & y-m \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \sum_{(i,j)=(0,0)}^{(m,n)}\binom{m}{i}\binom{n}{j}(s+j)^{m-i}(t+i)^{n-j}(x-j)^{i}(y-i)^{j}, \end{aligned} \tag{26}$$

where we view s, t as constants and x, y as variable. First, it is easy to compute that

$$f_{m,0}(x,y) = (s+x)^m \quad \text{and} \quad f_{0,n}(x,y) = (y+t)^n. \tag{27}$$

Note that

$$\frac{\partial^2}{\partial x\partial y}f_{m,n}(x,y) = \sum_{(i,j)=(0,0)}^{(m,n)} ij\binom{m}{i}\binom{n}{j}(s+j)^{m-i}(t+i)^{n-j}(x-j)^{i-1}(y-i)^{j-1}.$$

Hence Identity (25) implies that

$$(x+s)^m(y+t)^n = f_{m,n}(x,y) - \frac{\partial^2}{\partial x\partial y}f_{m,n}(x,y). \tag{28}$$

Let $h_{m,n}(x,y) = f_{m,n}(x,y) - (x+s)^m(y+t)^n$. Then

$$h_{m,n}(x,y) = mn(x+s)^{m-1}(y+t)^{n-1} + \frac{\partial^2}{\partial x\partial y}h_{m,n}(x,y).$$

Comparing with (28), we notice that $\frac{1}{mn}h_{m,n}$ satisfies the same differential equation as $f_{m-1,n-1}$.

Consider the solution for the homogeneous differential equation

$$F(x,y) = k\frac{\partial^2}{\partial x\partial y}F(x,y). \tag{29}$$

If $F(x,y) = \sum_{i,j} a_{i,j}x^iy^j$, then we have the relations that

$$a_{i,j} = \begin{cases} \dfrac{1}{k^j}\dfrac{a_{i-j,0}}{(i)_j j!} & \text{if } i \geq j, \\[2ex] \dfrac{1}{k^i}\dfrac{a_{j-i,0}}{(j)_i i!} & \text{if } i < j. \end{cases}$$

The only polynomial solution of (29) is $F(x, y) = 0$. Since both $h_{m,n}(x, y)$ and $f_{m-1,n-1}(x, y)$ are polynomials of x and y, we conclude that

$$h_{m,n}(x, y) = mnf_{m-1,n-1}(x, y).$$

Therefore we have the recurrence

$$f_{m,n}(x, y) = (x + s)^m (y + t)^n + mnf_{m-1,n-1}(x, y) \quad (m, n \geq 1). \tag{30}$$

Combining with the initial conditions (27), we obtain

$$f_{m,n}(x, y) = \sum_{k=0}^{\min(m,n)} (m)_k (n)_k (s + x)^{m-k} (t + y)^{n-k}. \tag{31}$$

Let

$$F(u, v) := \sum_{m,n\geq 0} f_{m,n} \frac{u^m}{m!} \frac{v^n}{n!}.$$

Then from (31) we have

$$F(u, v) = \sum_{k\geq 0} u^k v^k \sum_{m\geq k} \frac{[u(s + x)]^{m-k}}{(m - k)!} \sum_{n\geq k} \frac{[v(t + y)]^{n-k}}{(n - k)!} = \frac{e^{u(x+s)+v(y+t)}}{1 - uv}. \tag{32}$$

In terms of $\mathcal{A}_{m,n}$, we have that

Theorem 5.1.

$$\mathcal{A}_{m,n} \begin{pmatrix} s & t & x & y \\ 0 & 0 & 0 & 0 \end{pmatrix} = \sum_{k=0}^{\min(m,n)} (m)_k (n)_k (s + x + n)^{m-k} (t + y + m)^{n-k}, \tag{33}$$

and the generating function

$$\sum_{m,n\geq 0} \mathcal{A}_{m,n} \begin{pmatrix} s & t & x - n & y - m \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{u^m}{m!} \frac{v^n}{n!} = \frac{e^{u(x+s)+v(y+t)}}{1 - uv}.$$

Note the similarity of Formula (33) with Riordan's Abel identity of $A_n(x, y; 0, 0)$, which was also called Cauchy's formula and given in [18, Formula (24), Chapter 1]:

$$A_n(x, y; 0, 0) = \sum_{k=0}^{n} (n)_k (x + y + n)^{n-k}.$$

5.3. Formula for $\alpha = -1$ and $\beta = \gamma = \delta = 0$

Using the recurrence relations of $\mathcal{A}_{m,n}$ and Theorem 5.1, we can obtain the closed formula for another case, namely, when exactly one of $\alpha, \beta, \gamma, \delta$ is -1 and the others are 0. We work out the details with $\alpha = -1$ and $\beta = \gamma = \delta = 0$.

In (21), applying (24) to the second term, we have

$$\begin{aligned}
\mathcal{A}_{m,n} \begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} &= s\mathcal{A}_{m,n} \begin{pmatrix} s & t & x & y \\ \alpha - 1 & \beta & \gamma & \delta \end{pmatrix} + n\mathcal{A}_{m,n-1} \begin{pmatrix} s + 1 & t & x & y \\ \alpha - 1 & \beta & \gamma & \delta + 1 \end{pmatrix} \\
&= s\mathcal{A}_{m,n} \begin{pmatrix} s & t & x & y \\ \alpha - 1 & \beta & \gamma & \delta \end{pmatrix} \\
&\quad + n \left[y\mathcal{A}_{m,n-1} \begin{pmatrix} s + 1 & t & x & y \\ \alpha - 1 & \beta & \gamma & \delta \end{pmatrix} + m\mathcal{A}_{m-1,n-1} \begin{pmatrix} s + 1 & t & x & y + 1 \\ \alpha & \beta & \gamma & \delta \end{pmatrix} \right].
\end{aligned}$$

Hence

$$\begin{aligned}
&s\mathcal{A}_{m,n} \begin{pmatrix} s & t & x & y \\ \alpha - 1 & \beta & \gamma & \delta \end{pmatrix} + ny\mathcal{A}_{m,n-1} \begin{pmatrix} s + 1 & t & x & y \\ \alpha - 1 & \beta & \gamma & \delta \end{pmatrix} \\
&\quad = \mathcal{A}_{m,n} \begin{pmatrix} s & t & x & y \\ \alpha & \beta & \gamma & \delta \end{pmatrix} - mn\mathcal{A}_{m-1,n-1} \begin{pmatrix} s + 1 & t & x & y + 1 \\ \alpha & \beta & \gamma & \delta \end{pmatrix}.
\end{aligned} \tag{34}$$

When $\alpha = \beta = \gamma = \delta = 0$, using (33) one computes that

$$\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ 0 & 0 & 0 & 0 \end{pmatrix} - mn\mathcal{A}_{m-1,n-1}\begin{pmatrix} s+1 & t & x & y+1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (s+x+n)^m(t+y+m)^n.$$

Therefore (34) leads to the identity

$$s\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ -1 & 0 & 0 & 0 \end{pmatrix} + ny\mathcal{A}_{m,n-1}\begin{pmatrix} s+1 & t & x & y \\ -1 & 0 & 0 & 0 \end{pmatrix} = (s+x+n)^m(t+y+m)^n. \tag{35}$$

The general formula of $\mathcal{A}_{m,n}$ with $(\alpha, \beta, \gamma, \delta) = (-1, 0, 0, 0)$ can be obtained by iterating the recurrence (35) with the initial value

$$\mathcal{A}_{m,0}\begin{pmatrix} s & t & x & y \\ -1 & 0 & 0 & 0 \end{pmatrix} = s^{-1}(s+x)^m.$$

For example,

$$\mathcal{A}_{m,1}\begin{pmatrix} s & t & x & y \\ -1 & 0 & 0 & 0 \end{pmatrix} = (x+s+1)^m\left(\frac{t+y+m}{s} - \frac{y}{s(s+1)}\right).$$

$$\mathcal{A}_{m,2}\begin{pmatrix} s & t & x & y \\ -1 & 0 & 0 & 0 \end{pmatrix} = (s+x+2)^m\left(\frac{(t+y+m)^2}{s} - \frac{2y(t+y+m)}{s(s+1)} + \frac{y^2}{s(s+1)(s+2)}\right).$$

In general, we have

Theorem 5.2.

$$\mathcal{A}_{m,n}\begin{pmatrix} s & t & x & y \\ -1 & 0 & 0 & 0 \end{pmatrix} = (s+x+n)^m\left(\sum_{i=0}^{n}\frac{(-1)^i(n)_i}{(s+i)_{i+1}}(t+y+m)^{n-i}y^i\right). \tag{36}$$

When $s = 1$, the formula (36) is much simpler. In that case, $(s+i)_{i+1} = (1+i)!$, and hence

Corollary 5.3.

$$\begin{aligned}\mathcal{A}_{m,n}\begin{pmatrix} 1 & t & x & y \\ -1 & 0 & 0 & 0 \end{pmatrix} &= (1+x+n)^m\left(\sum_{i=0}^{n}\frac{(-1)^i(n)_i}{(1+i)!}(t+y+m)^{n-i}y^i\right) \\ &= \frac{(1+x+n)^m}{(n+1)y}((t+y+m)^{n+1} - (t+m)^{n+1}).\end{aligned}$$

Note that one cannot take $s = 0$ directly in (35) since s is not allowed to be zero when $\alpha = -1$ in the defining Eq. (18) of $\mathcal{A}_{m,n}$.

5.4. *Final remark*

We conclude the paper by briefly mentioning another interesting case where the matrix A is degenerate, i.e.,

$$A = \begin{pmatrix} a & ab \\ ca & cab \end{pmatrix}.$$

Applying the same techniques as in the preceding subsections, we obtain a simpler expression for the following double summation. Let

$$\mathcal{B}_{m,n}(s,t,x,y) = \sum_{(i,j)=(0,0)}^{(m,n)}\binom{m}{i}\binom{n}{j}(s+i+bj)^{m-i}(t+i+bj)^{n-j}$$
$$\cdot(x-i-bj)^i(y-i-bj)^{j-1}. \tag{37}$$

Then we have

$$\mathcal{B}_{m,n}(s,t,x,y;b) = (t+y)^n\left(\sum_{i=0}^{m}\frac{(-1)^i(m)_i}{(y)_{i+1}}(s+x)^{m-i}(y-x)^i\right). \tag{38}$$

Note that $\mathcal{B}_{m,n}(s,t,x,y;b)$ is independent of b. In particular, when $y = -1$, $(y)_{i+1} = (-1)^{i+1}(i+1)!$ Therefore we obtain the closed formula

$$\mathcal{B}_{m,n}(s,t,x,-1;b) = \frac{(t-1)^n}{(1+x)(1+m)}\left((s-1)^{m+1} - (s+x)^{m+1}\right). \tag{39}$$

We skip the details of computation.

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