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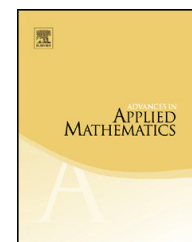
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Positive and negative chains in charged moon polyominoes

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ABSTRACT

We associate with a moon polyomino \mathcal{M} a charge function $C : \mathcal{M} \rightarrow \{\pm 1\}$ such that every cell of the polyomino is either positive or negative. The charge function induces naturally a sign on the northeast and southeast chains of length 2 in 01-fillings of the moon polyomino. We prove that the joint distribution of the numbers of positive chains and negative chains is symmetric and independent of the charge function. Our result reveals a deeper symmetry between the northeast and southeast chains of length 2 in 01-fillings of moon polyominoes, and gives a new family of Mahonian statistics for many combinatorial structures.

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1. Introduction

In the last five years there has been an increasing interest in studying crossings and nestings in various combinatorial structures, where crossings and nestings of two edges are natural analogs of inversions and coinversions of permutations. It has been observed in many cases that the numbers of crossings and nestings have a symmetric joint distribution, for example, in matchings and set partitions by de Sainte-Catherine [8] and Kasraoui and Zeng [12], in linked partitions by Chen, Wu and Yan [4], and in certain diagrams associated with permutations by Corteel [5]. The symmetry of the maximal crossing and maximal nesting was proved by Chen, Deng, Du, Stanley and Yan [1]. Klazar [13] further studied the distribution of crossings and nestings over the set of matchings obtained from a

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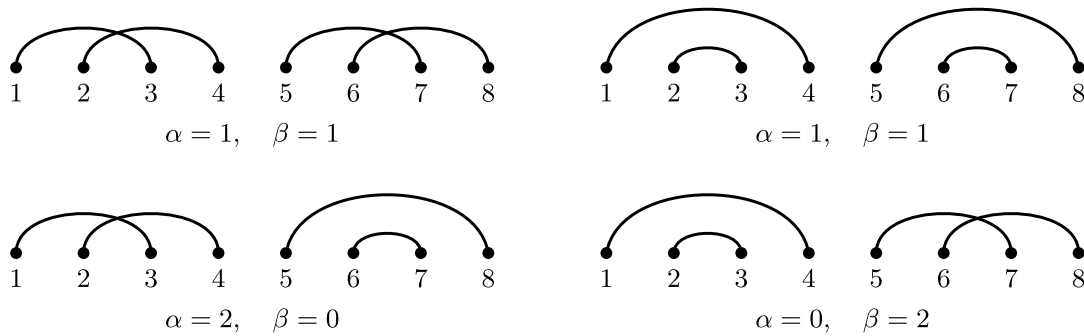


Fig. 1. An illustration of the statistics α and β .

given one by successfully adding edges, and the parallel results on set partitions were developed by Poznanović and Yan [15].

Many classical results on enumerative combinatorics can be put in the larger context of counting submatrices in fillings of polyominoes. For example, words and permutations can be represented as 01-fillings of rectangular boards, and graphs can be represented as fillings of arbitrary Ferrers shapes, which were studied by Krattenthaler [14] and de Mier [6,7]. Other extensions include stack polyominoes [9,10] and moon polyominoes [16,2]. In the language of fillings of polyominoes, crossings and nestings of two edges become northeast and southeast chains of length 2. A unified result on the numbers of northeast and southeast chains of length 2 was given by Kasraoui [11], who proved that this pair of statistics has a symmetric joint distribution over families of 01-fillings of moon polyominoes where either every row has at most one 1, or every column has at most one 1. Kasraoui also gave an explicit formula for the joint distribution: with prefixed row-sum and column-sum, the distribution is always a product of (p, q) -Gaussian coefficients. In [3] our collaborators and we revealed a stronger symmetry between northeast and southeast chains of length 2. Roughly speaking, we proved that one can mix the northeast and southeast chains according to the row position (or column position) of one of the cells in the chain, yet still preserve the symmetric joint distribution.

Recently the first author of the present paper discovered an interesting phenomenon for crossings and nestings of two edges over matchings and set partitions, which cannot be explained by the aforementioned results. Let us describe it for matchings of $[2n]$, which are partitions of the set $[2n]$ into n blocks of size 2. Each block is called an *edge* and is denoted (i, j) if $i < j$. We say that two edges (i_1, j_1) and (i_2, j_2) form a crossing if $i_1 < i_2 < j_1 < j_2$; they form a nesting if $i_1 < i_2 < j_2 < j_1$. Let $\text{cros}(P)$ and $\text{nest}(P)$ be the number of crossings and nestings of a matching P , respectively.

For a crossing with edges $(i_1, j_1), (i_2, j_2)$ where $i_1 < i_2$, we say it is of Type I if $i_2 + j_2 \leq 2n$; otherwise it is of Type II. For a nesting with edges $(i_1, j_1), (i_2, j_2)$ where $i_1 < i_2$, we say it is of Type I if $i_2 + j_1 \leq 2n$; otherwise it is of Type II.

Let P be a matching of $[2n]$. Consider two statistics

$$\alpha(P) = \#\{\text{crossings of Type I in } P\} + \#\{\text{nestings of Type II in } P\},$$

$$\beta(P) = \#\{\text{crossings of Type II in } P\} + \#\{\text{nestings of Type I in } P\}.$$

For small n we observed that the pair $(\alpha(P), \beta(P))$ is equally distributed to the pair $(\text{cros}(P), \text{nest}(P))$ over the set of matchings of $[2n]$ with given sets of minimal block elements and maximal block elements. The same result has been observed for set partitions as well. See Example 1 for an illustration. Trying to explain this intriguing symmetry is the motivation of the present paper.

Example 1. Fig. 1 lists all the matchings of $[8]$ with minimal block elements $\{1, 2, 5, 6\}$ and maximal block elements $\{3, 4, 7, 8\}$, and corresponding values of $\alpha(P)$ and $\beta(P)$. It is clear that $(\alpha(P), \beta(P))$ is equally distributed with $(\text{cros}(P), \text{nest}(P))$ over this set of matchings.

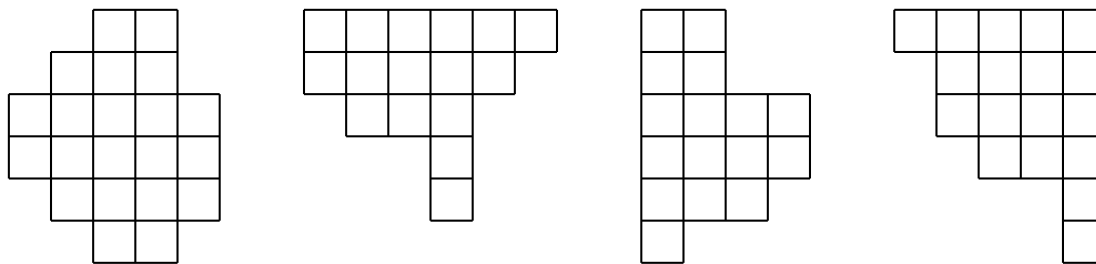


Fig. 2. A moon polyomino, a top-aligned and a left-aligned stack polyomino, and a top and right aligned Ferrers shape.

It is natural to consider the problem in the context of 01-fillings of moon polyominoes. In Section 2 we associate with a moon polyomino \mathcal{M} a charge function $C : \mathcal{M} \rightarrow \{-1, +1\}$, which assigns to each cell of the polyomino either a positive or a negative charge. The charge function induces a sign for any 2×2 submatrix of the polyomino \mathcal{M} , and hence divides the northeast and southeast chains of length 2 into two sets: the positive chains and the negative chains, which are counted by $\text{pos}_C(M)$ and $\text{neg}_C(M)$, respectively. We prove that for certain families of stack polyominoes, the pair of statistics $(\text{pos}_C(M), \text{neg}_C(M))$ has a stable symmetric distribution which is independent of the charge function. In particular, when the charge is always positive, we get $(\text{ne}(M), \text{se}(M))$, the numbers of northeast and southeast chains of length 2. This explains the new symmetry we found for matchings and set partitions.

Unfortunately with the above natural definition of positive/negative chains the symmetry does not hold in fillings of general moon polyominoes. In Section 3 we introduce the notions of restrictively positive/negative chains, which coincide with the positive/negative chains in stack polyominoes. We prove that the numbers of restrictively positive and negative chains are always equidistributed as the pair $(\text{ne}(M), \text{se}(M))$ over 01-fillings of any charged moon polyominoes. Since the model of 01-fillings of polyominoes contains many combinatorial structures, and northeast/southeast chains are natural analogues of the basic Mahonian (permutation) statistics inversions/coinversions, our results lead to a new family of Mahonian statistics for general combinatorial structures.

In Section 4 we present a new way to compute the symmetric joint distribution of $(\text{ne}(M), \text{se}(M))$ over 01-fillings of a moon polyomino. The idea is to transform the northeast/southeast chains in 01-fillings of a moon polyomino into positive/negative chains in fillings of a charged stack polyomino, and then to fillings in a Ferrers shape. Enumeration of the latter is equivalent to the enumeration of crossings and nestings in linked partitions, for which the formula has already been given in [4].

2. Positive chains and negative chains in stack polyominoes

In this section we introduce the notion of charged moon polyominoes, which induces a natural sign on the northeast and southeast chains of length 2. We show that the numbers of positive and negative chains have a stable and symmetric distribution over 01-fillings of top-aligned or left-aligned stack polyominoes. First we introduce the necessary background and definitions.

A *polyomino* is a finite subset of \mathbb{Z}^2 , where every element of \mathbb{Z}^2 is represented by a square cell. The polyomino is *convex* if its intersection with any column or row is connected. It is *intersection-free* if every two columns are comparable, i.e., the row-coordinates of one column form a subset of those of the other column. Equivalently, it is intersection-free if every two rows are comparable. A *moon polyomino* is a convex and intersection-free polyomino.

In this paper we always label the rows of a polyomino from top to bottom, and the columns from left to right. If the rows (resp. columns) of a moon polyomino \mathcal{M} are monotonically arranged, we call it a *stack polyomino*. In particular, we say it is *top-aligned* (*bottom-aligned*) if the rows are arranged from large to small (small to large). Similarly, we have *left-aligned* (*right-aligned*) stack polyomino if the columns are arranged from large to small (small to large). If both the rows and the columns are monotonically arranged, then the moon polyomino is a *Ferrers shape*. When needed, we will specify the alignment of the Ferrers shape. See Fig. 2 for an illustration.

Given a moon polyomino \mathcal{M} , we assign 0 or 1 to each cell of \mathcal{M} so that there is at most one 1 in each column. Throughout this paper we will simply use the term *filling* to denote such 01-fillings.

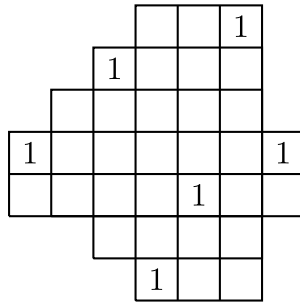


Fig. 3. A filling M with $\mathbf{s} = (1, 1, 0, 2, 1, 0, 1)$ and $\mathbf{e} = (1, 0, 1, 1, 1, 1, 1)$.

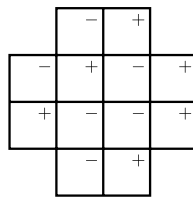


Fig. 4. A charged moon polyomino.

We say that a cell is empty if it is assigned 0, and it is a 1-cell otherwise. Assume the rows of \mathcal{M} are R_1, \dots, R_n from top to bottom, and the columns are C_1, \dots, C_m from left to right. The cell (i, j) refers to the cell lying at row R_i and column C_j . Let $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$ and $\mathbf{e} = (\varepsilon_1, \dots, \varepsilon_m) \in \{0, 1\}^m$, where

$$\sum_{i=1}^n s_i = \sum_{j=1}^m \varepsilon_j.$$

We denote by $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$ the set of fillings M of \mathcal{M} such that row R_i has exactly s_i 1's, and column C_j has exactly ε_j 1's, for $1 \leq i \leq n$ and $1 \leq j \leq m$. The vectors \mathbf{s} and \mathbf{e} are called the *row-sum* and *column-sum* vectors of the filling, respectively. Fig. 3 gives an example.

A 2×2 submatrix S of \mathcal{M} is a set of four cells in \mathcal{M} with the coordinates

$$S = \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2) \in \mathcal{M} : 1 \leq i_1 < i_2 \leq n, 1 \leq j_1 < j_2 \leq m\}. \quad (1)$$

A *northeast chain* of length 2 in a filling M consists of a 2×2 submatrix S as in (1) where (i_1, j_2) and (i_2, j_1) are 1-cells. There is no constraint on the filling of the other two cells, except that all four cells of S must be contained in \mathcal{M} . Similarly, a *southeast chain* of length 2 consists of a submatrix S where (i_1, j_1) and (i_2, j_2) are 1-cells. In both cases we say that the submatrix S is the *support* of the chain. In this paper we are only concerned with chains of length 2. Hence we simply refer to them as *chains*. Northeast (resp. southeast) chains are abbreviated as NE (resp. SE) chains. The number of NE (resp. SE) chains of M is denoted by $\text{ne}(M)$ (resp. $\text{se}(M)$).

A charged moon polyomino is a moon polyomino \mathcal{M} equipped with a charge function, denoted by C , where $C : \mathcal{M} \rightarrow \{\pm 1\}$ assigns each cell of \mathcal{M} either a positive or a negative charge. See Fig. 4 for an example, where + (resp. -) denotes the positive (resp. negative) charge of each cell.

For a 2×2 submatrix S of \mathcal{M} with cells as in (1), set the sign of S as the charge of its bottom right corner, that is,

$$\text{sgn}(S) = C((i_2, j_2)). \quad (2)$$

Definition 2. Let M be a 01-filling of a charged moon polyomino \mathcal{M} with charge function C . A chain with support matrix S is *positive* with respect to C if it is

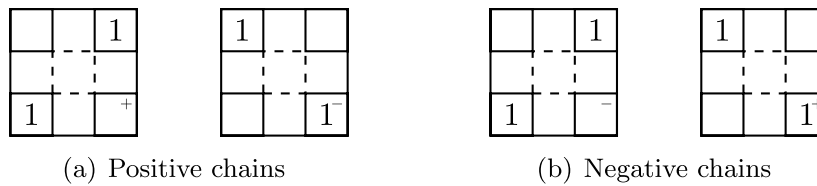


Fig. 5. Positive and negative chains.

1. a northeast chain with $\text{sgn}(S) = 1$, or
2. a southeast chain with $\text{sgn}(S) = -1$.

Otherwise, the chain is *negative*.

Fig. 5 shows the positive and negative chains. Note that the charge on the cells of S other than (i_2, j_2) does not affect the sign of S .

Given a moon polyomino \mathcal{M} with charge function C , for any filling M in $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$, denote by $\text{pos}_C(M)$ and $\text{neg}_C(M)$ the numbers of positive chains and negative chains of M with respect to C . Let $F_C(p, q)$ be the bi-variate generating function for the pair $(\text{pos}_C(M), \text{neg}_C(M))$, that is

$$F_C(p, q) = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\text{pos}_C(M)} q^{\text{neg}_C(M)}.$$

Write $C = +$ if all the cells in the polyomino are positively charged, and $C = -$ if all the cells are negatively charged. We observe that $\text{pos}_+(M) = \text{neg}_-(M) = \text{ne}(M)$, and $\text{neg}_+(M) = \text{pos}_-(M) = \text{se}(M)$. Our first result is the following theorem.

Theorem 3. Let \mathcal{M} be a top-aligned or left-aligned stack polyomino. Then the bi-variate generating function $F_C(p, q)$ does not depend on C . Consequently,

$$F_C(p, q) = F_+(p, q) = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\text{ne}(M)} q^{\text{se}(M)}.$$

As an immediate consequence, we have

Corollary 4. For a top-aligned or left-aligned stack polyomino \mathcal{M} , the distribution of the pair $(\text{pos}_C(M), \text{neg}_C(M))$ is symmetric over $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$.

Proof. Suppose C is a charge function of the given moon polyomino \mathcal{M} . Let $C' = -C$. It is easy to see that every positive (resp. negative) chain of $M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$ under C becomes a negative (resp. positive) chain under C' . It follows that

$$\text{pos}_C(M) = \text{neg}_{C'}(M) \quad \text{and} \quad \text{neg}_C(M) = \text{pos}_{C'}(M).$$

By Theorem 3

$$\sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\text{pos}_C(M)} q^{\text{neg}_C(M)} = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\text{pos}_{C'}(M)} q^{\text{neg}_{C'}(M)} = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\text{neg}_C(M)} q^{\text{pos}_C(M)},$$

which yields the desired symmetry. \square

To prove Theorem 3, we start with the case that the polyomino is a rectangle.

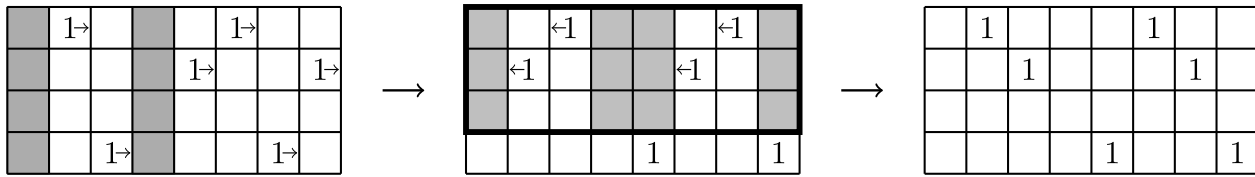


Fig. 6. An illustration of the second case, where X is the bottom right cell.

Lemma 5. Let \mathcal{R} be an $n \times m$ rectangle, and $X = (n, m)$ be the cell at the bottom right corner. Let C be the charge function

$$C((i, j)) = \begin{cases} -1 & \text{if } (i, j) = X, \\ +1 & \text{otherwise.} \end{cases}$$

Then there exists a bijection ρ from $\mathbf{F}(\mathcal{R}, \mathbf{s}, \mathbf{e})$ to itself such that

$$(\text{pos}_C(M), \text{neg}_C(M)) = (\text{ne}(\rho(M)), \text{se}(\rho(M))). \tag{3}$$

Consequently, $(\text{pos}_C(M), \text{neg}_C(M))$ has the same distribution as $(\text{ne}(M), \text{se}(M))$ over $\mathbf{F}(\mathcal{R}, \mathbf{s}, \mathbf{e})$, where \mathbf{s} and \mathbf{e} are given row-sum and column-sum vectors.

Proof. The bijection ρ is defined as follows. If $s_n = 0$ or $\epsilon_m = 0$, then $\rho(M) = M$ is the identity map. Otherwise, assume $s_n > 0$ and $\epsilon_m = 1$. Let M be a filling in $\mathbf{F}(\mathcal{R}, \mathbf{s}, \mathbf{e})$. We will keep all the empty rows and empty columns of M unchanged, and define $\rho(M)$ by the following steps.

1. If X is a 1-cell of M , then $\rho(M)$ is obtained from M by blocking all the empty columns and cyclically shifting every 1 horizontally to the next nonempty column on the right. That is, if i_1, i_2, \dots, m are the indices of the nonempty columns with $i_1 < i_2 < \dots < m$, then a 1 in the cell $Y = (k, i_t) \neq X$ is moved to the cell (k, i_{t+1}) , and the 1 in X is moved to (n, i_1) .
2. If X is empty but there is a 1-cell above X in the same column, then first block all the empty columns and cyclic-shift every 1 horizontally to the next column on the right, as in the previous step. Let \mathcal{R}' be the rectangle obtained from \mathcal{R} by removing the last row. Block all the empty columns of \mathcal{R}' and cyclic-shift every 1 of \mathcal{R}' horizontally to the next column on the left. See Fig. 6 for an illustration.

We check that ρ is a bijection. The case that $s_n = 0$ or $\epsilon_m = 0$ is trivial. If $s_n \epsilon_m \neq 0$, we observe that $\rho(M)$ is obtained by the first case if the lowest cell of the first nonempty column is a 1-cell, and by the second case if it is empty. In both cases, the procedure is easily reversed. So ρ defines a bijection on $\mathbf{F}(\mathcal{R}, \mathbf{s}, \mathbf{e})$.

To check Eq. (3), we compute that when X is a 1-cell of M ,

$$\text{pos}_C(M) = \text{ne}(M) + (s_1 + s_2 + \dots + s_{n-1}) = \text{ne}(\rho(M)),$$

where the second equation is true since the 1 in the cell X is moved to the very left, and hence forms NE chains with any 1 in the above rows. Similarly we have

$$\text{neg}_C(M) = \text{se}(M) - (s_1 + s_2 + \dots + s_{n-1}) = \text{se}(\rho(M)).$$

When X is empty in M , we claim

$$\text{pos}_C(M) = \text{ne}(M) - s_n = \text{ne}(\rho(M)).$$

The second equation is true since each 1 in the last row is moved one step to the right and hence decreases the number of NE chains by one. Similarly we have

$$\text{neg}_C(M) = \text{se}(M) + s_n = \text{se}(\rho(M)). \quad \square$$

Let \mathcal{M} be a top-aligned or left-aligned stack polyomino with charge function C such that $C \neq +$. We prove [Theorem 3](#) by showing that changing the charge of one cell from $-$ to $+$ does not change the distribution of $(\text{pos}_C(M), \text{neg}_C(M))$ over $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$. Let $X = (a, t)$ be the highest and the leftmost cell with a negative charge under C . That is, $C(X) = -1$ and $C((i, j)) = +1$ for any cell $(i, j) \in \mathcal{M}$ with $i < a$, or $i = a$ and $j < t$. Let C' be the charge function obtained from C by changing the charge of X to $+1$.

Claim. $F_C(p, q) = F_{C'}(p, q)$.

[Theorem 3](#) follows from the claim by changing the charge of the highest and the leftmost cell with a negative charge in succession. Eventually all the cells of the polyomino would have positive charges and the distribution of the pair of statistics $(\text{pos}_C(M), \text{neg}_C(M))$ remains the same in every step.

Proof of the claim. Let \mathcal{R}_X be the largest rectangle in \mathcal{M} whose bottom right corner is the cell $X = (a, t)$. More precisely, assume that the leftmost cell in row R_a is (a, s) and the highest cell in column C_t is (b, t) . Then

$$\mathcal{R}_X = \{(i, j) \in \mathcal{M} : b \leq i \leq a, s \leq j \leq t\}.$$

Note that the rectangle \mathcal{R}_X is unique and well-defined for any cell X when \mathcal{M} is a top or left aligned stack polyomino.

Let M be a filling in $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$ and ρ be the bijection defined in the proof of [Lemma 5](#). Define $f(M)$ by replacing $M \cap \mathcal{R}_X$ by $\rho(M \cap \mathcal{R}_X)$ and keeping the filling of $M \setminus \mathcal{R}_X$ unchanged. Clearly this is a bijection on $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$. We shall verify that

$$(\text{pos}_C(M), \text{neg}_C(M)) = (\text{pos}_{C'}(f(M)), \text{neg}_{C'}(f(M))). \quad (4)$$

First, by [Lemma 5](#),

$$\text{pos}_C(M \cap \mathcal{R}_X) = \text{ne}(f(M) \cap \mathcal{R}_X) = \text{pos}_{C'}(f(M) \cap \mathcal{R}_X).$$

Second, an NE or SE chain formed by two 1-cells outside \mathcal{R}_X is not changed, nor is the sign of its support. Hence such chains contribute equally to both pairs of statistics in (4). We only need to compare the contribution of chains in which there is exactly one relevant 1-cell outside \mathcal{R}_X . Note that for such a chain the bottom right corner of its support matrix S cannot be X , hence $\text{sgn}(S)$ is the same under C and C' .

The proof is complete by showing that for any 1-cell $Y \notin \mathcal{R}_X$, the number of NE (or SE) chains formed by Y and 1-cells in \mathcal{R}_X depends only on the row-sum or column-sum of $M \cap \mathcal{R}_X$, which is preserved under the map ρ . We provide details for the NE chains in a top-aligned stack polyomino. The cases for SE chains and left-aligned stack polyominoes are proved in a similar manner.

Let $Y = (u, v) \in \mathcal{M} \setminus \mathcal{R}_X$ be a 1-cell in the filling M (and hence $f(M)$). We divide the cases according to the location of Y with respect to the rectangle \mathcal{R}_X . See [Fig. 7](#).

- Case 1.** $u > a$, i.e., Y is in a row lower than \mathcal{R}_X . The number of NE chains formed by Y and 1-cells in \mathcal{R}_X equals the number of 1-cells in the columns $\{C_l \cap \mathcal{R}_X : v < l \leq t \text{ and } (u, l) \in \mathcal{M}\}$.
- Case 2.** $u \leq a$ and $v < s$, i.e., Y is on the left of \mathcal{R}_X . The number of NE chains formed by Y and 1-cells in \mathcal{R}_X equals the number of 1-cells in the rows $\{R_k \cap \mathcal{R}_X : k < u\}$.

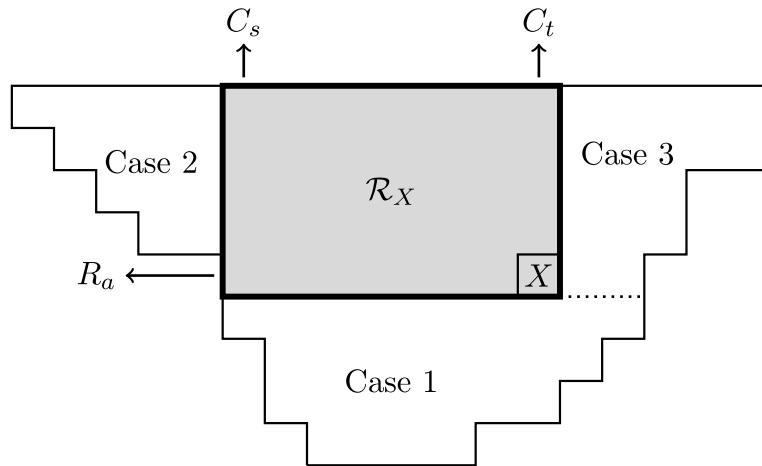


Fig. 7. The location of Y with respect to the rectangle \mathcal{R}_X .

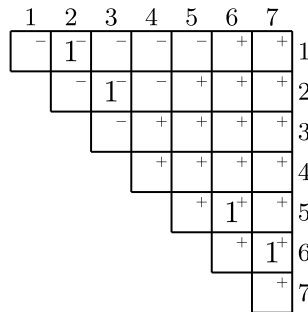


Fig. 8. A charged triangular shape.

Case 3. $u \leq a$ and $v > t$, i.e., Y is on the right of \mathcal{R}_X . The number of NE chains formed by Y and a 1-cell in \mathcal{R}_X equals the number of 1-cells in the rows $\{R_k \cap \mathcal{R}_X: k > u \text{ and } (k, v) \in \mathcal{M}\}$.

In each case, the number is the same in both M and $f(M)$. This completes the proof. \square

Theorem 3 provides an explanation for the symmetry we found for matchings and set partitions, as described in the Introduction. Take a matching P of $[2n]$. Consider a triangular shape whose rows are of length $2n - 1, 2n - 2, \dots, 1$ that is top-aligned and right-aligned. Represent the edge (i, j) of P by assigning a 1 in the cell $(i, j - 1)$. In the triangular board, let a cell (i, j) be positive if and only if $i + j \geq 2n - 1$. Then the statistics $(\alpha(P), \beta(P))$ are exactly the number of positive and negative chains with respect to this charge function. Fig. 8 gives the corresponding charged triangular shape and filling for the first matching in Example 1.

Remark.

- Theorem 3** also holds for the set $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$, i.e., the 01-fillings where there is at most one 1 in every row, but can have an arbitrary number of 1's in a column. To prove it, one notes that the reflection with respect to the diagonal (a line of slope -1) maps a top-aligned stack polyomino to a left-aligned stack polyomino while exchanging rows and columns. But such a reflection preserves NE/SE chains, as well as the sign of any 2×2 submatrix in the polyomino.
- In this paper we would only consider fillings that either the row-sum or the column-sum is a 01-vector. The reason is that with no restrictions on the row-sum or the column-sum, the joint distribution of the statistics (ne, se) is not always symmetric, that is, (ne, se) and (se, ne) may have different distributions. See Kasraoui [11, Sec. 6].

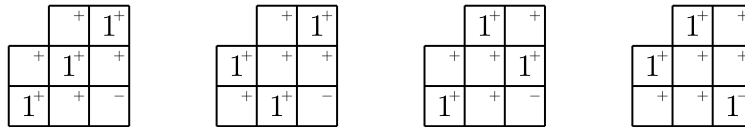


Fig. 9. A counterexample.

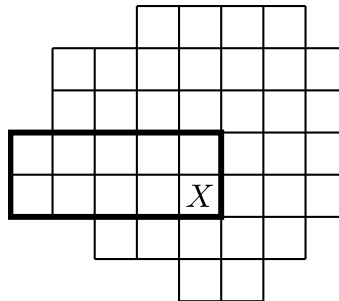


Fig. 10. The box \mathcal{R}_X .

3. Extensions to general moon polyominoes

For general moon polyominoes [Theorem 3](#) may not hold. The following is a counterexample.

Example 6. Consider the 01-fillings in [Fig. 9](#) where every row and every column has exactly one 1, and all but the bottom right cell have positive charges. We have

$$\sum_M p^{\text{ne}(M)} q^{\text{se}(M)} = p^2 + 2pq + q^2, \quad \text{while} \quad \sum_M p^{\text{pos}_C(M)} q^{\text{neg}_C(M)} = 2p^2 + 2q^2.$$

In this section we introduce the notions of restrictively positive and negative chains, which coincide with [Definition 2](#) for top or left aligned stack polyominoes. We prove that the joint distribution of restrictively positive and negative chains is stable and independent of the charge function for an arbitrary moon polyomino.

We say that a rectangle \mathcal{R} in a polyomino \mathcal{M} is *maximal* if there is no rectangle $\mathcal{R}' \subseteq \mathcal{M}$ such that \mathcal{R} is a proper subset of \mathcal{R}' . For a cell $X \in \mathcal{M}$, define \mathcal{R}_X , the *box of X*, to be the widest maximal rectangle contained in \mathcal{M} whose bottom right corner is X . In particular, \mathcal{R}_X contains all the cells that are in the same row and on the left of X . [Fig. 10](#) gives an example where \mathcal{R}_X is depicted by thick lines.

Definition 7. A 2×2 submatrix $S \in \mathcal{M}$ is said to be *restrictive* if and only if $S \subseteq \mathcal{R}_{(i_2, j_2)}$, where (i_2, j_2) is the bottom right corner of S . We say that an NE/SE-chain is restrictive if its support matrix is restrictive.

Definition 8. Let M be a 01-filling of a charged moon polyomino \mathcal{M} with charge function C . A chain with support matrix S is *restrictively positive* with respect to C if it is

1. a northeast chain with $\text{sgn}(S) = 1$, or
2. a northeast chain with $\text{sgn}(S) = -1$ that is not restrictive, or
3. a southeast chain with $\text{sgn}(S) = -1$ that is restrictive.

Otherwise, the chain is *restrictively negative*. Explicitly, a chain with support matrix S is *restrictively negative* if it is

1. a southeast chain with $\text{sgn}(S) = 1$, or

2. a southeast chain with $\text{sgn}(S) = -1$ that is not restrictive, or
3. a northeast chain with $\text{sgn}(S) = -1$ that is restrictive.

If \mathcal{M} is a top or left aligned stack polyomino, then any 2×2 submatrix S of \mathcal{M} is completely contained in the box of its bottom right cell, i.e., S is always restrictive. Hence Definition 8 coincides with Definition 2 for those special polyominoes.

Given a charged moon polyomino \mathcal{M} with charge function C , for any filling M in $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$, denote by $\overline{\text{pos}}_C(M)$ and $\overline{\text{neg}}_C(M)$ the numbers of restrictively positive chains and restrictively negative chains of M with respect to C . Let $\overline{F}_C(p, q)$ be the bi-variate generating function for the pair $(\overline{\text{pos}}_C(M), \overline{\text{neg}}_C(M))$, that is

$$\overline{F}_C(p, q) = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\overline{\text{pos}}_C(M)} q^{\overline{\text{neg}}_C(M)}.$$

Theorem 9. *The bi-variate generating function $\overline{F}_C(p, q)$ does not depend on the charge function. Consequently,*

$$\overline{F}_C(p, q) = \overline{F}_+(p, q) = \sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{\text{ne}(M)} q^{\text{se}(M)}.$$

Proof. We adapt the proof of Theorem 3 by showing $\overline{F}_C(p, q) = \overline{F}_{C'}(p, q)$, where C is any charge function that is not always positive, $X = (a, t)$ is the highest and the leftmost cell with $C(X) = -1$, and C' is obtained from C by changing the charge of X to $+1$.

Given a filling $M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$, let \mathcal{R}_X be the box of X . Define $f(M)$ by replacing $M \cap \mathcal{R}_X$ by $\rho(M \cap \mathcal{R}_X)$ and keeping the other cells unchanged, where ρ is the bijection defined in Lemma 5. Since \mathcal{R}_X depends only on the shape of the moon polyomino \mathcal{M} , but not on the filling, this is a bijection on $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$. We verify that

$$(\overline{\text{pos}}_C(M), \overline{\text{neg}}_C(M)) = (\overline{\text{pos}}_{C'}(f(M)), \overline{\text{neg}}_{C'}(f(M))). \tag{5}$$

Similarly to the proof of Theorem 3, the contributions of chains formed by two 1-cells both inside \mathcal{R}_X , or both outside \mathcal{R}_X are the same for the two pairs of statistics in (5). We only need to compare the contribution of chains with exactly one relevant 1-cell outside \mathcal{R}_X .

Suppose that the four corners of \mathcal{R}_X are (b, s) , (b, t) , (a, s) and $(a, t) (= X)$ respectively with $b < a$ and $s < t$. Then $(a, s - 1) \notin \mathcal{M}$ and

$$\mathcal{R}_X = \{(i, j): b \leq i \leq a, s \leq j \leq t\}.$$

Let $Y = (u, v) \in \mathcal{M} \setminus \mathcal{R}_X$ be a 1-cell in the filling M and hence $f(M)$. We shall prove for each Y that the number of restrictively positive (negative) chains formed by Y and 1-cells in \mathcal{R}_X depends only on the row-sum or column-sum of $M \cap \mathcal{R}_X$, which are the same for M and $f(M)$. It is verified case-by-case according to the position of Y with respect to \mathcal{R}_X . In the following we shall give all details for NE chains only. Fig. 11 shows the different positions of Y . The cases for SE chains are similar.

Case 1. $v < s$, i.e., Y lies on the left of \mathcal{R}_X . Since \mathcal{M} does not contain the cell $(a, s - 1)$, Y could form NE chains with 1-cells in \mathcal{R}_X only when $b < u < a$, and the number of NE chains formed by Y and 1-cells in \mathcal{R}_X equals the number of 1-cells in the rows $\{R_k \cap \mathcal{R}_X: b \leq k < u \text{ and } (k, v) \in \mathcal{M}\}$. All such chains have positive support under both charge functions C and C' , hence are always restrictively positive.

Case 2. $s \leq v \leq t$. There are two cases.

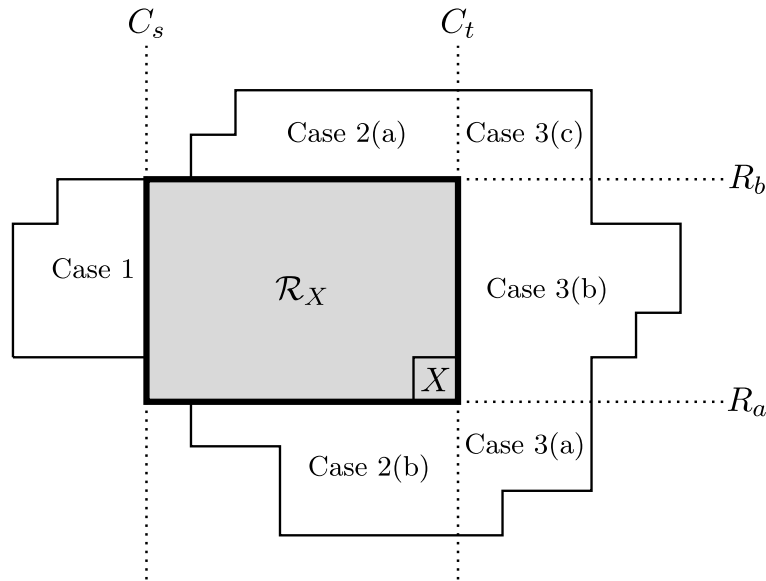


Fig. 11. Various positions of Y .

- (a) $u < b$, i.e., Y is on top of \mathcal{R}_X . The number of NE chains formed by Y and 1-cells in \mathcal{R}_X equals the number of 1-cells in the columns $\{C_l \cap \mathcal{R}_X: s \leq l < v \text{ and } (u, l) \in \mathcal{M}\}$. If $v \neq t$, all such chains have positive support. When $v = t$, all such chains have positive support under C' ; while under C , such chains either have positive support, or are not restrictive. In any case, all the NE chains are restrictively positive for both M and $f(M)$.
- (b) $u > a$, i.e., Y is below \mathcal{R}_X . The number of NE chains formed by Y and 1-cells in \mathcal{R}_X equals the number of 1-cells in the columns $\{C_l \cap \mathcal{R}_X: v < l \leq t \text{ and } (u, l) \in \mathcal{M}\}$. For such a chain with a 1-cell in $C_l \cap \mathcal{R}_X$, the sign of its support is $C(Z) = C'(Z)$ where $Z = (u, l)$. Whether it is restrictive depends only on the shape of \mathcal{M} , but not the charge function or the filling: If the cell $(u, s - 1) \in \mathcal{M}$, then $\mathcal{R}_Z \cap \mathcal{R}_X = \emptyset$ and the chain is not restrictive; if $(u, s - 1) \notin \mathcal{M}$, then \mathcal{R}_Z contains $C_l \cap \mathcal{R}_X$ and the chain is restrictive.

Case 3. $v > t$.

- (a) If $u \geq a$, there are no NE chains formed by Y and 1-cells in \mathcal{R}_X .
- (b) $b \leq u < a$, i.e., Y is on the right of \mathcal{R}_X . For $u < k \leq a$, the number of NE chains formed by Y and 1-cells in row $R_k \cap \mathcal{R}_X$ equals the number of 1-cells in $R_k \cap \mathcal{R}_X$ if $(k, v) \in \mathcal{M}$, and is 0 otherwise. For each row R_k such that $Z = (k, v) \in \mathcal{M}$, the charge of Z and the box \mathcal{R}_Z are not changed. Hence all the NE chains formed by Y and 1-cells in $R_k \cap \mathcal{R}_X$ have the same (restrictive) parity under C and C' .
- (c) $u < b$, i.e., Y is in the northeast of \mathcal{R}_X . Let $Z = (a, v)$ and $W = (u, s)$. Since moon polyominoes are intersection-free, at least one of W and Z is contained in \mathcal{M} . If $W \in \mathcal{M}$, the proof is the same as the case 3(b), where the row index k ranges from b to a . Otherwise, assume $Z \in \mathcal{M}$ but $W \notin \mathcal{M}$, the number of NE chains formed by Y and 1-cells in \mathcal{R}_X equals the number of 1-cells in the columns $\{C_l \cap \mathcal{R}_X: s < l \leq t \text{ and } (u, l) \in \mathcal{M}\}$. For each cell (i, l) in this set, the 2×2 submatrix consisting of cells $\{(u, l), (u, v), (i, l), (i, v)\}$ is not restrictive since $W \notin \mathcal{M}$. Hence all such NE chains are restrictively positive in both M and $f(M)$. \square

Remark.

1. There can be many ways to define the *restrictively positive/negative* chains. For example, we can replace \mathcal{R}_X by \mathcal{R}'_X , which is the longest maximal rectangle in \mathcal{M} whose bottom right corner is X . In other words, if $X = (a, t)$ and (c, t) is the top cell in column C_t , then \mathcal{R}'_X is the maximal rectangle contained in \mathcal{M} whose last column consists of cells $\{(i, t): c \leq i \leq a\}$. All the results and their proofs would hold suitably adapted.

- Following the preceding remark, [Theorem 9](#) also holds for the set $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$, i.e., 01-fillings that the number of 1's in a row is at most 1, and the number of 1's in a column can be any non-negative integer. To see this, one simply reflects the polyomino and its fillings with respect to the diagonal (a line of slope -1) to map fillings from $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ to $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$ and \mathcal{R}_X to \mathcal{R}'_X , while keeping NE/SE chains and the sign of any 2×2 submatrix.

In the following we discuss the relations of the positive/negative chains with the mixed statistics introduced in [\[3\]](#).

For a moon polyomino \mathcal{M} , let S be a subset of rows. An NE (SE) chain is called a *top S-NE (SE) chain* if its top 1-cell is in S ; similarly, it is a *bottom S-NE/SE chain* if the lower 1-cell of the chain is in S .

Consider the set of fillings $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$. (In [\[3\]](#) the set $\mathbf{F}(\mathcal{M}, \mathbf{e}, \mathbf{s})$ is used. By the symmetric shape of the moon polyomino, it is equivalent to $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$. So we will use the latter to keep notations consistent in this paper.) Let $\bar{S} = \mathcal{M} \setminus S$ be the complement of S . Given a filling $M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$, define the *top-mixed statistic* $\alpha(S; M)$ and the *bottom-mixed statistic* $\beta(S; M)$ with respect to S as

$$\alpha(S; M) = \#\{\text{top } S\text{-NE chains of } M\} + \#\{\text{top } \bar{S}\text{-SE chains of } M\},$$

$$\beta(S; M) = \#\{\text{bottom } S\text{-NE chains of } M\} + \#\{\text{bottom } \bar{S}\text{-SE chains of } M\}.$$

Analogously one defines the left-mixed statistic $\gamma(T; M)$ and right-mixed statistic $\delta(T; M)$ with respect to a subset T of columns.

The main result of [\[3\]](#) is the following. For a general moon polyomino \mathcal{M} , let $\lambda(A; M)$ be any of the four statistics $\alpha(S; M)$, $\beta(S; M)$, $\gamma(T; M)$, $\delta(T; M)$. Then the joint distribution of the pair $(\lambda(A; M), \lambda(\bar{A}; M))$ is symmetric and independent of the subsets S, T . In particular, the joint distribution is the same as the distribution of $(\text{ne}(M), \text{se}(M))$.

The mixed statistics and the positive/negative chains considered here are complementary to each other. There are some common cases. For example, to get the bottom-mixed statistic $\beta(S; M)$, let C be the charge function on \mathcal{M} such that

$$C((i, j)) = \begin{cases} 1 & \text{if } i \in S, \\ -1 & \text{if } i \notin S. \end{cases}$$

Then clearly

$$\beta(S; M) = \text{pos}_C(M), \quad \beta(\bar{S}; M) = \text{neg}_C(M),$$

and hence the stable distribution of $(\beta(S; M), \beta(\bar{S}; M))$ for top or left aligned stack polyominoes is a special case of [Theorem 3](#), so is the result for right-mixed statistic. For other mixed-statistics, we can revise formula [\(2\)](#) of the sign of the 2×2 submatrix by using the charge of another corner of the matrix, so that [Theorem 3](#) implies the stable distribution of each family of mixed statistics with their matching stack polyominoes.

However, in a general moon polyomino the numbers of positive/negative chains do not have a stable symmetric distribution, nor can the mixed statistics be interpreted as the numbers of restrictively positive/negative chains. Together they reveal a subtle balance between the shape of the polyomino, the charge function, and how restrictive the positivity is. In summary, we have that for the numbers of positive/negative chains in 01-fillings of a polyomino,

- for a general moon polyomino and an arbitrary charge function, the symmetry holds only for the restrictively positive and negative chains;
- for an arbitrary charge function and no restrictions on the positivity of chains, the symmetry holds only for fillings on stack polyominoes whose shapes are compatible with the sign of 2×2 submatrices;

- for a general moon polyomino and no restrictions on the positivity of chains, the symmetry holds only for charge functions which are constant on each row, or on each column.

How to put the above results into a unified theory is still an open problem. Also we want to mention that recently, Yan and Yeh proposed a new family of polyominoes—layer polyominoes, which are only row-convex and row-intersection-free [17]. Layer polyominoes admit a free action by the symmetric group. The notions of northeast and southeast chains can be extended to layer polyominoes, and it is proved that the symmetry still holds for $(\text{ne}(M), \text{se}(M))$. The relation between charge functions and layer polyominoes is currently under investigation.

4. Formula of the joint distribution

By Theorem 9 the joint distribution of $(\overline{\text{pos}}_C(M), \overline{\text{neg}}_C(M))$ over $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$ is the same as that of $(\text{ne}(M), \text{se}(M))$, for which an explicit formula was given by Kasraoui [11] as a product of (p, q) -Gaussian coefficients. Kasraoui's proof consists of several technical steps: first one defines an ordering on the rows of \mathcal{M} , then maps each filling in $\mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$ bijectively to a product of integer compositions. The generating function is then obtained by analyzing the contribution of each composition. In this section we give a new way to obtain the joint distribution of $(\text{ne}(M), \text{se}(M))$ by reducing the problem to a known result following these steps:

1. Transform the moon polyomino into a Ferrers shape \mathcal{F} with the same set of row-lengths and column-lengths, so that one just needs to compute the distribution of (ne, se) over fillings of \mathcal{F} .
2. Reduce the enumeration of (ne, se) for 01-fillings of a Ferrers shape \mathcal{F} to the enumeration of crossings and nestings over the set of linked partitions with a given degree sequence. The latter is already known in [4].

4.1. Reduce the moon polyomino to a Ferrers shape

Let \mathcal{M} be a moon polyomino with a positive charge in each cell and \mathcal{N} be obtained from \mathcal{M} by rearranging the rows from longest to shortest. That is, \mathcal{N} is a top-aligned stack polyomino.

Given a filling $M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$, we introduce an algorithm that generates a filling N of \mathcal{N} such that

$$(\text{ne}(M), \text{se}(M)) = (\text{pos}_C(N), \text{neg}_C(N))$$

under a suitably defined charge function $C = C_{\mathcal{N}}$. Let $|R_i|$ be the number of cells in the i -th row, and $\kappa(\mathcal{M})$ be such that

$$|R_1| \leq |R_2| \leq \dots \leq |R_{\kappa(\mathcal{M})}| > |R_{\kappa(\mathcal{M})+1}| \geq \dots \geq |R_n|.$$

Algorithm τ for transforming \mathcal{M} into a top-aligned stack polyomino \mathcal{N} with charge function $C_{\mathcal{N}}$

1. Set $\mathcal{M}' = \mathcal{M}$ and $C' = +$.
2. If \mathcal{M}' is top-aligned, go to Step 5.
3. If \mathcal{M}' is not top-aligned, find the largest rectangle \mathcal{B} completely contained in \mathcal{M}' such that (1) \mathcal{B} contains the first row R_1 of \mathcal{M}' , and (2) \mathcal{B} does not contain any row R_i with $|R_i| = |R_1|$ and $i > \kappa(\mathcal{M}')$.
4. Update C' by changing the charge of every cell in R_1 to negative, and update \mathcal{M}' by moving R_1 to the bottom of the rectangle \mathcal{B} . Go to Step 2.
5. Set $\mathcal{N} = \mathcal{M}'$ and $C_{\mathcal{N}} = C'$.

Given a filling M of the moon polyomino \mathcal{M} , let N be the filling of $\mathcal{N} = \tau(\mathcal{M})$ obtained by applying the algorithm τ to M and keeping the filling of each row when the row is moved. Denote $N = \tau(M)$.

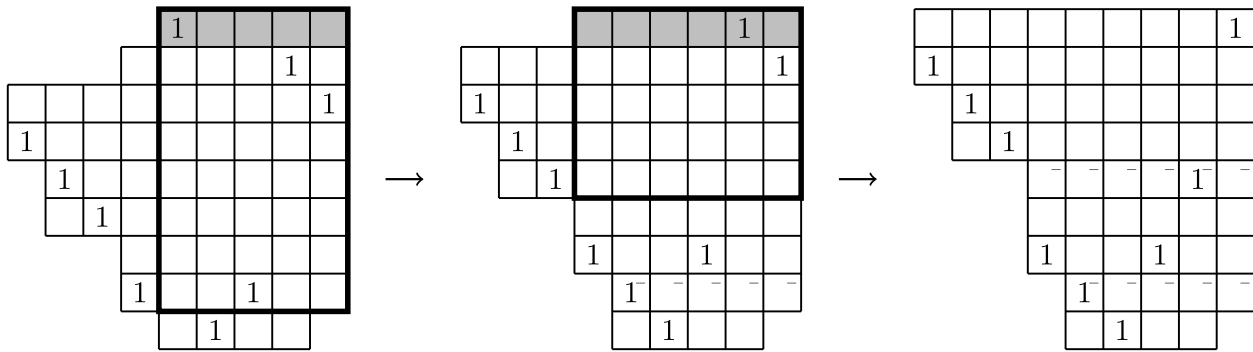


Fig. 12. The algorithm τ , where the rows that are moved downward are highlighted.

It is easy to see that the set of NE (resp. SE) chains of M is in one-to-one correspondence to the set of positive (resp. negative) chains of N . See Fig. 12 for an illustration.

Next we use the algorithm τ to reduce the enumeration of $(ne(M), se(M))$ over a moon polyomino to that over a Ferrers shape with the same sets of row-lengths and column-lengths.

1. Given a filling $M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})$, apply the algorithm τ to obtain a filling $N \in \mathbf{F}(\mathcal{N}, \mathbf{s}', \mathbf{e})$, where \mathcal{N} is a top-aligned stack polyomino with the charge function $C_{\mathcal{N}}$ and \mathbf{s}' is the sequence obtained from \mathbf{s} in the same way as the rows of \mathcal{N} are obtained from the rows of \mathcal{M} . Then

$$(ne(M), se(M)) = (pos_{C_{\mathcal{N}}}(N), neg_{C_{\mathcal{N}}}(N)),$$

and hence

$$\sum_{M \in \mathbf{F}(\mathcal{M}, \mathbf{s}, \mathbf{e})} p^{ne(M)} q^{se(M)} = \sum_{N \in \mathbf{F}(\mathcal{N}, \mathbf{s}', \mathbf{e})} p^{pos_{C_{\mathcal{N}}}(N)} q^{neg_{C_{\mathcal{N}}}(N)}.$$

2. By Theorem 3,

$$\sum_{N \in \mathbf{F}(\mathcal{N}, \mathbf{s}', \mathbf{e})} p^{pos_{C_{\mathcal{N}}}(N)} q^{neg_{C_{\mathcal{N}}}(N)} = \sum_{N \in \mathbf{F}(\mathcal{N}, \mathbf{s}', \mathbf{e})} p^{ne(N)} q^{se(N)}.$$

Hence we can ignore $C_{\mathcal{N}}$, or equivalently, reset $C_{\mathcal{N}} = +$ in \mathcal{N} .

3. For a filling of \mathcal{N} (with the charge function $+$), rotating it 90 degrees counterclockwise, applying the algorithm τ , and then rotating 90 degrees clockwise, we get a filling $F \in \mathbf{F}(\mathcal{F}, \mathbf{s}', \mathbf{e}')$ of a top and right aligned Ferrers shape \mathcal{F} , together with a charge function C_1 on \mathcal{F} , where \mathbf{e}' is the sequence obtained from \mathbf{e} in the same way as the columns of \mathcal{F} are obtained from the columns of \mathcal{N} . Now we have

$$\sum_{N \in \mathbf{F}(\mathcal{N}, \mathbf{s}', \mathbf{e})} p^{ne(N)} q^{se(N)} = \sum_{F \in \mathbf{F}(\mathcal{F}, \mathbf{s}', \mathbf{e}')} p^{pos_{C_1}(F)} q^{neg_{C_1}(F)} = \sum_{F \in \mathbf{F}(\mathcal{F}, \mathbf{s}', \mathbf{e}')} p^{ne(F)} q^{se(F)}.$$

Fig. 13 shows an example of transforming a stack polyomino into a Ferrers shape.

4.2. Equivalence with crossings and nestings of linked partitions

Now it is sufficient to compute the distribution of $(ne(F), se(F))$ over fillings in $\mathbf{F}(\mathcal{F}, \mathbf{s}', \mathbf{e}')$ where \mathcal{F} is a Ferrers shape with the same row/column lengths as \mathcal{M} , and $\mathbf{s}' \in \mathbb{N}^n$, $\mathbf{e}' \in \{0, 1\}^m$ are given vectors.

It is first observed by de Mier [7] that fillings of Ferrers shapes with prescribed row and column sums are in one-to-one correspondence with loop-less graphs on $[n]$ with given left–right degree sequence. In a graph G on $[n]$, the left (resp. right) degree of a vertex i is the number of edges that join i

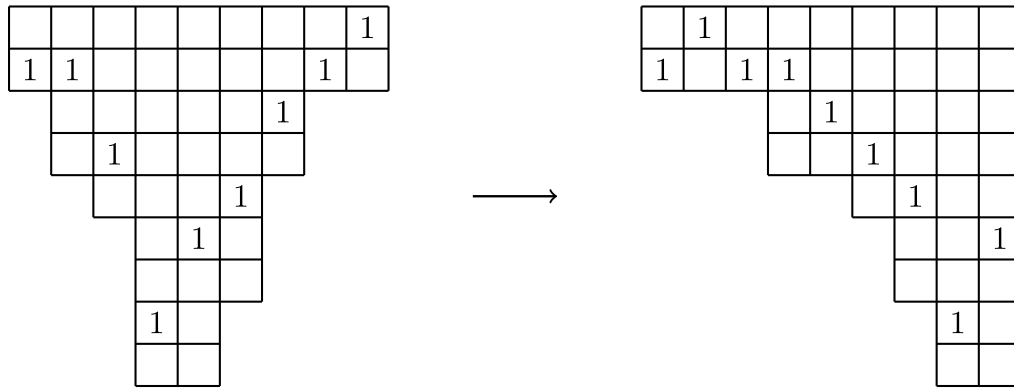


Fig. 13. Reduce a stack polyomino to a Ferrers shape.

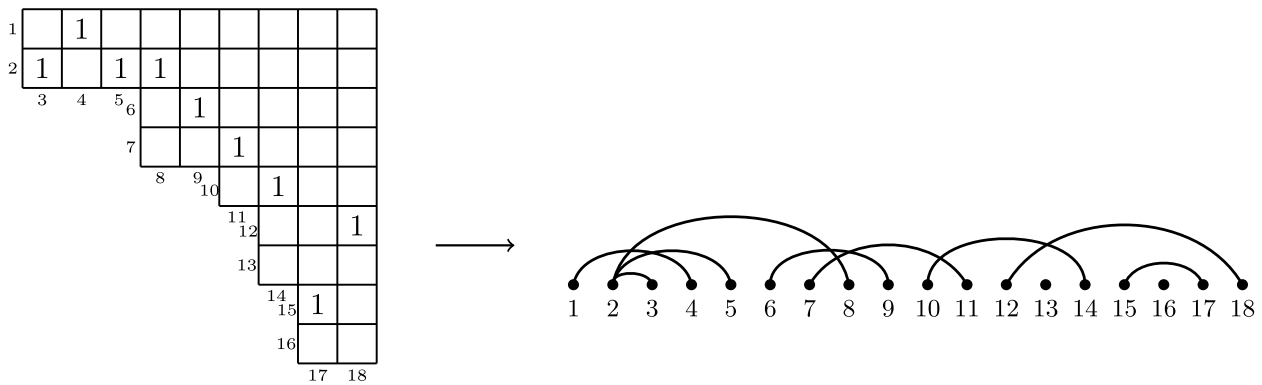


Fig. 14. Correspondence between fillings of Ferrers diagram and graphs on $[n]$.

to a vertex j with $j < i$ (resp. $j > i$). The left–right degree sequence of G is the sequence $(l_i, r_i)_{1 \leq i \leq n}$, where l_i is the left-degree of i and r_i is the right degree of i . In particular, if in each pair (l_i, r_i) , either l_i or r_i is 0, the graph is called a *left–right graph*. A bijection is described by de Mier [7] between the fillings of Ferrers shapes to left–right graphs, which can be stated as follows.

Let \mathcal{F} be a Ferrers shape with n rows and m columns, which is aligned on the top and right. Starting from the point on the top left corner of \mathcal{F} , travel along the boundary of the polyomino using south step $(0, -1)$ and east step $(1, 0)$ until the bottom right corner of \mathcal{F} is reached, and label the steps from 1 to $n + m$. Assume that the south steps are labeled $\{a_1, \dots, a_n\}$ while the east steps are labeled $\{b_1, \dots, b_m\}$. For $1 \leq k \leq n + m$, let $u_k = (0, s'_i)$ if $k = a_i$; otherwise $u_k = (e'_j, 0)$ if $k = b_j$.

For a filling $F \in \mathbf{F}(\mathcal{F}, \mathbf{s}', \mathbf{e}')$, define a graph $G(F)$ on the vertex set $[n + m]$ by letting a_i adjacent to b_j if and only if there is a 1 in the cell lying in the row labeled by a_i and the column labeled by b_j . Fig. 14 shows the graph corresponding to the filling on the right of Fig. 13.

Clearly $G(F)$ is a left–right graph where the left–right degree of the vertex k is u_k . This gives a one-to-one correspondence between the set $\mathbf{F}(\mathcal{F}, \mathbf{s}', \mathbf{e}')$ and the set $G[n + m; \mathbf{u}]$ that consists of graphs on $[n + m]$ with the left–right degree sequence $\mathbf{u} = (u_1, \dots, u_{n+m})$. Under this correspondence the NE/SE chains in a filling become nestings and crossings formed by two edges in a graph.

Note that in a graph in $G[n + m; \mathbf{u}]$, the left-degree of any vertex is either 0 or 1. Such graphs have been studied in [4] under the name *linked partitions*. Using the notation of [4, Section 3], $G[n + m; \mathbf{u}]$ is exactly the set $LP_{n+m}(S, T)$ of linked partitions on $[n + m]$ with the multisets $S = \{a_1^{s'_1}, \dots, a_n^{s'_n}\}$ and $T = \{b_j: e'_j \neq 0\}$. The joint distribution of the numbers of crossings and nestings is given in Theorem 3.5 of [4]. Translating to fillings of \mathcal{F} , we obtain the following theorem, which agrees with Kasraoui's on moon polyominoes [11, Theorem 2.2].

Let $[r]_{p,q}$ be the p, q -integer

$$[r]_{p,q} = \frac{p^r - q^r}{p - q} = p^{r-1} + p^{r-2}q + \dots + pq^{r-2} + q^{r-1}.$$

The p, q -factorial $[r]_{p,q}!$ is defined as $[r]_{p,q}! = \prod_{i=1}^r [i]_{p,q}$, and the (p, q) -Gaussian coefficient $\begin{bmatrix} h \\ s \end{bmatrix}_{p,q}$ is given by

$$\begin{bmatrix} h \\ s \end{bmatrix}_{p,q} = \frac{[h]_{p,q}!}{[s]_{p,q}![h-s]_{p,q}!}.$$

Theorem 10. For each $1 \leq i \leq n$, let $h_i = |\{j \in T: j > a_i\}| - |\{j \in S: j > a_i\}|$. Then

$$\sum_{F \in \mathcal{F}(\mathcal{F}, \mathbf{s}', \mathbf{e}')} p^{\text{ne}(F)} q^{\text{se}(F)} = \prod_{i=1}^n \begin{bmatrix} h_i \\ s_i \end{bmatrix}_{p,q}. \tag{6}$$

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