Supplement: On the equation in Corollary 4.7

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In Corollary 4.7 of [3] we obtained the following interesting equation of the Narayana numbers $N(n,k) = \frac{1}{k}\binom{n}{k}\binom{n}{k-1}$.

$$\sum_{p\geq 0} N(1+p+q, 1+p)z^p = \frac{N_q(z)}{(1-z)^{2q+1}}, \tag{1}$$

where $N_q(z) = \sum_{k=1}^{n} N(n,k)z^{k-1}$.

In a talk presented at the International Conference on Enumerative Combinatorics and Applications Virtual (ICECA 2023, Sep 4–6, 2023), I asked whether anyone has seen this equation or something similar before. Thanks for replies from Mireille Bousquet-Melou, Richard Stanley, and Sergi Elizalde, who provided valuable information on this equation.

1 Parallelogram polyomino

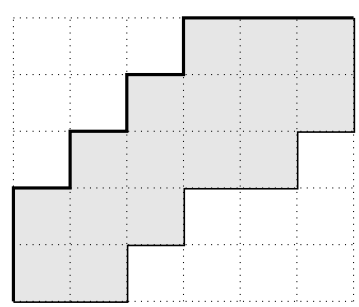

Figure 1: A parallelogram polyomino having a 6×5 bounding box.

Equation (1) has been proved by Mireille Bousquet-Melou [1, 2] as the generating function of the number of parallelogram polyominoes having a fixed width.

A *parallelogram polyomino having an $m \times n$ bounding box* is a polyomino in a rectangle consisting of $m \times n$ cells that is formed by cutting out two (possibly empty) non-touching Young diagrams which have corners at $(0, n)$ and $(m, 0)$. See Figure 1 for a parallelogram polyomino having a 6×5 bounding box.

In [2, Eq.(19)] Bousquet-Melou proved that $P_{n,m}$, the number of parallelogram polyominoes within a $p \times q$ bounding box, is the Narayana numner $N(p+q-1,p)$. In an accompanying paper [1] she derived explicit expressions for the generating function of parallelogram polyominoes, according lo their height, width and area. A specialization of the generating function of parallelogram polyominoes with width n, given in Equations (23)–(25) in [1], can be stated as

$$P(y) := \sum_m y^m P_{n,m} = y \frac{N_{n-1}(y)}{(1-y)^{2n-1}}.$$

Given a parallelogram polyomino with an $m \times n$ bounding box, removing the two cells at the lower-left corner and the upper-right corner, we obtain two lattice paths in the boundary of the polyomino, namely, L_1 from $(1,0)$ to $(m, n-1)$ and L_2 from $(0,1)$ to $(m-1,n)$. Moving L_1 one unit to the left and L_2 one unit down, we obtain a pair of non-crossing lattice paths from $(0,0)$ to $(m-1,n-1)$. Figure 2 shows the non-crossing lattice paths corresponding to the parallelogram polyomino in Figure 1. This above process is invertable and gives a bijection between parallelogram polyominoes with an $m \times n$ bounding box and pairs of non-crossing lattice paths with the box $(m-1) \times (n-1)$. The latter correspond to the set of $(m-1, n-1)$-parking functions. Changing variables, we see that Corollary 4.7 in our paper matches exactly the results of Bousquet-Melou.

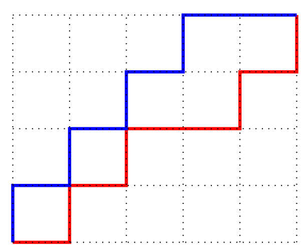

Figure 2: A pair of non-crossing lattice paths in a 5×4 box.

2 Order polynomial and P-Eulerian polynomial

This interpretation of (1) is communicated by Richard Stanley.

In [3, Prop. 4.3, Cor. 4.4] we obtained that $N(1+p+q, 1+p)$ is number of multichains of length $p+1$ in the poset $\mathbf{2} \times \boldsymbol{q}$, which also counts the number of order preserving maps from $\mathbf{2} \times \boldsymbol{q}$ to $\boldsymbol{p} + \mathbf{1}$. Thus the left side of Equation (1) is the generating function for the order polynomial of the poset $\mathbf{2} \times \boldsymbol{q}$ (up to a factor z).

From [5, Theorem 4.5.14], the generating function of the order polynomial of a poset P can be expressed as $P(X)/(1-z)^{n+1}$, where $n = |P|$ and $P(x)$ is the P-Eulerian polynomial.

Some background on P-Eulerian polynomial: Let $(P, \preceq)$ be a poset on the vertex set $[n]$. The Jordan-Hölder set of P is the set of linear extensions of P:

$$\mathcal{L}(P) := \{\pi \in \mathfrak{S}_n : \text{ if } \pi_i \preceq \pi_j, \text{ then } i \leq j \text{ for all } i, j \in [n]\},$$

where each permutation $\pi = \pi_1\pi_2\cdots\pi_n$ in the symmetric group $\mathfrak{S}_n$ is written in one-line notation. The P-Eulerian polynomial is defined by

$$A_P(x) := \sum_{\pi\in\mathcal{L}(P)} x^{\mathrm{des}(\pi)+1}. \tag{2}$$

When $P = \mathbf{2}\times\boldsymbol{q}$, we can assume that the vertex set is $[2q]$ and the partial order $\preceq$ is given by $1 \preceq 2 \preceq \cdots \preceq q$, $q+1 \preceq q+2 \preceq \cdots \preceq 2q$ and $i \preceq q+i$ for $i \in [q]$. Hence each linear extension of P corresponds to a Dyck path of semilength q, and a descent corresponds to a valley of the Dyck path. Since the number of valleys is one less than the number of peaks, we have that the P-Eulerian polynomial is the generating function of Dyck paths by the number of peaks, which is $zN_q(z)$.

The above argument gives the equation

$$\sum_{p\geq 0} N(1+p+q, 1+p)z^{p+1} = \frac{zN_q(z)}{(1-z)^{2q+1}},$$

which is equivalent to Equation (1).

Remark. For any finite poset P, the order polytope $O(P)$, and a related chain polytope $C(P)$, have the property that their Ehrhart polynomial is the order polynomial of P (shifted by 1). This implies that the h^*-vector of $O(P)$ is the P-Eulerian polynomial. See [4, Section 4].

Equivalently, $N_q(z)$ is the h^*-vector of the order polytope of $\mathbf{2}\times\boldsymbol{q}$. By the hook-content formula, the coefficients of the generating function are

$$(p+1)(p+2)^2(p+3)^2...(p+q)^2(p+q+1)/(12^2...q^2(q+1)) = N(1+p+q, 1+p),$$

agreeing with our result.

3 A bijective proof

The following bijective proof of (1) is constructed by Sergi Elizalde.

The coefficient of z^p in the left-hand side of Equation (1) counts Dyck paths of semilength $p+q+1$ with $p+1$ peaks.

The coefficient of z^p in the right-hand side counts Dyck paths of semilength q (given by the numerator), together with a sequence of nonnegative integers $(a_0, a_1, \ldots, a_{2q})$ (given by the denominator). If the path has k peaks, then we must have $a_0 + a_1 + \cdots + a_{2q} = p+1-k$, to get a z^p. We can think of $(a_0, \ldots, a_{2q})$ as labels on the $2q+1$ vertices of the path.

Now here is a bijection between the two sides. Take a Dyck path of semilength $p+q+1$ with $p+1$ peaks. Remove the $p+1$ peaks UD of this path, and put a label on each vertex indicating how many peaks were removed from that location.

This gives a bijection to Dyck paths of semilength q together with a sequence of nonnegative integers $(b_0, b_1, \ldots, b_{2q})$ indicating how many peaks were removed at each vertex, with the condition that $b_0 + \cdots + b_{2q} = p+1$, and that each vertex on top of a peak must have at a label $b_i > 0$ (otherwise it would have been a peak of the original path and it would have been removed).

Now, subtract one from each label at a peak.

This gives a bijection to Dyck paths of semilength q together with a sequence of nonnegative integers $(a_0, a_1, \ldots, a_{2q})$ such that $a_0 + \cdots + a_{2q} = p + 1 - k$, where k is the number of peaks of the path.

References

[1] Mireille Bousquet-Melou. Convex polyominoes and heaps of segments. J. Phys. A: Math. Gen. 25 (1992), 1925–1934.

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[3] Lauren Snider and Catherine Yan. U-Parking Functions and (p, q)-Parking Functions. Adv. Appl. Math. vol.134, 2022, 102309.

[4] Richard P. Stanley. Two poset polytopes. Discrete Comput. Geom. 1(1986), 9–23. `https://math.mit.edu/~rstan/pubs/pubfiles/66.pdf`.

[5] R.P. Stanley. Enumerative Combinatorics, Vol. 1. Cambridge University Press, 1997.