Chapter 2. Matching theory.

2.1. What is matching theory?

An answer to this question can be found in the survey paper [HR] of L.H. Harper and G.-C. Rota:

'Roughly speaking, matching theory is concerned with the possibility and the number of ways of covering a large, irregularly shaped combinatorial object with several replicas of a given small regularly shaped object, subject usually to the requirement that the small objects shall not overlap and that the small objects be 'lined up' in some sense or other.'

An elementary example of matching theory is the following puzzle (popularized by R. Gomory). Given a standard 8×8 chessboard, can it always be covered by dominoes (a piece consisting of two squares) if one arbitrary black square and one arbitrary white square are deleted? The solution is given in the exercises.

Another puzzle which involves the idea of matchings is the following Putnam problem from 1979. Let n *red* points $r_1, r_2, \ldots, r_n$ and n *blue* points $b_1, b_2, \ldots, b_n$ be given in the Euclidean plane. Show that there exists a permutation π of $\{1, 2, \ldots, n\}$, matching the red points with the blue points, so that no pair of finite line segments $\overline{r_i \ b_{\pi(i)}}$ and $\overline{r_j \ b_{\pi(j)}}$ intersect. Although this puzzle is about matchings, it is perhaps not matching theory in our sense. The reason – a hint for its solution – is that it is really a *geometry* problem. [1]

Our third example is indisputably a theorem in matching theory. [2] This theorem settles the question, often asked when one is first told that left and right cosets of a subgroup may not be the same, whether there exists a set of elements acting as simultaneous left and right coset representatives.

2.1.1. König's theorem. Let S be a set of size mn. Suppose that S is partitioned into m subsets, all having size n, in two ways, $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$. Then there exist m distinct elements $a_1, a_2, \ldots, a_m$ and a permutation π of $\{1, 2, \ldots, m\}$ such that for all i, $a_i \in A_i \cap B_{\pi(i)}$.

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[1] The problem and its solution can be found in P. Winkler, *Mathematical Puzzles,* A.K. Peters, Wellesley MA, 2004.

[2] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, *Math. Ann.* 77 (1916) 453–465. This theorem was cited by Philip Hall, for example, as a motivation for the marriage theorem, in spite of the fact that in this paper, König has also proved the "König-Egerváry theorem", a more general theorem than the marriage theorem. Such historical anomalies occur rather often in matching theory.

Exercises.

2.1.1. (a) The answer to Gomory's puzzle is 'yes'. One way to prove it to find a "rook tour" of the chessboard where one steps from one square to an adjacent square, going through every square exactly once, and ending at the starting square.

(b) Can an 8×8 chessboard be covered by dominoes if two arbitrary black squares and two arbitrary white squares are removed? (Hint. The answer is "almost always" and the exceptions can be described precisely.)

(c) Study Gomory's puzzle for $m \times n$ chessboards.

(d) Develop a theory for domino-coverings of $m \times n$ chessboards with some squares removed. [3]

2.1.2. Give an elementary proof (that is, a proof not using the marriage theorem) of König's theorem.

2.2. The marriage theorem

A subrelation M of a relation $R : S \to X$ is a *partial matching* of R if no two edges (or ordered pairs) in M have an element in common. A partial matching M is a *matching* if $|M| = |S|$, or equivalently, the subrelation $M : S \to T$ is a one-to-one function.

Observe that a relation $R : S \to X$ is determined, up to a permutation of S, by the list of subsets $R(a)$, where a ranges over the set S. This gives another way of looking at matchings. A *transversal* or *system of distinct representatives* of a list $X_1, X_2, \ldots, X_n$ of subsets of a set X is a labeled set of n (distinct) elements $\{a_1, a_2, \ldots, a_n\}$ of X such that for each i, $a_i \in X_i$. Thus, transversals are, more or less, ranges of matchings. A transversal exists for the list $X_1, X_2, \ldots, X_n$ of subsets if and only if the relation $R : \{1, 2, \ldots, n\} \to X$ with $R(i) = X_i$ has a matching.

Suppose a matching M exists in $R : S \to X$. If $A \subseteq S$, then $M(A) \subseteq R(A)$. Since M is a one-to-one function, $|M(A)| = |A|$ and hence, $|R(A)| \geq |A|$. Thus, the condition,

$$\text{for all subsets } A \text{ in } S,\ |R(A)| \geq |A|,$$

is a necessary condition for the existence of a matching. This condition is called the *(Philip) Hall condition*. The fundamental result in matching theory is that the Hall condition is also sufficient.

2.2.1. The marriage theorem. A relation $R : S \to X$ has a matching if and only if for every subset A in S, $|R(A)| \geq |A|$.

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[3] Such a theory is developed in the survey paper [B].

An equivalent way to state the marriage theorem is in terms of transversals of subsets.

2.2.2. Theorem. Let $X_1, X_2, \ldots, X_n$ be a list of subsets of a set X. Then there exists a transversal if and only if for all subsets $A \subseteq \{1, 2, \ldots, n\}$,

$$\left| \bigcup_{i:\, i \in A} X_i \right| \geq |A|.$$

We shall give several proofs, each with a different idea. Since necessity has already been proved, we need only prove sufficiency.

First proof of the marriage theorem. [4] We proceed by induction, observing that the case $|S| = 1$ obviously holds. We distinguish two cases.

Case 1. For every non-empty subset A strictly contained in S,

$$|R(A)| > |A|.$$

Choose any edge (a, x) in the relation R and consider the relation R' obtained by restricting R to the sets $S \backslash \{a\}$ and $X \backslash \{x\}$. If $A \subseteq S \backslash \{a\}$, $|R'(A)|$ equals $|R(A)|$ or $R(A) - 1$, depending on whether x is in $R(A)$. Thus, as $|R(A)| > |A|$, the smaller relation R' satisfies the Hall condition and has a matching by induction. Adding the ordered pair (a, x) to the matching for R', we obtain a matching for R.

Case 2. There is a non-empty subset A strictly contained in S such that $|R(A)| = |A|$. Then the restriction $R|_A : A \to R(A)$ of R to the subsets A and $R(A)$ has a matching by induction.

Let $R'' : S \backslash A \to X \backslash R(A)$ be the relation obtained by restricting R to $S \backslash A$ and $X \backslash R(A)$. Consider a subset B in the complement $S \backslash A$. Then $R(A \cup B)$ is the union of the disjoint sets $R(A)$ and $R''(B)$. Since A and B are disjoint, R satisfies the Hall condition, and $|R(A)| = |A|$,

$$\begin{aligned} |R''(B)| &= |R(A \cup B)| - |R(A)| \\ &\geq |A \cup B| - |A| \\ &= |B|. \end{aligned}$$

Hence, R'' satisfies the Hall condition and has a matching by induction. The matchings for $R|_A$ and R'' are disjoint. Taking their union, we obtain a matching for R. □

[4] T.E. Easterfield, A combinatorial algorithm, *J. London Math. Soc.* 21 (1946) 219–226; P.R. Halmos and H.E. Vaughan, The marriage problem, *Amer. J. Math.* 72 (1950) 214–215. This paper of Halmos and Vaughan popularized the matrimonial interpretation of Theorem 2.1.

The argument in the proof of Easterfield, Halmos and Vaughan can be modified to prove the following theorem of Marshall Hall. [5]

2.2.3. Marshall Hall's theorem. Let $R : S \to X$ be a relation satisfying the Hall condition. Suppose that, in addition, every element in S is related to at least k elements in X. Then R has at least $k!$ matchings.

Proof. We proceed by induction on S and k, the case $|S| = 1$ being obvious.
As in the previous proof, we distinguish two cases.

Case 1. For all non-empty proper subsets A in S, $|R(A)| > |A|$. Choose an element a in S and let $R(a) = \{x_1, x_2, \ldots, x_m\}$. Consider the relations R_i obtained by restricting R to $S\backslash\{a\}$ to $X\backslash\{x_i\}$. In R_i, every element in $S\backslash\{a\}$ is related to at least $k-1$ elements in $X\backslash\{x_i\}$. By induction, the relation R_i has at least $(k-1)!$ matchings. Adding (a, x_i) to any matching of R_i yields a matching of R. Doing this for all the relations R_i, we obtain $m(k-1)!$ matchings. Since $m \geq k$, there are at least $k(k-1)!$ matchings in R.

Case 2. There exists a non-empty proper subset A in S so that $|A| = |R(A)|$. As in the earlier proof, we can break up R into two smaller matchings $R|_A$ and R'', both satisfying the Hall condition. In the relation $R|_A : A \to R(A)$, every element in A is related to at least k elements in $R(A)$. Hence, by induction, $R|_A$ has at least $k!$ matching. Taking the union of a fixed matching of R'' and a matching in $R|_A$, we obtain at least $k!$ matchings for R. □

2.2.4. Corollary. Let $R : S \to X$ be a relation and suppose that every element in S is related to at least k elements in X. Then R has a matching implies that R has at least $k!$ matchings.

Note that an easy argument shows that the number of matchings in $R : S \to X$ is bounded above by $\prod_{i=1}^{n} |R(i)|$, with equality if and only if the sets $R(i)$ are pairwise disjoint.

Second proof of the marriage theorem. [6]
This proof is based on the following lemma, which allows edges to be removed while preserving the Hall condition.

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[5] M. Hall, Distinct representatives of subsets, *Bull. Amer. Math. Soc.* 54 (1948) 922–926. Hall's argument is similar to the one given here. See also Exercise 2.2.3.

[6] R. Rado, Note on the transfinite case of Hall's theorem on representatives, *J. London Math. Soc.* 42 (1967) 321–324.

2.2.5. Rado's Lemma. Let $R : S \to X$ be a relation satisfying the Hall condition. If (c, x) and (c, y) are two edges in R, then at least one of the relations $R\backslash\{(c, x)\}$ and $R\backslash\{(c, y)\}$ satisfies the Hall condition.

Proof. Suppose the lemma is false. Let $R' = R\backslash\{(c, x)\}$ and $R'' = R\backslash\{(c, y)\}$. Then we can find subsets A and B in S such that $c \notin A$, $c \notin B$,

$$|R'(A \cup \{c\})| < |A \cup \{c\}|,$$

and

$$|R''(B \cup \{c\})| < |B \cup \{c\}|.$$

Since $R'(A) = R(A)$ and $R''(B) = R(B)$, it follows that $R'(c) \subseteq R(A)$ and $R''(c) \subseteq R(B)$. We conclude that

$$|R'(A \cup \{c\})| = |R(A)| = |A|$$

and

$$|R''(B \cup \{c\})| = |R(B)| = |B|.$$

Using these two equalities, we have

$$\begin{aligned}
|A| + |B| &= |R'(A \cup \{c\})| + |R''(B \cup \{c\})| \\
&= |R'(A \cup \{c\}) \cup R''(B \cup \{c\})| + |R'(A \cup \{c\}) \cap R''(B \cup \{c\})| \\
&= |R(A \cup B \cup \{c\})| + |R(A) \cap R(B)| \\
&\geq |R(A \cup B \cup \{c\})| + |R(A \cap B)| \\
&\geq |A \cup B \cup \{c\}| + |A \cap B| \\
&\geq |A \cup B| + 1 + |A \cap B| \\
&> |A| + |B| + 1,
\end{aligned}$$

a contradiction. □

We can now prove the marriage theorem by removing edges until we reach a matching.

Third proof of the marriage theorem. [7]

The third proof combines ideas from the first and second proof. This proof begins with an easy lemma, which we state without proof.

2.2.6. Lemma. Let $R : S \to X$ be a relation satisfying the Hall condition. Suppose that A and B are subsets of S such that $|A| = |R(A)|$ and $|B| = |R(B)|$. Then $|A \cup B| = |R(A \cup B)|$ and $|A \cap B| = |R(A \cap B)|$.

Choose an element a in S. Suppose that for some x in $R(a)$, the relation $R_x : S\backslash\{a\} \to X\backslash\{x\}$ obtained by restricting R to $S\backslash\{a\}$ and $X\backslash\{x\}$ satisfies the Hall condition. Then by induction, R_x has a matching and adding (a, x) to that matching, we obtain a matching for R.

[7] C.J. Everett and G. Whaples, Representations of sequences of sets, *Amer. J. Math.* 71 (1949) 287–293.

Thus, we need only deal with the case $(*)$: for all x in $R(a)$, the relation R_x does not satisfy the Hall condition, that is to say, for each x in $R(a)$, there exists a subset B_x of $S\backslash\{a\}$ such that $|B_x| < |R_x(B_x)|$. Since R satisfies the Hall condition, $|R(B_x)| \geq |B_x|$. In addition,

$$R_x(B_x) \subseteq R(B_x) \subseteq R_x(B_x) \cup \{x\}.$$

Thus, $R(B_x) = R(B_x) \cup \{x\}$, $x \in R(B_x)$, and $|R(B_x)| = |R_x(B_x)| + 1$. Since R satisfy the Hall condition, it follows that for all x in $R(a)$,

$$|R(B_x)| = |B_x|.$$

Let

$$B = \bigcup_{x\colon x \in R(a)} B_x.$$

By Lemma 2.2.6, $|B| = |R(B)|$. However, $R(a) \subseteq R(B)$. Together, this implies that

$$|B \cup \{a\}| > |R(B)|$$

contradicting the assumption that R satisfies the Hall condition. Thus, the case $(*)$ is impossible and the proof is complete. □

Fourth proof of the marriage theorem. [8] This is Philip Hall's proof. Let $R : S \to X$ be a relation with at least one matching. Let

$$H(R) = \bigcap_M M(S),$$

where the intersection ranges over all matchings M in R; in other words, $H(R)$ is the subset of elements in X which must occur in the range of a matching of R. The set $H(R)$ may be empty.

2.2.7. Lemma. Assume that $R : S \to X$ has a matching. If $A \subseteq S$ and $|A| = |R(A)|$, then $R(A) \subseteq H(R)$.

Proof. Suppose $|A| = |R(A)|$. Then for any matching M in R, $M(A) = R(A)$. Hence, $R(A)$ occurs as a subset in the range of every matching and, assuming R has a matching, $R(A) \subseteq H(R)$. □

2.2.8. Lemma. Let R have a matching, M be a matching of R and $I = \{a : M(a) \subseteq H(R)\}$. Then $R(I) = H(R)$. In particular, $|H(R)| = |I|$.

Proof. We will use an alternating path argument. [9] An *alternating path (relative to the matching M)* is a sequence $x, a_1, x_1, a_2, \ldots, x_{l-1}, x_l$ such that the edges

$$(x, a_1), (a_1, x_1), (x_1, a_2), (a_2, x_3), \ldots, (a_{l-1}, x_{l-1}), (x_{l-1}, a_l), (a_l, x_l),$$

[8] P. Hall, On representatives of subsets, *J. London Math. Soc.* 10 (1935) 26–20. Although citing the paper of Hall is obligatory, Hall's proof seems not to have appeared in any textbook. For example, Mirsky described it as "comparatively difficult" in [My2]. Part of the blame lies in the expository style of Hall's paper.

[9] Alternating paths first appeared in König's 1916 paper.

are all in R, the edges (a_i, x_i) are in the matching M, and except for the head x and the tail x_l, which may be equal, all the elements in the path are distinct. If x is not in the range $M(S)$, then it is easy to check that the matching M^Δ defined by

$$M^\Delta = (M \backslash \{(a_1, x_1), (a_2, x_2), \ldots, (a_l, x_l)\}) \cup \{(x, a_1), (x_1, a_2), (x_2, a_3), \ldots, (x_{l-1}, a_l)\},$$

is a matching of R and x_l is not in the range $M^\Delta(S)$.

Let $H'(M)$ be the subset of elements x in X connected by an alternating path to an element y in $H(R)$. Then $H(R) \subseteq H'(M)$. We shall show that in fact,

$$H(R) = H'(M).$$

To do this, we first show that $H'(M) \subseteq M(S)$. Suppose that $x \in H'(M)$ but $x \notin M(S)$. Then y is not in the range of the new matching M^Δ defined by the alternating path from x to y, contradicting the assumption that $y \in H(R)$.

Next, let $J = \{a : M(a) \subseteq H'(M)\}$. If $a \in J$ and $M(a) = \{x\}$, then there is an alternating path P starting with an edge (x, b), where $b \in S$ and $b \neq a$, and ending at a vertex in $H(R)$. If $y \in R(a)$ and $y \neq x$, then (y, a) is an edge in R, (a, x) is an edge in M, and y, a, P is an alternating path ending at a vertex in $H(R)$. We conclude that if $a \in J$, then $R(a) \subseteq H'(M)$. In particular, $R(J) \subseteq H'(M)$. Thus, we have

$$M(J) \subseteq R(J) \subseteq H'(M) = M(J).$$

From this, it follows that $|J| = |R(J)|$. By Lemma 2.2.7, $H'(M) \subseteq H(R)$. We conclude that $H(R) = H'(M)$. In particular, $I = J$ and $H(R) = R(I)$. □

We now proceed by induction on n. Let $R : S \to X$ be a relation satisfying the Hall condition and let $b \in S$. Since the Hall condition holds for the restriction $R|_{S\backslash\{b\}} : S\backslash\{b\} \to X$, the smaller relation $R|_{S\backslash\{b\}}$ has a matching M by induction. Let $I = \{a : M(a) \subseteq H(R|_{S\backslash\{b\}})\}$. If $R(b) \subseteq H(R|_{S\backslash\{b\}})$, then

$$R(I \cup \{b\}) = H(R|_{S\backslash\{b\}}),$$

and $|R(I \cup \{b\})| = |I| < |I| + 1$, contradicting the assumption that the Hall condition holds. Thus it must be the case that $R(b)$ is not contained in $H(R|_{S\backslash\{b\}})$. For any element x in $R(a)$ not in $H(R|_{S\backslash\{b\}})$, the union $M \cup \{(a, x)\}$ is a matching for R. □

We end this section with two easy but useful extensions of the marriage theorem.

2.2.9. Theorem. [10] A relation $R : S \to X$ has a partial matching of size $|S| - d$ if and only if R satisfies the *Ore condition*:

for every subset A in S, $|R(A)| \geq |A| - d$.

[10] O. Ore, Graphs and matching theorems, *Duke Math. J.* 22 (1955), 625–639; see Exercise 2.4.4 for Ore's proof.

Proof. Let D be a set of size d disjoint from T and $R' : S \to X \cup D$ be the relation $R \cup (S \times D)$. Then it is easy to check the following statements hold:

- R' satisfies the Hall condition if and only if R satisfies the Hall condition.
- R' has a matching if and only if R has a partial matching of size $|S| - d$.

We can now complete the proof using the marriage theorem. □

Restating Theorem 2.2.9, we obtain a result of Ore.

2.2.10. Defect form of the marriage theorem. Let $R : S \to X$ be a relation and τ be the maximum size of a partial matching in R. Then

$$\tau = |S| + \min\{|R(A)| - |A| : A \subseteq S\}.$$

Exercises.

2.2.1. *The Hungarian method.* [11]

Use alternating paths (defined in the fourth proof) to obtain a proof of the marriage theorem. (Comment. This has become the standard proof in introductory books. Among many advantages, alternating paths give a polynomial-time algorithm for finding a maximum size partial matching. See [LP].)

2.2.2. *Cosets of groups and König's theorem.* [12]

(a) Let H be a subgroup of a group G such that the index $|G|/|H|$ is finite. Then there are coset representatives which are simultaneously right and left coset representatives. (Hint. Let $R_1, R_2, \ldots, R_i$ be the right cosets and $L_1, L_2, \ldots, L_i$ be the left cosets of H in G. Define the relation $I : \{R_1, R_2, \ldots, R_i\} \to \{L_1, L_2, \ldots, L_i\}$ by the condition:

$$(R_j, L_k) \in I \text{ whenever } R_j \cap L_k \neq \emptyset.$$

Using the fact that the collections $\{R_j\}$ and $\{L_k\}$ of cosets partition G, show that I satisfies the Hall condition.

(b) Use the method in (a) to prove König's theorem in Section 2.1.

(c) Conclude that if H and K are subgroups with the same finite index, then there exist one system which is a system of coset representives for both H and K.

(d) Let $S_1, S_2, \ldots, S_m$ and $T_1, T_2, \ldots, T_m$ be two partitions of a set S. Find nice conditions for the conclusion in König's theorem to hold.

2.2.3. *Regular relations.*

Let $R : S \to X$ be a relation in which every element a in S is related to the same number k of elements in X and every element in X is related to the same number l in S. Show that the Hall condition holds and hence, R contains a matching.

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[11] H.W. Kuhn, The Hungarian method for the assignment problem, *Naval Res. Logistics Quart.* 2 (1955) 83–97. The name is chosen in honor of D. König.

[12] O. Ore, On coset representatives in groups, *Proc. Amer. Math. Soc.* 9 (1958) 665–670.

2.2.4. *Extending Latin rectangles.* [13]

Let $\mathcal{A}$ be an alphabet with n letters. If $1 \leq r, s \leq n$, an $r \times s$ array with entries from $\mathcal{A}$ is a *Latin rectangle* if each letter occurs at most once in any row or column. An $n \times n$ Latin rectangle is called a *Latin square*.

(a) Show that if $m \leq n-1$, every $m \times n$ Latin rectangle can be extended to an $(m+1) \times n$ Latin rectangle. In particular, every $m \times n$ Latin rectangle can be extended to a Latin square.

(b) Use Theorem 2.2.2 to show that there exists at least

$$n!(n-1)!(n-2)!\cdots(n-m+1)!$$

$m \times n$ Latin rectangles.

(c) Let $1 \leq r, s \leq n$. Show that an $r \times s$ Latin rectangle L can be extended to an Latin square if and only if every letter in $\mathcal{A}$ occurs at least $r+s-n$ times in L.

2.2.5. *The number of matchings.* [14]

Let $R : \{1, 2, \ldots, n\} \to X$ be a relation such that $|R(i)| \leq |R(j)|$ whenever $i < j$. (This condition can always be achieved by rearranging $\{1, 2, \ldots, n\}$.) Show that if R has at least one matching, then the number of matchings in R is at least

$$\prod_{i=1}^{n} \max(1, |R(i)| - i + 1),$$

with equality holding if $R(i) \subseteq R(j)$ whenever $i < j$.

2.2.6. *Easterfield's problem.*

Find all possible ways of obtaining the minimum sum of n entries of a given $n \times n$ matrix of positive integers, with no two entries from the same row or column. (Comment. Easterfield gives an algorithm for doing this. In the discussion of this algorithm, he discovered the marriage theorem. There may be a more efficient algorithm than Easterfield's.)

2.3. Free and incidence matrices

Let $R : S \to X$ be a relation. A matrix (c_{ax}) with row set S and column set X is *supported by* the relation R if the entry c_{ax} is non-zero if and only if $(a, x) \in R$.

There are several ways to choose the non-zero entries. One way is to create independent variables (or elements transcendental over some ground field) $X_{a,x}$, one for each edge (a, x) in the relation R, and set $c_{a,x} = X_{a,x}$. This choice of

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[13] Marshall Hall, paper cited earlier; H.J. Ryser, A combinatorial theorem with an application to Latin rectangles, *Proc. Amer. Math. Soc.* 2 (1951) 550–552. The bound given in (b) for the number of $m \times n$ Latin rectangles is far from sharp. For better bounds, see M.R. Murty and Y.-R. Liu, Sieve methods in combinatorics, *J. Combin. Theory Ser. A* 111 (2005) 1–23, and the references cited in that paper.

[14] R. Rado, On the number of systems of distinct representatives of sets, *J. London Math. Soc.* 42 (1967) 107–109; P.A. Ostrand, Systems of distinct representatives. II, *J. Math. Anal. Appl.* 32 (1970) 1–4.

non-zero entries gives the *free* or *generic matrix* of the relation R. At the other extreme, one can set $c_{a,x} = 1$. This gives the $(0,1)$-*incidence matrix* of R.

We begin by considering determinants of free matrices.

2.3.1. Lemma. Let $R : S \to X$ be a relation with $|S| = |X|$.

(a) If C is a matrix supported by R, then $\det C \neq 0$ implies that R has a matching.

(b) If C is the free matrix of R, then the number of matchings in R equals the number of non-zero monomials in the expansion of the determinant of the free matrix C of R. In particular, $\det C \neq 0$ if and only if R has a matching.

Proof. Expanding the determinant of C, we have

$$\det C = \sum_{\sigma} \prod_{a:\, a \in S} c_{a,\sigma(a)},$$

where the sum ranges over all bijections $\sigma : S \to X$. If $\det C \neq 0$, then there is a non-zero product in the expansion. The bijection σ giving that product is a matching of R. This proves Part (a). Part (b) follows since independent variables have no algebraic relations. $\square$

Since the determinant of a square matrix is non-zero if and only if it has full rank, we have the following corollary.

2.3.2. Corollary. Let $R : S \to X$ be a relation and C be its free matrix. Then the following are equivalent:

(a) R has a matching of size $|S| - d$.

(b) The free matrix C has rank $|S| - d$.

(c) There are subsets $S' \subseteq S$ and $X' \subseteq X$, both having size $|S| - d$, such that the $(|S| - d) \times (|S| - d)$ square submatrix $C[S'|X']$ formed by restricting C to the rows in S' and the columns in X' has non-zero determinant.

Using Lemma 2.3.1 and its corollary, we can reformulate the marriage theorem as a result about the rank of free matrices.

2.3.3. Edmonds' theorem. [15] Let $m \leq n$ and C be a free $m \times n$ matrix of a relation R. Then C has rank strictly less than m if and only if there exists a set H of h rows such that the submatrix of C consisting of the rows in H has $h - 1$ or fewer non-zero columns.

Proof. Suppose that C contains a $h \times (n - h + 1)$ submatrix with all entries zero. Then every square $m \times m$ submatrix C' of C contains a $h \times (m - h + 1)$ zero submatrix. In the expansion of the determinant of C', every product $\prod c_{a,\sigma(a)}$ has at least one term $c_{b,\sigma(b)}$ in the $h \times (m - h + 1)$ zero submatrix. Thus, the

[15] J. Edmonds, Systems of distinct representatives and linear algebra, *J. Res. Nat. Bur. Standards* 71B (1967) 241–245.

determinant of every square $m \times m$ submatrix is zero, and C has rank strictly less than m.

Now suppose that C has rank strictly less than m. Then its rows are linearly dependent. Let H be a minimal linearly dependent set of rows. Relabeling, we may assume that $H = \{1, 2, \ldots, h\}$. Let $C[H]$ be the $h \times n$ submatrix consisting of the rows in H. Since H is a minimal linearly dependent set, the submatrix $C[H]$ has rank $h-1$ and there exists a set K of $r-1$ columns such that the $h \times h-1$ submatrix $C[H|K]$, obtained by restricting C to the rows in H and columns in K, has rank $h-1$. By relabeling, we may assume that $K = \{1, 2, \ldots, r-1\}$.

We shall show that if a column in $C[H]$ is not in K, then every entry in it is zero. Consider the $h \times h$ square matrix obtained by adding a column l in $C[H]$ not in C to $C[H|K]$. Since $C[H]$ has rank $h-1$, the square matrix is singular and its determinant equals zero. Taking the Laplace expansion along the added column, we obtain

$$\sum_{i=1}^{h} (-1)^i \det C[H \backslash \{i\}|K] c_{i,l} = 0. \qquad \text{(Det)}$$

As H is a minimal linearly dependent set of rows, *every* subdeterminant in the left hand sum in the equation is non-zero. Hence, each subdeterminant is a non-zero polynomial in the variables X_{ij} with $1 \leq j \leq r-1$. If $l \geq r$ and $c_{i,l}$ is non-zero for some i, then Equation (Det) gives a non-trivial polynomial relation amongst the variables $X_{ij}, 1 \leq j \leq r-1$ and the variables X_{il}, $(i,l) \in R$, contradicting the declaration that the variables X_{ij} are independent. We conclude that every entry in every column in $C[H]$ but not in K is zero. □

Note that by Corollary 2.3.2, Edmonds' theorem is equivalent to the marriage theorem.

Let C is a matrix be a matrix on row set S and column set X. A pair (A, Y), where $A \subseteq S$ and $Y \subseteq X$, *covers* C if for every non-zero entry $c_{a,x}$ in C, $a \in A$ or $x \in Y$.

2.3.3. The König-Egerváry theorem. Let C be a matrix supported by the relation R and let τ be the maximum size of a partial matching in R. Then

$$\tau = \min\{|A| + |Y| : (A, Y) \text{ covers } C\}.$$

In particular, if C is the free matrix of R, then

$$\mathrm{rk}(C) = \min\{|A| + |Y| : (A, Y) \text{ covers } C\}.$$

Proof. Let (A, Y) be a cover of C. If (a, x) is an edge in a partial matching, then $a \in A$, $x \in Y$, or both. Hence, $|A| + |Y| \geq \tau$.

On the other hand, $(S \backslash A, R(A))$ is a cover of C. Hence, by the defect form of the marriage theorem (2.2.10),

$$\begin{aligned} \tau &= \min\{|R(A)| + (|S| - |A|) : A \subseteq S\}. \\ &\geq \min\{|Y| + |A| : (A, Y) \text{ covers } C\}. \end{aligned}$$

□

Other proofs will be given in the exercises. The König-Egerváry theorem says for a rank-m free matrix C, we can permute the rows and columns so that M has the following form:

$$\begin{pmatrix} ? & ? & ? & ? & ? & \bullet & ? & \dots & ? \\ ? & & & ? & \bullet & ? & & & ? \\ ? & & ? & \bullet & 0 & 0 & 0 & \dots & 0 \\ ? & ? & \bullet & ? & 0 & & & & 0 \\ ? & \bullet & ? & ? & 0 & & & & 0 \\ \bullet & ? & & ? & 0 & & & & 0 \\ ? & & & ? & 0 & & & & 0 \\ \vdots & & & \vdots & \vdots & & & & \vdots \\ ? & ? & ? & ? & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where the entries $c_{m,1}, c_{m-1,2}, \dots, c_{1,m}$ in the sub-antidiagonal are non-zero and there are non-negative integers a and b, $a + b = m$, such that all the non-zero entries in C lie in the union of the two rectangles $\{1, 2, \dots, a\} \times \{1, 2, \dots, n\}$ and $\{1, 2, \dots, m\} \times \{1, 2, \dots, b\}$. In the example above, $m = 6$, $a = 2$, $b = 4$, a "•" is a non-zero entry (in a maximum size partial matching), a "?" is an entry which may be zero or non-zero, and a "0" is, naturally, a zero entry.

Exercises.

2.3.1. *Graph-theoretic form of the König-Egerváry theorem.*

Let G be a graph. A *partial matching* M in G is a subset of edges, with no two edges in M sharing a vertex. A *vertex cover* C is a subset of vertices such that every edge is incident on at least one vertex in C. Show that the König-Egerváry theorem is equivalent to *König's minimax theorem*: In a bipartite graph G, the maximum size of a partial matching equals the minimum size of a vertex cover. Find graph-theoretic proofs of König's minimax theorem. (Hint. One way is to show that a bipartite graph minimal under edge-deletion with respect to having a partial matching of size τ is a graph consisting of τ disjoint edges and isolated vertices. [16] Other proofs can be found in [LP].)

2.3.2. *Bapat's extension of the König-Egervàry theorem.* [17]

Let M be a matrix with rows labelled by S and columns labelled by T. If $A \subseteq S$ and $B \subseteq T$, then let $M[B|A]$ be the submatrix labelled by rows in A and columns in A. A submatrix $M[B|A]$ has *zero type* if for every $b \in B$ and $a \in A$,

$$\operatorname{rank} M[B|A] = \operatorname{rank} M[B\backslash\{b\}|A\backslash\{a\}].$$

[16]L. Lovász, Three short proofs in graph theory, *J. Combin. Theory Ser. B* 19 (1975) 95–98.

[17]R.B. Bapat, König's theorem and bimatroids, *Linear Alg. Appl.* 212/213 (1994) 353–365.

A submatrix with all entries zero has zero type. Another example is $2 \times n$ matrix (with $n > 1$) where the second row is a non-zero multiple of the first.

(a) Show that if $\operatorname{rank} M[T|S] < \min\{|T|, |S|\}$, then then there exists a submatrix $M[B|A]$ of zero type such that

$$|T| + |S| - \operatorname{rank} M[T|S] = |B| + |A| - \operatorname{rank} M[B|A].$$

(Hint. Use induction and the matirx submodular inequality in Section 2.8.)

(b) Show that if M is the free matrix of a relation, then a submatrix N has zero type if and only if every entry in N is zero. Hence, deduce the König-Egerváry theorem from Bapat's theorem.

2.3.3. *Frobenius' irreducibility theorem.* [18]

Let M be the $n \times n$ square free matrix of the complete relation $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. A *general matrix functions* $G(X_{ij})$ over the field $\mathbb{F}$ is a polynomial of the form

$$\sum_{\pi} c_\pi X_{1,\pi(1)} X_{2,\pi(2)} \cdots X_{n,\pi(n)},$$

the sum ranging over all permutations π of $\{1, 2, \ldots, n\}$ and c_π is in $\mathbb{F}$.

(a) Show that a general matrix function factors in the form

$$G = PQ,$$

where P and Q has positive degrees m and $n - m$, if and only if there are square submatrices M_1 and M_2 of size m and $n - m$ and P and Q are general matrix functions of M_1 and M_2. (Hint. It is easier to prove a more general result first. Let $p(X_1, X_2, \ldots, X_n)$ be a *multilinear* polynomial, that is, each variable X_i occurs at most once in each monomial in $p(X_1, X_2, \ldots, X_n)$. Then $p = p_1 p_2$ if and only if there is a partition of $\{1, 2, \ldots, n\}$ into two subsets A and B such that p_1 is a multilinear polynomial in the variables X_i, $i \in A$, and p_2 is a multilinear polynomial in the variables X_i, $i \in B$.)

A matrix M is said to be *decomposable* or *reducible* if there exist permutations of the row and column sets so that

$$M = \begin{pmatrix} M_1 & U \\ 0 & M_2 \end{pmatrix},$$

where the matrix in the lower left corner is a zero matrix with at least one row and one column.

(b) Let M be a $n \times n$ square free matrix (of an arbitrary relation). Then the determinant of M factors non-trivially if and only if M is decomposable.

2.3.4. *Converting determinants to permanents.* [19]

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[18] G. Frobenius, Über zerlegbare Determinanten, *Sitzungsber. Preuss. Akad. Wiss. Berlin 1917* 274–277. Frobenius proved his theorem for determinants. The more general form stated here is from Brualdi-Ryser [BR], p. 295. A good account of the Frobenius' work on combinatorial matrix theory can be found in H. Schneider, The concepts of irreducibility and full indecomposability of a matrix in the works of Frobenius, König and Markov, *Linear Algebra Appl.* 18 (1977) 139–162.

[19] P.M. Gibson, Conversion of the permanent into the determinant, *Proc. Amer. Math. Soc.* 27 (1971) 471–476.

Let (a_{ij}) be a $(0,1)$-matrix such that $\operatorname{per}(a_{ij}) > 0$. If there exists a matrix (b_{ij}) such that $b_{ij} = \epsilon_{ij} a_{ij}$, where ϵ_{ij} equals -1 or 1, and $\operatorname{per} A = \det B$, then (a_{ij}) has at most $(n^2 + 3n - 2)/2$ non-zero entries.

2.3.5. *The bipartite graph case of Kasteleyn's theorem.* If the bipartite graph associated with the relation is planar, then there exists a assignment of signs to the incidence matrix so that the determinant of the incidence matrix equals the number of matchings. [20]

2.4. Submodular functions and independent matchings

Let S be a set and $\rho : 2^S \to \mathbb{R}$ be a function from the subsets of S to the real numbers. The function ρ is *submodular* if for all subsets A and B of S,

$$\rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B).$$

Note that if $A \subseteq B$ or $B \subseteq A$, then the inequality holds trivially as an equality. The function ρ is *increasing* if $\rho(B) \leq \rho(A)$ whenever $B \subseteq A$. Submodular functions need not be increasing. (There is an example on a set of size 2.)

Submodular functions occur naturally from valuations when the intersection is smaller than expected. For example, let $R : S \to X$ be a relation. Then,

$$R(A \cap B) \subseteq R(A) \cap R(B)$$

and it is possible that strict containment occurs. Thus, the function

$$\rho : 2^S \to \{0, 1, 2, \ldots\}, \; A \mapsto |R(A)|$$

is not a valuation in general. However, it is a submodular function.

Another example arises from matrices. Let M be a matrix with column set S. The *(column) rank function* rk is the function defined on S sending a subset A in S to the rank of the submatrix of M formed by the columns in A.

2.4.1. Lemma. Let M be a matrix with column set S. Then the column rank function rk is a submodular function.

Proof. We shall use *Grassmann's identity* from linear algebra. If U and V are subspaces of a vector space, then

$$\dim(U) + \dim(V) = \dim(U \vee V) + \dim(U \cap V),$$

where $U \vee V$ is the subspace spanned by the vectors in the union $U \cup V$. Since $\operatorname{rk}(A)$ equals the dimension of the subspace spanned by the column vectors in A, we have

$$\operatorname{rk}(A) + \operatorname{rk}(B) = \operatorname{rk}(A \cup B) + \dim(\overline{A} \cap \overline{B}),$$

where $\overline{A}$ and $\overline{B}$ are the subspaces spanned by A and B. If a spanning set of the subspace intersection $\overline{A} \cap \overline{B}$ happens to be in the set intersection $A \cap B$, then

$$\operatorname{rk}(A) + \operatorname{rk}(B) = \operatorname{rk}(A \cup B) + \operatorname{rk}(A \cap B).$$

[20] P.W. Kasteleyn, Dimer statistics and phase transitions, *J. Math. Physics* 4 (1963) 287–293.

However, in general, we only have the submodular inequality

$$\mathrm{rk}(A) + \mathrm{rk}(B) \geq \mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B).$$

□

A matrix rank function rk satisfies two additional conditions:

Normalization. $\mathrm{rk}(\emptyset) = 0$.

Unit increase. For a subset A and an element a in X,

$$\mathrm{rk}(A) \leq \mathrm{rk}(A \cup \{a\}) \leq \mathrm{rk}(A) + 1.$$

In his 1935 paper "On the abstract properties of linear dependence," [21] H. Whitney defined a *(matroid) rank function* to be an integer-valued normalized submodular function satisfying the unit-increase condition. A matroid rank function defines a *matroid* on the set X. Whitney's intuition was that the three axioms for a matroid rank function captures all the combinatorics or "abstract properties" of rank functions of matrices. His intuition was confirmed by several independent rediscovery of the matroid axioms. Many concepts and results in elementary linear algebra extend to matroids. For example, one can extend the notion of linear independence by defining a subset I to be *independent* (relative to the rank function rk) if $|I| = \mathrm{rk}(I)$.

Let $R : S \to X$ be a relation, let C be its free matrix, and rk the matrix rank function on the columns in C. Then by Lemma 2.3.2, if $Y \subseteq X$, $\mathrm{rk}(Y)$ is the maximum size of a partial matching in the relation $R_Y : S \to Y$ obtained by restricting R to Y, so that $R_Y(a) = R(a) \cap Y$. Thus, by the defect form of the marriage theorem (2.2.10),

$$\mathrm{rk}(Y) = |S| - \min\{|R(A) \cap Y| - |A| : A \subseteq S\}. \tag{TM}$$

The matroid defined in this way is the *transversal matroid defined by* R on X.

There are two operations on matroids, restriction and contraction. Let $\mathrm{rk} : 2^X \to \{0, 1, 2, \ldots\}$ be a matroid rank function on X. If $Y \subseteq X$, then the *restriction* of rk to Y is the restriction $\mathrm{rk} : 2^Y \to \{0, 1, 2, \ldots\}$ of rk to the subsets in Y. If $Z \subseteq X$, then the rank function rk_Z obtained by *contracting* Z is the function $2^{X \backslash Z} \to \{0, 1, 2, \ldots\}$ defined as follows: for $B \subseteq X \backslash Z$,

$$\mathrm{rk}_Z(B) = \mathrm{rk}(B \cup Z) - \mathrm{rk}(Z).$$

It is routine to show that rk_Z is a matroid rank function.

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[21] H. Whitney, On the abstract properties of linear dependence, *Amer. J. Math.* 57 (1935) 509–533. There are many ways to do matroid theory. Three different accounts of matroids can be found in H.H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory. Combinatorial Geometries,* Preliminary edition, M.I.T. Press, Cambridge MA, 1970, J.G. Oxley, *Matroid Theory,* Oxford University Press, Oxford 1992, J.P.S. Kung, Matroid theory, in *Handbook of Algebra, Vol. 1*, North-Holland, Amsterdam, pp. 157-184. There are strong connections between matching theory and matroids. See, for example, Chapter 12 of Oxley's book.

A partial matching M of $R : S \to X$ is *independent* if its range $M(S)$ is an independent set. An independent matching is an independent partial matching of size $|S|$.

2.4.3. The marriage theorem for matroids. [22] Let $R : S \to X$ be a relation and X be equipped with a matroid rank function rk. Then R has a independent matching if and only if R satisfies the Rado-Hall condition:

$$\text{for all subsets } A \text{ in } S,\ \text{rk}(R(A)) \geq |A|.$$

Proof. As in the marriage theorem, necessity is clear. To prove sufficiency, we use a modification of the argument of Easterfield, Halmos and Vaughan (see Section 1). We proceed by induction, observing that the theorem holds for $|S| = 1$.

Case 1. For every non-empty proper subset A of S,

$$\text{rk}(R(A)) > |A|.$$

Choose an element a in S. Since $\text{rk}(R(a)) > 1$, there exists an element x in $R(a)$ such that $\text{rk}(\{x\}) = 1$. Consider the relation R' obtained by restricting R to $S\backslash\{a\} \to X\backslash\{x\}$ and let $X\backslash\{x\}$ be equipped with the contraction rank function $\text{rk}_{\{x\}}$. By induction, the relation R' has an independent matching M' relative to the contraction rank function $\text{rk}_{\{x\}}$. Adding the edge (a, x) to M' yields a matching M of R. The matching M is independent because

$$\begin{aligned} \text{rk}(M(S)) &= \text{rk}(M(S\backslash\{x\}) \cup \{x\}) \\ &= \text{rk}_{\{x\}}(M(S\backslash\{x\})) + 1 \\ &= (|S| - 1) + 1. \end{aligned}$$

Case 2. There is a proper non-empty subset A such that

$$\text{rk}(R(A)) = |A|.$$

Consider the restriction $R|_A : A \to R(A)$ equipped with the restriction of rk to the subset $R(A)$ of X. This restriction satisfy the Rado-Hall condition and, by induction, $R|_A$ has an independent matching $M|_A$ of size $|A|$. Its range is $R(A)$.

Let $R'' : S\backslash A \to X\backslash R(A)$ be the restriction of R to the sets indicated, equipped with the contraction rank function $\text{rk}_{R(A)}$ defined by

$$\text{rk}_{R(A)}(Y) = \text{rk}(Y \cup R(A)) - \text{rk}(R(A)),$$

when Y is a subset of $X\backslash R(A)$. Consider a subset B in $S\backslash A$. Then, as in the Halmos-Vaughan proof,

$$R(A \cup B) = R''(B) \cup R(A).$$

[22] R. Rado, A theorem on independence relations, *Quart. J. Math. Oxford* 13 (1942) 83–89.

Since R satisfies the Hall-Rado condition and $\mathrm{rk}(R(A)) = |A|$, we have

$$\begin{aligned} \mathrm{rk}_{R(A)}(R''(B)) &= \mathrm{rk}(R''(B) \cup R(A)) - \mathrm{rk}(R(A)) \\ &= \mathrm{rk}(R(A \cup B)) - \mathrm{rk}(R(A)) \\ &\geq |A \cup B| - |A| \\ &= |B|. \end{aligned}$$

Hence, R'' satisfies the Hall-Rado condition and has a matching M'' by induction. Taking the union of the matchings $M|_A$ and M'', we obtain a matching for R. This is an independent matching because

$$\begin{aligned} |S \backslash A| &= \mathrm{rk}_{R(A)}(M''(S \backslash A)) \\ &= \mathrm{rk}(M''(S \backslash A) \cup R(A)) - |A|, \end{aligned}$$

and hence,

$$\mathrm{rk}(M''(S \backslash A) \cup M|_A(A)) = |S \backslash A| + |A| = |S|.$$

□

2.4.4. Corollary. Let $R : S \to X$ be a relation and X be equipped with a matroid rank function rk. Then R has a independent partial matching of size $|S| - d$ if and only if for all subsets A in S, $\mathrm{rk}(R(A)) \geq |A| - d$. In particular, the maximum size of an independent partial matching equals

$$|S| + \min\{\mathrm{rk}(R(A)) - |A| : A \subseteq S\}.$$

Sketch of Proof. Apply Theorem 2.4.3 to the relation $R' : S \to X \cup D$, where D is a set of size d disjoint from S, $P' = R \cup (S \times D)$, and $X \cup D$ is equipped with the rank function: if $A \subseteq X \cup D$, then

$$\mathrm{rk}(A) = \mathrm{rk}(A \cap S) + |A \cap D|.$$

(For matroid theorists, the elements of D are added as isthmuses or coloops.) □

Two relations $P : S \to X$ and $Q : T \to X$ have a *common partial transversal of size* τ if there is a subset Y of size τ in X such that both the relations $P' : S \to Y$ and $Q' : T \to Y$ (obtained by restricting the relations P and Q to Y) have a partial matching of size τ. If $|S| = |T|$ and P and Q has a partial matching of size $|S|$, then P and Q have a *common transversal.*

2.4.4. The Ford-Fulkerson common transversal theorem. [23] The relations $P : S \to X$ and $Q : T \to Y$ have a common partial transversal of size τ if and only if for all pairs $A \subseteq S$ and $B \subseteq T$,

$$|P(A) \cap Q(B)| \geq |A| + |B| + \tau - |S| - |T|.$$

[23] L.R. Ford and D.R. Fulkerson, Network flows and systems of representatives, *Canad. J. Math.* 10 (1958, 78–85. Ford and Fulkerson give a proof using the maximum-flow minimum-cut theorem. Our proof is from L. Mirsky and H. Perfect, Applications of the notion of independence to problems of combinatorial analysis, *J. Combinatorial Theory* 2 (1967) 327–357.

In particular,

$$\tau = |S| + |T| + \min\{|P(A) \cap Q(B)| - |A| - |B|\}. \qquad \text{(FF)}$$

Proof. It suffices to prove (FF). Let rk be the rank function of the transversal matroid on X defined by the relation Q. Then a common partial transversal of size τ exists if and only if an independent partial matching of size τ exists for P relative to rk. By Rado's theorem, this occurs if and only if for all subsets $A \subseteq S$,

$$\mathrm{rk}(P(A)) \geq |A| - (|S| - \tau).$$

To obtain equation (FF), use the fact that, by (TM),

$$\mathrm{rk}(P(A)) = |T| - \min\{|Q(B) \cap P(A)| - |B| : B \subseteq T\}.$$

□

Exercises.

2.4.1. *Submodular functions and matrodis.*

Prove the following results.

(a) If ρ and σ are submodular functions and α and β are non-negative real numbers, then $\alpha\rho + \beta\sigma$ is also a submodular functions.

(b) Let $\rho : 2^X \to \mathbb{Z}$ be an integer-valued, increasing, submodular function. Then the subsets I such that for all non-empty subsets $J \subseteq I$, $|J| \leq \rho(J)$, are the independent sets of a matroid $M(\rho)$ on X.

(c) If ρ is assumed to be normalized, that is, $\rho(\emptyset) = 0$, then the matroid rank function rk of $M(\rho)$ is given by

$$\mathrm{rk}(A) = \min\{\rho(B) + |A \backslash B| : B \subseteq A\}.$$

(d) Let $R : S \to X$ be a relation and ρ the submodular function on S defined by $A \mapsto |R(A)|$. Characterize such submodular functions ρ and the matroids $M(\rho)$ constructed from them.

(Hint: Parts (b) and (c) are difficult theorems and requires some matroid theory. They are due to J. Edmonds and G.-C. Rota. An account can be found in the books by Crapo and Rota, and Oxley cited earlier. The theory of submodular functions and their polyhedra can be viewed as a generalization of matching theory. A key paper is J. Edmonds, Submodular functions, matroids, and certain polyhedra, In *Combinatorial Structures and their Applications,* R. Guy and others, eds., Gordon and Breach, New York, 1970, 69–87.

2.4.2. *Welsh's version of the matroid marriage theorem.* [24]

Adapt Rado's proof (the second proof in Section 2.2) of the marriage theorem to the matroid case. This will prove the following more general form of the matroid marriage theorem. Let $R : S \to X$ and let ρ be a non-negative, integer-valued, increasing, submodular function defined on X. Then there exists

[24] D.J.A. Welsh, Generalized versions of Hall's theorem, *J. Combin. Theory Ser. B* 10 (1971) 95–101.

a matching M in R with $\rho(M(T)) \geq |T|$ for all subsets T in S if and only if $\rho(R(T)) \geq |T|$ for all subsets T in S.

2.4.3. Adapt other proofs of the marriage theorem to the matroid case. Is there a matroid analog of Theorem 2.2.2 ?

2.4.4. *Contractions and Gaussian elimination.*

Suppose that rk is the rank function of a matrix M with column set X and a a non-zero column of M. Construct the matrix M' as follows. Choose a row index i such that the ia-entry is non-zero. Subtracting a suitable multiple of row i from the other rows, reduce the matrix M so that the only non-zero entry in column a is the ia-entry. Let M' be the matrix obtained from the reduced matrix by deleting row i and column a. Show that the rank function $\mathrm{rk}_{\{a\}}$ obtained by contracting $\{a\}$ is the rank function on the matrix M'.

2.4.5. Prove the matroid marriage theorem with rk a matrix rank function using the Binet-Cauchy formula for determinants and the idea in Edmonds' proof.

2.4.5. *Common transversals, compositions of relations, and matrix products.*

(a) Let $P : S \to X$ and $Q : S \to Y$ be relations. Show that P and Q have a common partial transversal of size τ if and only if the composition $PQ^{-1} : Y \to X$ has a partial matching of size τ.

(b) Let C and D be the free matrices of P and Q. Then P and Q have a common partial transversal of size τ if and only if the matrix product $D^T C$ has rank τ. (Hint: Use the Binet-Cauchy theorem for determinants. Note that the $D^T C$ is not the free matrix of PQ^{-1}.)

(b) Find an analog of the König-Egerváry theorem for the product of two free matrices?

2.4.6. *Ore's excess function.* [25]

Let $R : S \to X$ be a relation. The *excess function* η is defined by

$$\eta(A) = |R(A)| - |A|.$$

(a) Prove that η is a submodular function.

(b) Observe that the Hall condition are equivalent to $\eta(A) \geq 0$ for every subset A in S.

(c) let $\eta_0(R) = \min\{\eta(A) : A \subseteq S\}$. Note that $\eta_0(R)$ may be negative. Define a subset A of S to be *critical* if $\eta(A) = \eta_0$. Show that if A and B are critical, then $A \cup B$ and $A \cap B$ are also critical.

(d) Define the *core* of the relation R to be the intersection of all the critical subsets of S, or, equivalently, the minimum critical subset. Let C be the core of R. Show (without the assumption that R satisfies the Hall condition) that the restriction $R : S \backslash C \to X$ of R to the complement of C always satisfies the Hall condition.

(e) Every element x in $R(C)$ is related to at least two elements of C.

[25] O. Ore, Graphs and matching theorems, *Duke Math. J.* 22 (1955), 625–639; Theory of Graphs, Amer. Math. Soc., Providence RI, 1962, Chapter 10.

(f) Let $a \in S$ and let R' be the relation obtained from R by removing a and all ordered pairs (a, x) in R containing a. Show that

$$\eta_0(R) \leq \eta_0(R') \leq \eta_0(R) + 1.$$

Indeed, show that if $a \notin C$, then $\eta_0(R') = \eta_0(R)$ and if $a \in C$, then $\eta_0(R') = \eta_0(R) + 1$.

(g) Show that if $\eta_0(R) < 0$, then there exist $-\eta_0(R)$ elements in S such that if we remove them and all ordered pairs containing them, we obtain a relation satisfying the Hall condition. Using the marriage theorem, deduce Ore's theorem (Theorem 2.2.8): Let $R : S \to X$ be a relation. Then the maximum size of a partial matching is $|S| + \eta_0(R)$. (Ore gave an independent proof of the marriage theorem using the excess function η. However, this proof is quite similar to the Easterfield-Halmos-Vaughan proof.)

(h) Let $R^{-1} : X \to S$ be the "reverse" relation. Show that

$$\operatorname{core} R \cap R^{-1}(\operatorname{core} R^{-1}) = \emptyset.$$

2.4.7. *Rota's basis conjecture.* [26]

Let $B_1, B_2, \ldots, B_n$ be bases of an n-dimensional vector space (or more generally, a rank-n matroid). Then the n^2 vectors occurring in the bases can be arranged in an $n \times n$ square so that each row and each column is a basis. Such a square can be considered a vector-analog of a Latin square.

[26] R. Huang and G.-C. Rota, On the relations of various conjectures on Latin squares and straightening coefficients, *Discrete Math.* 128 (1994) 225–236.