Monomial ideals in affine semigroup rings

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Goal and monomial ideal

- **Goal:** Understand the monomial ideal in the affine semigroup ring in a combinatorial way.
- **Motivation:** Both monomial ideals and affine semigroup rings are rich subject for combinatorial study.

**Definition (Monomial and its ideal.)**

A *monomial* denotes a polynomial with one term over a field $\mathbb{K}$. A *monomial ideal* is an ideal generated by monomials.

**Example**

$$x^2yz^3 \in \mathbb{K}[x, y, z] \rightarrow x^{(2,1,3)} \in \mathbb{K}[x_1, x_2, x_3]$$

It is natural to depict a monomial as a lattice point in $\mathbb{Z}^d$ ($\mathbb{Z}^d$-graded).
Affine semigroup ring

- **Affine semigroup**: $\mathbb{NA} := \{ A \cdot u : u \in \mathbb{N}^n \}$ where $A := \{ a_1, \cdots, a_n \} \subset \mathbb{Z}^d$ as a $d \times n$ matrix;

- **Affine semigroup ring**: $\mathbb{K}[\mathbb{NA}] := \mathbb{K}[t^{a_1}, \cdots, t^{a_n}]$ as a subring of the Laurent polynomial ring $\mathbb{K}[t_1^{\pm}, \cdots, t_d^{\pm}]$ ($\mathbb{Z}^d$-graded)

- **Monomial ideal**: a homogeneous ideal in $\mathbb{K}[\mathbb{NA}]$.

- $\mathbb{K}[\mathbb{NA}]$ is *normal* if $\mathbb{NA} = \mathbb{R}_{\geq 0} A \cap \mathbb{Z} A$.

Figure: (L) \[ \mathbb{K}[x, y] \begin{cases} I = \langle x^3 y^1, xy^2 \rangle \end{cases} \]
(R) \[ \mathbb{K}[s, st, st^2] \begin{cases} I = \langle s^2 t^2, s^3 t \rangle \end{cases} \]
Non-normal example

\[ A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \end{pmatrix}, \ \mathbb{K}[\mathbb{N}A] \subset \mathbb{K}[x, y, z], \ l = \langle x, xyz, xyz^2 \rangle: \]
Facts for a monomial ideal \( I \subset K[\mathbb{N}A] \) (Helm and Miller, 2005; Miller and Sturmfels, 2005)

- A monomial prime ideal \( \leftrightarrow \) A face of \( \mathbb{R}_{\geq 0}A \).
- Irreducible decomposition exists.
- If \( K[\mathbb{N}A] \) is normal, \( \exists \) an algorithmic irreducible decomposition and irreducible resolution.

*Standard pairs* can be used to generalizes the above to the nonnormal case.
Standard Pairs

- **F**: *face* of $A$ if $F = A \cap H$ for a face $H$ of $\mathbb{R}_{\geq 0}A$.
- $(a, F)$: *proper pair* if $(a + \mathbb{N}F) \cap I = \emptyset$.
- $(a, F) < (b, G)$ if $a + \mathbb{N}F \subseteq b + \mathbb{N}G$.
- $(a, F)$ is *standard* if maximal w.r.t. $<$.
- $(a, F)$ *divides* $(b, G)$ if $\exists c \in \mathbb{N}A \text{ s.t. } a + c + \mathbb{N}F \subset b + \mathbb{N}G$
- $(a, F)$ and $(b, G)$ *overlap* if they divide each other. (Equivalence relation)
Example of Standard Pairs

\[ A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \end{pmatrix}, \ K[N\Lambda] \subset K[x, y, z], I = \langle x, xyz, xyz^2 \rangle : \]
Example of Standard Pairs

Figure: Standard pairs of $I = \langle x, xyz, xyz^2 \rangle$ in $K[NA]$
Example of Standard Pairs: Overlap class

\[ A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad I = \langle x^{(2,0,2)}, x^{(2,1,2)}, x^{(2,1,1)} \rangle, \quad F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \]

Standard Pairs: \((0, F)\), \((x^{(0,0,1)}, F)\), and \((x^{(0,1,1)}, F)\).

Overlap happens between \((x^{(0,0,1)}, F)\) and \((x^{(0,1,1)}, F)\).
Main Result of (Matusevich and Yu, 2020)

Given a monomial ideal $I$ in $\mathbb{K}[NA]$, 

- $I$ is **primary** iff all standard pairs of $I$ correspond to a same face.
- $I$ is **irreducible** iff $I$ is primary and has the unique maximal overlap classes of the standard pairs w.r.t. divisibility.
- $I$ has associated prime $P_F$ iff $I$ has a standard pair $(a, F)$.
- The **multiplicity** of $P_F = \#$ of overlap classes of $I$ whose face belongs to $F$.
- $\#$ of maximal (w.r.t divisibility) overlap classes of $I = \#$ of components of an irreducible irredundant decomposition of $I$. 
Example of Irreducible Decomposition

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \ K[\mathbb{N}A] = K[s, st, st^2, st^3], \ l = \langle s^3, s^2t, s^2t^4 \rangle. \]

Standard Pairs: two red lines and four red circles.
Example of Irreducible Decomposition

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \mathbb{K}[\mathbb{N}A] = \mathbb{K}[s, st, st^2, st^3], l = \langle s^3, s^2 t, s^2 t^4 \rangle.$$ 
Maximal Overlap Classes
Example of Irreducible Decomposition

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \quad K[NA] = K[s, st, st^2, st^3], \quad l = \langle s^3, s^2 t, s^2 t^4 \rangle. \]

\[ l = \langle s, st \rangle \cap \langle s^2, s^2 t, s^2 t^2, s^2 t^4, s^2 t^5, s^2 t^6 \rangle \cap \langle st^3, s^2 t, s^2 \rangle \cap \langle st, st^2, st^3, s^3 \rangle. \]
Example of Irreducible Decomposition

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \quad \mathbb{K}[\mathbb{N}A] = \mathbb{K}[s, st, st^2, st^3], \quad l = \langle s^3, s^2 t, s^2 t^4 \rangle. \]

\[ l = \langle s, st \rangle \cap \langle s^2, s^2 t, s^2 t^2, s^2 t^4, s^2 t^5, s^2 t^6 \rangle \cap \langle st^3, s^2 t, s^2 \rangle \cap \langle st, st^2, st^3, s^3 \rangle. \]
Computation of monomial ideals

- Polynomial ring case is known and adopted into Macaulay 2. (Eisenbud et al., 2002)
- General Affine semigroup: `stdPairs.spyx` (Yu, 2020)
  - Library in a SageMath.
  - Compatible with Macaulay2.
  - One can save and load his/her computation on the monomial ideal.
Thank you for listening!


