Chapter 3

Eigenvalues and Eigenvectors

In this chapter we begin our study of the most important, and certainly the most dominant aspect, of matrix theory. Called spectral theory, it allows us to give fundamental structure theorems for matrices and to develop power tools for comparing and computing with matrices. We begin with a study of norms on matrices.

3.1 Matrix Norms

We know M_n is a vector space. It is most useful to apply a metric on this vector space. The reasons are manifold, ranging from general information of a metrized system to perturbation theory where the "smallness" of a matrix must be measured. For that reason we define metrics called matrix norms that are regular norms with one additional property pertaining to the matrix product.

Definition 3.1.1. Let $A \in M_n$. Recall that a *norm*, $\|\cdot\|$, on any vector space satifies the properties:

(i) $\|A\| \geq 0$ and $|A\| = 0$ if and only if $A = 0$

(ii) $\|cA\| = |c|\|A\|$ for $c \in R$

(iii) $\|A + B\| \leq \|A\| + \|B\|$.

There is a true vector product on M_n defined by matrix multiplication. In this connection we say that the norm is **submultiplicative** if

(iv) $\|AB\| \leq \|A\|\|B\|$

In the case that the norm $\|\cdot\|$ satifies all four properties (i) - (iv) we call it a **matrix norm**.

Here are a few simple consequences for matrix norms. The proofs are straightforward.

Proposition 3.1.1. *Let $\|\cdot\|$ be a matrix norm on M_n, and suppose that $A \in M_n$. Then*

(a) $\|A\|^2 \leq \|A\|^2$, $\|A^p\| \leq \|A\|^p$, $p = 2, 3, \ldots$

(b) If $A^2 = A$ then $\|A\| \geq 1$

(c) If A is invertible, then $\|A^{-1}\| \geq \frac{\|I\|}{\|A\|}$

(d) $\|I\| \geq 1$.

Proof. The proof of (a) is a consequence of induction. Supposing that $A^2 = A$, we have by the submultiplicativity property that $\|A\| = \left\|A^2\right\| \leq \|A\|^2$. Hence $\|A\| \geq 1$, and therefore (b) follows. If A is invertible, we apply the submultiplicativity again to obtain $\|I\| = \left\|AA^{-1}\right\| \leq \|A\| \left\|A^{-1}\right\|$, whence (c) follows. Finally, (d) follows because $I^2 = I$ and (b) applies. □

Matrices for which $A^2 = A$ are called **idempotent**. Idempotent matrices turn up in most unlikely places and are useful for applications.

Examples. We can easily apply standard vector space type norms, i.e. ℓ_1, ℓ_2, and ℓ_∞ to matrices. Indeed, an $n \times n$ matrix can clearly be viewed as an element of $\mathbb{C}_{n^2}$ with the coordinate stacked in rows of n numbers each. The trick is usually to verify the submultiplicativity condition (iv).

1. ℓ_1. Define

$$\|A\|_1 = \sum_{i,j} |a_{ij}|$$

The usual norm conditions (i)–(iii) hold. To show submultiplicativity we write

$$\begin{aligned}
\|AB\|_1 &= \sum_{ij} \left| \sum_k a_{ik} b_{kj} \right| \\
&\leq \sum_{ij} \sum_k |a_{ik}| |b_{kj}| \\
&\leq \sum_{ijkm} |a_{ik}| |b_{mj}| = \sum_{i,k} |a_{ik}| \sum_{mj} |b_{mj}| \\
&= \|A\|_1 \|B\|_1.
\end{aligned}$$

Thus $\|A\|_1$ is a matrix norm.
2. ℓ_2. Define

$$\|A\|_2 = \left(\sum_{i,j} a_{ij}^2\right)^{1/2}$$

conditions (i)–(iii) clearly hold. $\|A\|_2$ is also a matrix norm as we see by application of the Cauchy–Schwartz inequality. We have

$$\begin{aligned}
\|AB\|_2^2 = \sum_{ij}\left(\sum_k a_{ik}b_{kj}\right)^2 &\le \sum_{i,j}\left(\sum_k a_{ik}^2\right)\left(\sum_m b_{jm}^2\right) \\
&= \left(\sum_{i,k} |a_{ik}|^2\right)\left(\sum_{j,m} |b_{jm}|^2\right) \\
&= \|A\|_2^2\|B\|_2^2.
\end{aligned}$$

This norm has three common names: The (a) Frobenius norm, (b) Schur norm, and (c) Hilbert–Schmidt norm. It has considerable importance in matrix theory.
3. ℓ_∞. Define for $A \in M_n(R)$

$$\|A\|_\infty = \sup_{i,j} |a_{ij}| = \max_{i,j} |a_{ij}|.$$

Note that if $J = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, $\|J\|_\infty = 1$. Also $J^2 = 2J$. Thus $\|J^2\| = 2\|J\| = 1 \not\le \|J\|^2$. So $\|A\|_\infty$ is not a matrix norm, though it is a vector space norm. We can make it into a matrix norm by

$$\|A\| = n\|A\|_\infty.$$

Note

$$\begin{aligned}
|||AB||| &= n \max_{i,j}\left|\sum_k a_{ik}b_{kj}\right| \\
&\le n \max_{ij} n \max_k |a_{ik}||b_{kj}| \\
&\le n^2 \max_{i,k} |a_{ik}| \max_{k,j} |b_{kj}| \\
&= |||A||| \;\; |||B|||.
\end{aligned}$$

In the inequalities above we use the fundamental inequality

$$\sum_k |c_k d_k| \le \max_k |d_k| \sum_k |c_k|$$

(See Exercise 4.) While these norms have some use in general matrix theory, most of the widely applicable norms are those that are subordinate to vector norms in the manner defined below.

Definition 3.1.2. Let $\|\cdot\|$ be a vector norm on R_n (or $\mathbb{C}_n$). For $A \in M_n(R)$ (or $M_n(\mathbb{C})$) we define the norm $\|A\|$ on M_n by

$$\|A\| = \max_{\|x\|=1} \|Ax\|. \tag{$\star$}$$

and call $\|A\|$ the norm **subordinate** to the vector norm. Note the use of the same notation for both the vector and subordinate norms.

Theorem 3.1.1. *The subordinate norm is a matrix norm and* $\|Ax\| \le \|A\|\|x\|$.

Proof. We need to verify conditions (i)–(iv). Conditions (i) and (ii) are obvious and are left to the reader . To show (iii), we have

$$\begin{aligned} \|A+B\| &= \max_{\|x\|=1} \|(A+B)x\| \le \max_{\|x\|=1} (\|Ax\| + \|Bx\|) \\ &\le \max_{\|x\|=1} \|Ax\| + \max_{\|x\|=1} \|Bx\| \\ &= \|A\| + \|B\|. \end{aligned}$$

Note that

$$\begin{aligned} \|Ax\| &= \|x\| A \left(\frac{x}{\|x\|} \right) \\ &\le \|A\| \; \|x\| \end{aligned}$$

since $\left\| \frac{x}{\|x\|} \right\| = 1$. Finally, it follows that for any $x \in R_n$

$$\|ABx\| \le \|A\|\|Bx\| \le \|A\|\|B\|\|x\|$$

and therefore $\|AB\| \le \|A\|\|B\|$. ☐

Corollary 3.1.1. *(i)* $\|I\| = 1$.

(ii) If A is invertible, then

$$\|A^{-1}\| \geq (\|A\|)^{-1}.$$

Proof. For (i) we have

$$\|I\| = \max_{\|x\|=1} \|Ix\| = \max_{\|x\|=1} \|x\| = 1.$$

To prove (ii) begin with $A^{-1}A = I$. Then by the submultiplicativity and (i)

$$1 = \|I\| \leq \|A^{-1}\|\|A\|$$

and so $\|A^{-1}\| \geq 1/\|A\|$. □

There are many results connected with matrix norms and eigenvectors that we shall explore before long. The relation between the norm of the matrix and its inverse is important in computational linear algebra. The quantity $\|A^{-1}\|\,\|A\|$ the **condition number** of the matrix A. When it is very large, the solution of the linear system $Ax = b$ by general methods such as Gaussian elimination may produce results with considerable error. The condition number, therefore, tips off investigators to this possibility. Naturally enough the condition number may be difficult to compute accurately in exactly these circumstances. Alternative and very approximate methods are often used as reliable substitutes for the condition number.

A special type of matrix, one for which $\|Ax\| = \|x\|$ for every $x \in \mathbb{C}$, is called an **isometry**. Such matrices which do not "stretch" any vectors have remarkable spectral properties and play an important roll in spectral theory.

3.2 Convergence and perturbation theory

It will often be necessary to compare one matrix with another matrix that is *nearby* in some sense. When a matrix norm at is hand it is possible to measure the proximity of two matrices by computing the norm of their difference. This is just as we do for numbers. We begin with this study by showing that if the norm of a matrix is less than one, then its difference with the identity is invertible. Again, this is just as with numbers; that is, if $|r| < 1$, $\frac{1}{1-r}$ is defined. Let us assume $R \in M_n$ and $\|\quad\|$ is some norm on M_n. We want to show that if $\|R\| < 1$ then $(I - R)^{-1}$ exists. Toward this end we prove the following lemma.

Lemma 3.2.1. *For every $R \in M_n$*

$$(I - R)(I + R + R^2 + \cdots + R_n) = I - R^{n+1}.$$

Proof. This result for matrices is the direct analog of the result for numbers $(1-r)(1+r+r^2+\cdots+r^n) = 1-r^{n+1}$, also often written as $1+r+r^2+\cdots+r^n = \frac{1-r^{n+1}}{1-r}$. We prove the result inductively. If $n = 1$ the result follows from direct computation, $(I - R)(I + R) = I - R^2$. Assume the result holds up to $n - 1$. Then

$$\begin{aligned}(I - R)(I + R + R^2 + \cdots + R^{n-1} + R^n) &= (I - R)(I + R + \cdots + R^{n-1}) \\ &\quad + (I - R)R^n \\ &= (I - R^n) + R^n - R^{n+1} = I - R^{n+1}\end{aligned}$$

by our inductive hypothesis. This calculation completes the induction, and hence the proof. □

Remark 3.2.1. Sometimes the proof is presented in a "quasi-inductive" manner. That is, you will see

$$\begin{aligned}(I - R)(I + R + R^2 + \cdots + R^n) &= (I + R + R^2 + \cdots + R^n) \\ &\quad - (R + R^2 + \cdots + R^{n+1}) \qquad (*) \\ &= I - R^{n+1}\end{aligned}$$

This is usually considered acceptable because the correct induction is transparent in the calculation.

Below we will show that if $\|R\| = \lambda < 1$, then $(I + R + R^2 + \cdots) = (I - R)^{-1}$. It would be *incorrect* to apply the obvious fact that $\|R^{n+1}\| < \lambda^{n+1} \to \infty$ to draw the conclusion from the equality $(*)$ above without first establishing convergence of the series $\sum\limits_0^\infty R^k$. A crucial step in showing that an infinite series is convergent is showing that its partial sums satisfy the Cauchy criterion: $\sum_{k=1}^\infty a_k$ converges if and only if for each $\varepsilon > 0$, there exists an integer N such that if $m, n > N$, then $\left|\sum_{k=m+1}^n a_k\right| < \varepsilon$. (See Appendix A.) There is just one more aspect of this problem. While it is easy to establishe the Cauchy criterion for our present situation, we still need to resolve the situation between *norm convergence* and *pointwise convergence*. We need to conclude that if $\|R\| < 1$ then $\lim_{n\to\infty} R^n = 0$, and by this expression we mean that $(R^n)_{ij} \to 0$ for all $1 \le i, j \le n$.

Lemma 3.2.2. *Suppose that the norm $\|\cdot\|$ is a subordinate norm on M_n and $R \in M_n$.*

(i) If $\|R\| < \varepsilon$, then there is a constant M such that $|p_{ij}| < M\varepsilon$.

(ii) If $\lim_{n\to\infty} \|R^n\| = 0$, then $\lim_{n\to\infty} R^n = 0$.

Proof. (i) If $\|R\| < \epsilon$, if follows that $\|Rx\| < \epsilon$ for each vector x, and by selecting the standard vectors e_j in turn, it follows that from which it follows that $\|r_{*j}\| < \epsilon$, where r_{*j} denotes the j^{th} column of R. By Theorem 1.6.2 all norms are equivalent. It follows that there is a fixed constant M independent of ϵ and R such that $|r_{ij}| < M\epsilon$.
(ii) Suppose for some increasing subsequence of powers $n_k \to \infty$ it happens that $\left|(R^{n_k})_{ij}\right| \geq r$. Select the standard unit vector e_j. A little computation shows that $\|R^{n_k} e_j\| \geq r$, whence $\|R^{n_k}\| \geq r$, contradicting the known limit $\lim_{n\to\infty} \|R^n\| = 0$. The conclusion $\lim_{n\to\infty} R^n = 0$ follows. □

Lemma 3.2.3. *Suppose that the norm $\|\cdot\|$ is a subordinate norm on M_n. If $\|R\| = \lambda < 1$, then $I + R + R^2 + \cdots + R^k + \cdots$ converges.*

Proof. Let $P_n = I + R + \cdots + R^n$. To show convergence we establish that $\{P_n\}$ is a Cauchy sequence. For $n > m$ we have

$$P_n - P_m = \sum_{k=m+1}^{n} R^k$$

Hence

$$\begin{aligned}
\|P_n - P_m\| &= \left\| \sum_{k=m+1}^{n} R^k \right\| \\
&\leq \sum_{k=m+1}^{n} \|R^k\| \\
&\leq \sum_{k=m+1}^{n} \|R\|^k \\
&= \sum_{m+1}^{n} \lambda^k = \lambda^{m+1} \sum_{j=0}^{n-m-1} \lambda^j \\
&\leq \lambda^{m+1}(1-\lambda)^{-1} \to 0
\end{aligned}$$

where in the second last step we used the inequality, which is valid for $0 \leq \lambda \leq 1$. $\sum_{0}^{n-m-1} \lambda^j \leq \sum_{0}^{\infty} \lambda^j < (1-\lambda)^{-1}$. We conclude by Lemma 3.2.2 that the individual matrix entries of the parital sums converge and thus the series itself converges. □

Note that this result is independent of the particular norm. In practice it is often necessary to select a convenient norm to actually carry out or verify particular computations are valid. In the theorem below we complete the analysis of the matrix version of the geometric series, stating that when the norm of a matrix is less than one, the geometric series based on that matrix converges and the inverse of the difference with the identity exists.

Theorem 3.2.1. *If $R \in M_n(F)$ and $\|R\| < 1$ for some norm, then $(I-R)^{-1}$ exists and*

$$(I-R)^{-1} = I + R + R^2 + \cdots = \sum_{k=0}^{\infty} R^k.$$

Proof. Apply the two previous lemmas. □

The perturbation result alluded to above can now be stated and easily proved. In words this result states that if we begin with an invertible matrix and additively perturb it by a sufficiently small amount the result remains invertible. Overall, this is the first of a series of results where what is proved is that some property of a matrix is preserved under additive perturbations.

Corollary 3.2.1. *If $A, B \in M_n$ and A is invertible, then $A + \lambda B$ is invertible for sufficiently small $|\lambda|$ (in R or $\mathbb{C}$).*

Proof. As above we assume that $\|\cdot\|$ is a norm on $M_n(F)$. It is any easy computation to see that

$$A + \lambda B = A(I + \lambda A^{-1}B).$$

Select λ sufficiently small so that $\|\lambda A^{-1}B\| = |\lambda|\|A^{-1}B\| < 1$. Then by the theorem above, $I + \lambda A^{-1}B$ is invertible. Therefore

$$(A + \lambda B)^{-1} = (I + \lambda A^{-1}B)^{-1}A^{-1}$$

and the result follows. □

Another way of stating this is to say that if $A, B \in M_n$ and A has a nonzero determinant, then for sufficiently small λ the matrix $A + \lambda B$ also has a nonzero determinant. This corollary can be applied directly to the identity matrix itself being perturbed by a rank one matrix. In this case the λ can be specified in terms of the two vectors comprising the matrix. (Recall Theorem 2.3.1(8).)

Corollary 3.2.2. *Let $x, y \in R_n$ satisfy $|\langle x, y\rangle| = |\lambda| < 1$. Then $I + xy^T$ is invertible and*

$$(I + xy^T)^{-1} = I - xy^T(1 + \lambda)^{-1}.$$

Proof. We have that $(I + xy^T)^{-1}$ exists by selecting a norm $\|\cdot\|$ consistent with the inner product $\langle\cdot,\cdot\rangle$. (For example, take $\|A\| = \sup_{\|x\|_2=1} \|Ax\|_2$, where $\|\cdot\|_2$ is the Euclidean norm.) It is easy to see that $(xy^T)^k = \lambda^{k-1}xy^T$. Therefore

$$\begin{aligned}(I + xy^T)^{-1} &= I - xy^T + (xy^T)^2 - (xy^T)^3 + \cdots \\ &= I - xy^T + \lambda xy^T - \lambda^2 xy^T + \cdots \\ &= I - xy^T \left(\sum_{k=0}^{n} (-\lambda)^k\right).\end{aligned}$$

Thus

$$(I + xy^T)^{-1} = I - xy^T(1 + \lambda)^{-1}$$

and the result is proved. □

In words we conclude that the perturbation of the identity by a small rank 1 matrix has a *computable* inverse.

3.3 Eigenvectors and Eigenvalues

Throughout this section we will consider only matrices $A \in M_n(\mathbb{C})$ or $M_n(R)$. Furthermore, we suppress the field designation unless it is relevant.

Definition 3.3.1. If $A \in M_n$ and $x \in \mathbb{C}_n$ or R_n. If there is a constant $\lambda \in \mathbb{C}$ and a vector $x \neq 0$ for which

$$Ax = \lambda x$$

we call λ an **eigenvalue** of A and x its corresponding **eigenvector**. Alternatively, we call x the eigenvector *pertaining* to the eigenvalue λ, and vice-versa.

Definition 3.3.2. For $A \in M_n$, define

(1) $\sigma(A) = \{\lambda \mid Ax = \lambda x$ has a solution for a nonzero vector $x\}$. $\sigma(A)$ is called the **spectrum** of A.

(2) $\rho(A) = \sup\limits_{\lambda\in\sigma(A)} |\lambda| \quad \left(\text{or equivalently } \max\limits_{\lambda\in\sigma(A)} |\lambda|\right)$. $\rho(A)$ is called the **spectral radius**.

Example 3.3.1. Let $A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$. Then $\lambda = 1$ is an eigenvalue of A with eigenvector $x = [-1, 1]^T$. Also $\lambda = 3$ is an eigenvalue of A with eigenvector $x = (1, 1)^T$. The spectrum of A is $\sigma(A) = \{1, 3\}$ and the spectral radius of A is $\rho(A) = 3$.

Example 3.3.2. The 3×3 matrix $B = \begin{bmatrix} -3 & 0 & 6 \\ -12 & 9 & 26 \\ 4 & -4 & -9 \end{bmatrix}$ has eigenvalues:

$-1, -3, 1$. Pertaining to the eigenvalues are the eigenvectors

$$\left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow -3 \left\{\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}\right\} \leftrightarrow -1$$

The characteristic polynomial

To say that $Ax = \lambda x$ has a nontrivial solution ($x \neq 0$) for some $\lambda \in \mathbb{C}$ is the same as the assertion that $(A - \lambda I)x = 0$ has a nontrivial solution. This means that

$$\det(A - \lambda I) = 0$$

or what is more commonly written

$$\det(\lambda I - A) = 0.$$

From the original definition (Definition 2.5.1) the determinant is sum of products of individual matrix entries. Therefore, $\det(\lambda I - A)$ must be a polynomial in λ. This makes the definition:

Definition 3.3.3. Let $A \in M_n$. The determinant

$$p_A(\lambda) = \det(\lambda I - A)$$

is called the **characteristic polynomial** of A. Its zeros[1] are the called the **eigenvalues** of A. The set $\sigma(A)$ of all eigenvalues of A is called the **spectrum** of A.

A simple consequence of the nature of the determinant of $\det(\lambda I - A)$ is the following.

Proposition 3.3.1. *If $A \in M_n$, then $p_A(\lambda)$ has degree exactly n.*

See Appendix A for basic information on solving polynomials equations $p(\lambda) = 0$. We may note that even though $A \in M_n(\mathbb{C})$ has n^2 entries in its definition, its spectrum is completely determined by the n coefficients of $p_A(\lambda)$.

Procedure. The basic procedure of determining eigenvalues and eigenvectors is this: (1) Solve $\det(\lambda I - A) = 0$ for the eigenvalues λ and (2) for any given eigenvalue λ solve the system $(A - \lambda I)\, x = 0$ for the pertaining eigenvector(s). Though this procedure is not practical in general it can be effective for small sized matrices, and for matrices with special structures.

Theorem 3.3.1. *Let $A \in M_n$. The set of eigenvectors pertaining to any particular eigenvalue is a subspace of the given vector space $\mathbb{C}_n$.*

Proof. Let $\lambda \in \sigma(A)$. The set of eigenvectors pertaining to λ is the null space of $(A - \lambda I)$. The proof is complete by application of Theorem 2.3.1 that states the null space of any matrix is a subspace of the underlying vector space. □

In light of this theorem the following definition makes sense and is a most important concept in the study of eigenvalues and eigenvectors.

Definition 3.3.4. Let $A \in M_n$ and let $\lambda \in \sigma(A)$. The null space of $(A - \lambda I)$ is called the **eigenspace** of A pertaining to λ.

Theorem 3.3.2. *Let $A \in M_n$. Then*

(i) Eigenvectors pertaining to different eigenvalues are linearly independent.

[1]The *zeros* of a polynomial (or more generally a function) $p(\lambda)$ are the solutions to the equation $p(\lambda) = 0$. A solution to $p(\lambda) = 0$ is also called a *root* of the equation.

(ii) Suppose $\lambda \in \sigma(A)$ with eigenvector x is different from the set of eigenvalues $\{\mu_1, \ldots, \mu_k\} \subset \sigma(A)$ and V_μ is the span of the pertaining eigenspaces. Then $x \notin V_\mu$.

Proof. (i) Let $\mu, \lambda \in \sigma(A)$ with $\mu \neq \lambda$ pertaining eigenvectors x and y resepectively. Suppose these vectors are linearly dependent; that is, $y = cx$. Then

$$\begin{aligned} \mu y &= \mu c x = c\mu x \\ &= cAx = A(cx) \\ &= Ay \\ &= \lambda y \end{aligned}$$

This is a contradiction, and (i) is proved.
(ii) Suppose the contrary holds, namely that $x \in V_\mu$. Then $x = a_1 y_1 + \cdots + a_k y_m$ where $\{y_1, \cdots, y_m\}$ are linearly independent vectors of V_μ. Each of the vectors y_i is an eigenvector, we know. Assume the pertaining eigenvalues denoted by μ_{j_i}. That is to say, $Ay_i = \mu_{j_i} y_i$, for each $i = 1, \ldots, m$. Then

$$\begin{aligned} \lambda x &= Ax = A(a_1 y_1 + \cdots + a_m y_m) \\ &= a_1 \mu_{j_1} y_1 + \cdots + a_k \mu_{j_m} y_m \end{aligned}$$

If $\lambda \neq 0$ we have

$$x = a_1 y_1 + \cdots + a_m y_m = a_1 \frac{\mu_{j_1}}{\lambda} y_1 + \cdots + a_k \frac{\mu_{j_m}}{\lambda} y_m$$

We know by the previous part of this result that at least two of the coefficients a_i must be nonzero. (Why?) Thus we have two different representations of the same vector by linearly independent vectors, which is impossible. On the other hand, if $\lambda = 0$ then $a_1 \mu_1 y_1 + \cdots + a_k \mu_k y_k = 0$, which is also impossible. Thus, (ii) is proved.

□

The examples below will illustrate the spectra of various matrices.

Example 3.3.1. Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Then $p_A(\lambda)$ is given by

$$\begin{aligned} p_A(\lambda) = \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} \\ &= (\lambda - a)(\lambda - d) - bc \\ &= \lambda^2 - (a + d)\lambda + ad - bc. \end{aligned}$$

The eigenvalues are the roots of $p_A(\lambda) = 0$

$$\begin{aligned}\lambda &= \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} \\ &= \frac{a+d \pm \sqrt{(a-d)^2 + 4bc}}{2}.\end{aligned}$$

For this quadratic there are three possibilities:

(a) Two real roots ↗ different values, ↘ equal values

(b) Two complex roots

Here are three 2×2 examples that illustrate each possibility. The reader should compute the characteristic polynomials to verify these computation.

$$\begin{aligned}
B_1 &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \quad \text{Then } p_{B_1}(\lambda) = \lambda^2 \qquad \lambda = 0, 0 \\
B_2 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad \text{Then } p_{B_2}(\lambda) = \lambda^2 + 1 \qquad \lambda = \pm i \\
B_3 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad \text{Then } p_{B_3}(\lambda) = \lambda^2 - 1 \qquad \lambda = \pm 1.
\end{aligned}$$

Example 3.3.2. Consider the rotation in the x, z-plane through an angle θ

$$B = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

The characteristic polynomial is given by

$$\begin{aligned}
p_B(\lambda) &= \det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\right) \\
&= -1 + (1 + 2\cos\theta)\lambda - (1 + 2\cos\theta)\lambda^2 + \lambda^3
\end{aligned}$$

The eigenvalues are $1, \cos\theta + \sqrt{\cos^2\theta - 1}, \cos\theta - \sqrt{\cos^2\theta - 1}$. When θ is not equal to an even multiple of π, exactly two of the roots are complex numbers. In fact, they are complex conjugate pairs, which can also be written as $\cos\theta + i\sin\theta, \cos\theta - i\sin\theta$. The magnitude of each eigenvalue is 1, which means all three eigenvalues lie on the unit circle in the complex plane. An interesting observation is that the characteristic polynomial and hence the eigenvalues are the same regardless of which pair of axes (x-z, x-y, or y-z) is selected for the rotation. Matrices of the form B are actually called rotations. In two dimensions the counter-clockwise rotations through the angle θ are given by

$$B_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The eigenvalues for all θ is not equal to an even multiple of π are $\pm i$. (See Exercise 2.)

Example 3.3.3. If T is upper triangular with diag $T = [t_{11}, t_{22}, \ldots, t_{nn}]$. Then $\lambda I - T$ is upper triangular with diag$[\lambda - t_{11}, \lambda - t_{22}, \ldots, \lambda - t_{nn}]$. Thus the determinant of $\lambda I - T$ gives the characteristic polynomial of T to be

$$p_T(\lambda) = \prod_{i=1}^{n} (\lambda - t_{ii})$$

The eigenvalues of T are the diagonal elements of T. By expanding this product we see that

$$p_T(\lambda) = \lambda^n - (\Sigma t_{ii})\lambda^{n-1} + \text{lower order terms.}$$

The constant term of $p_T(\lambda)$ is $(-1)^n \prod_{i=1}^{n} t_{ii} = (-1)^n \det T$. We define

$$\text{tr } T = \sum_{i=1}^{n} t_{ii}$$

and call it the **trace** of T. The same statements apply to lower triangular matrices. Moreover, the trace definition applies to all matrices, not just to triangular ones, and the result will be the same.

Example 3.3.4. Suppose that A is rank 1. Then there are two vectors $w, z \in \mathbb{C}_n$ for which

$$A = wz^T.$$

To find the spectrum of A we consider the equation

$$Ax = \lambda x$$

or

$$\langle z, x \rangle w = \lambda x.$$

From this we see that $x = w$ is an eigenvector with eigenvalue $\langle z, w \rangle$. If $z \perp x$, x is an eigenvector pertaining to the eigenvalue 0. Therefore,

$$\sigma_A = \{\langle z, w \rangle, 0\}.$$

The characteristic polynomial is

$$p_\lambda(\lambda) = (\lambda - \langle z, w \rangle)\lambda^{n-1}.$$

If w and z are orthogonal then

$$p_A(\lambda) = \lambda^n.$$

If w and z are not orthogonal though there are just two eigenvalues, we say that 0 is an eigenvalue of *multiplicity* $n - 1$, the order of the factor $(\lambda - 0)$. Also $\langle z, w \rangle$ has multiplicity 1. This is the subject of the next section.

For instance, suppose $w = (1, -1, 2)$ and $z = (0, 1, -3)$. Then spectrum of the matrix $A = wz^T$ is given by $\sigma(A) = \{-7, 0\}$. The eigenvalue pertaining to $\lambda = -7$ is w and we may take $x = (0, 3, 1)$ and $(c, 3, 1)$, for any $c \in \mathbb{C}$, to be eigenvectors pertaining to $\lambda = 0$. Note that $\{w, (0, 3, 1), (1, 3, 1)\}$ form a basis for R_3.

To complete our discussion of characteristic polynomials we prove a result that every n^{th} degree polynomial with lead coefficient one is the characteristic polynomial of some matrix. You will note the similarity of this result and the analogous result for differential systems.

Theorem 3.3.3. *Every polynomial of n^{th} degree with lead coefficient 1, that is*

$$q(\lambda) = \lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n$$

is the characteristic polynomial of some matrix.

Proof. We consider the $n \times n$ matrix

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ -b_n & -b_{n-1} & & & -b_1 \end{bmatrix}$$

Then $\lambda I - B$ has the form

$$\lambda I - B = \begin{bmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & & 0 \\ \vdots & & \lambda & \ddots & \\ b_n & b_{n-1} & & & \lambda + b_1 \end{bmatrix}$$

Now expand in minors across the bottom row to get

$$\begin{aligned} \det(\lambda I - B) &= b_n (-1)^{n+1} \det \begin{bmatrix} -1 & 0 & \cdots & 0 \\ \lambda & -1 & & 0 \\ & \lambda & \ddots & \end{bmatrix} \\ &\quad + b_{n-1} (-1)^{n+2} \det \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & -1 & & 0 \\ \vdots & \lambda & \ddots & \end{bmatrix} + \cdots \\ &\quad + b_1 (-1)^{n+n} \det \begin{bmatrix} \lambda & -1 & 0 & \cdots \\ 0 & \lambda & -1 & \\ \vdots & & \lambda & \ddots \end{bmatrix} \\ &= b_n (-1)^{n+1} (-1)^{n-1} + b_{n-1} (-1)^{n+2} \lambda (-1)^{n-2} + \cdots \\ &\quad + (\lambda + b_1) (-1)^{n+n} \lambda^{n-1} \\ &= b_n + b_{n-1}\lambda + \cdots + b_1 \lambda^{n-1} + \lambda^n \end{aligned}$$

which is what we set out to prove. (The reader should check carefully the term with b_{n-2} to fully understand the nature of this proof.) □

Multiplicity

Let $A \in M_n(\mathbb{C})$. Since $p_A(\lambda)$ is a polynomial of degree exactly n, it must have exactly n eigenvectors (i.e. roots) $\lambda_1, \lambda_2, \ldots, \lambda_n$ counted according to multiplicity. Recall that the multiplicity of an eigenvalue is the number of times the monomial $(\lambda - \lambda_i)$ is repeated in the factorization of $p_A(\lambda)$. For example the multiplicity of the root 2 in the polynomial $(\lambda - 2)^3(\lambda - 5)$ is 3. Suppose $\mu_1, \ldots, \mu_k$ are the distinct eigenvalues with multiplicities $m_1, m_2, \ldots, m_k$ respectively. Then the characteristic polynomial can be rewritten as

$$\begin{aligned} p_A(\lambda) = \det(\lambda I - A) &= \prod_1^n (\lambda - \lambda_i) \\ &= \prod_1^k (\lambda - \mu_i)^{m_i}. \end{aligned}$$

More precisely, the multiplicities $m_1, m_2, \ldots, m_k$ are called the *algebraic multiplicities* of the respective eigenvalues. This factorization will be very useful later.

We know that for each eigenvalue λ of any multiplicity m, there must be at least *one* eigenvector pertaining to λ. What is desired, but not always possible, is to find μ linearly independent eigenvectors corresponding to the eigenvalue λ of multiplicity m. This state of affairs makes matrix theory at once much more challenging but also much more interesting.

Example 3.3.3. For the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{7}{4} & \frac{1}{4}\sqrt{3} \\ 0 & \frac{1}{4}\sqrt{3} & \frac{5}{4} \end{bmatrix}$$

the characteristic polynomial is $det(\lambda I - A) = p_A(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4$, which can be factored as $p_A(\lambda) = (\lambda - 1)(\lambda - 2)^2$ We see that the eigenvalues are 1, 2, and 2. So, the multiplicity of the eigenvalue $\lambda = 1$ is 1, and the multiplicity of the eigenvalue $\lambda = 2$ is 2. The eigenspace pertaining to the eigenvalue $\lambda = 1$ is generated by the vector $\begin{bmatrix} 0 \\ -\frac{1}{3}\sqrt{3} \\ 1 \end{bmatrix}$, and the dimension of this eigenspace is one. The eigenspace pertaining to $\lambda = 2$ is generated by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ \frac{1}{3}\sqrt{3} \end{bmatrix}$. (That is, these two vectors form a basis of the eigenspace.) To summarize, for the given matrix there is one eigenvector for the eigenvalue $\lambda = 1$, and there are two linearly independent eigenvectors for the eigenvalue $\lambda = 2$. The dimension of the eigenspace pertaining to $\lambda = 1$ is one, and the dimension of the eigenspace pertaining to $\lambda = 2$ is two.

Now contrast the above example where the eigenvectors span the space $\mathbb{C}_3$ and the next example where we have an eigenvalue of multiplicity three but the eigenspace is of dimension one.

Example 3.3.4. Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

The characteristic polynomial is given by $(\lambda - 2)^3$. Hence the eigenvalue $\lambda = 2$ has multiplicity three. The eigenspace pertaining to $\lambda = 2$ is generated by the single vector $[0, 0, 1]^T$ To see this we solve

$$\begin{aligned} (A - 2I)\, x &= \left(\begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) x \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x = 0 \end{aligned}$$

The row reduced echelon form for $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. From this it is apparent that we may take $x_3 = t$, but that $x_1 = x_2 = 0$. Now assign $t = 1$ to obtain the generating vector $[0, 0, 1]^T$. This type of example and its consequences seriously complexifies the study of matrices.

Symmetric Functions

Definition 3.3.5. Let n be a positive integer and $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be given numbers. Suppose that k is a positive integer with $1 \leq k \leq n$. The k^{th} **elementary symmetric function** on the Λ is defined by

$$S_k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \lambda_{i_j}.$$

It is easy to see that

$$\begin{aligned} S_1(\lambda_1, \ldots, \lambda_n) &= \sum_1^n \lambda_i \\ S_n(\lambda_1, \ldots, \lambda_n) &= \prod_1^n \lambda_i. \end{aligned}$$

For a given matrix $A \in M_n$ there are n symmetric functions defined with respect to its eigenvalues $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The symmetric functions are sometimes called the invariants of matrices as they are invariant under similarity transformations that will be in Section 3.5. They also furnish directly the coefficients of the characteristic polynomial. Thus specifying the n symmetric functions of an $n \times n$ matrix is sufficient to determine its eigenvalues.

Theorem 3.3.4. *Let $A \in M_n$ have symmetric functions S_k, $k = 1, 2, \ldots, n$. Then*

$$det(\lambda I - A) = \prod_{i=1}^{n}(\lambda - \lambda_i) = \lambda^n + \sum_{k=1}^{n}(-1)^k S_k \lambda^{n-k}.$$

Proof. The proof is a consequence of actually expanding the product $\prod_{i=1}^{n}(\lambda - \lambda_i)$. Each term in the expansion has exactly n terms multiplied together that are combinations of the factor λ and the $-\lambda_i's$. For example, for the power λ^{n-k} the coefficient is obtained by computing the total number of products of k "different" $-1\lambda_i's$. (The term different is in quotes because it refers to different indices not actual values.) Collecting all these terms is accomplished by addition. Now the number of ways we can obtain products of these k different $(-1)\lambda_i's$ is easily seen to be the number of sequences in the set $\{1 \leq i_1 < \cdots < i_k \leq n\}$. The sum of these is clearly $(-1)^k S_k$, with the $(-1)^k$ factor being the collected product of k -1's. □

Two of the symmetric functions are familiar. In the following we restate this using familiar terms.

Theorem 3.3.5. *Let $A \in M_n(\mathbb{C})$. Then*

$$p_A(\lambda) = \det(\lambda I - A) = \sum_{k=0}^{n} p_k \lambda^{n-k}$$

where $p_0 = 1$ and

(i) $p_1 = -tr(A) = -\sum_{1}^{n} a_{ii}$

(ii) $p_n = -\det A$.

(iii) $p_k = (-1)^k S_k$, *for* $1 < k < n$.

Proof. Note that $p_A(0) = -\det(A) = p_n$. This gives (ii). To establish (i), we consider

$$\det \begin{bmatrix} \lambda \cdot a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & & -a_{2n} \\ \vdots & & \ddots & \\ -a_{n1} & \dots & & \lambda - a_{nn} \end{bmatrix}.$$

Clearly the product $\prod_1^n (\lambda - a_{ii})$ is one of the selections of products in the calculation process. In *every* other product there must be no more than $n-2$ diagonal terms. Hence

$$p_A(\lambda) = \det(\lambda I - A) = \prod_1^n (\lambda - a_{ii}) + p_{n-2}(\lambda),$$

where $p_{n-2}(\lambda)$ is a polynomial of degree $n-2$. The coefficient of λ^{n-1} is $-\sum_1^n a_{ii}$ by the Theorem 3.3.4, and this is (i). □

As a final note, observe that the characteristic polynomial is defined by knowing the n symmetric functions. However, the matrix itself has n^2 entries. Therefore, one may expect that knowing the only characteristic polynomial of a matrix is insufficient to characterize it. This is correct. Many matrices having rather different properties can have the same characteristic polynomial.

3.4 The Hamilton-Cayley Theorem

The Hamilton-Cayley Theorem opens the doors to a finer analysis of a matrix through the use of polynomials, which in turn is an important tool of spectral analysis. The results states that any square matrix satisfies its own charactertic polynomial, that is $p_A(A) = 0$. The proof is not difficult, but we need some preliminary results about factoring matrix-valued polynomials. Preceding that we need to consider matrix polynomials in some detail.

Matrix polynomials

One of the very important results of matrix theory is the Hamilton-Cayley theorem which states that a matrix satisfies its only characteristic equation. This implies we need the notion of a matrix polynomial. It is an easy idea

— just replace the coefficients of any polynomial by matrices — but it bears some important consequences.

Definition 3.4.1. Let $A_0, A_1, \ldots, A_m$ be square $n \times n$ matrices. We can define the polynomial with matrix coefficients

$$A(\lambda) = A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_{m-1}\lambda + A_m.$$

The *degree* of $A(\lambda)$ is m, provided $A_0 \neq 0$. $A(\lambda)$ is called regular if $\det A_0 \neq 0$. In this case we can construct an equivalent monic[2] polynomial.

$$\begin{aligned}\tilde{A}(\lambda) &= A_0^{-1}A(\lambda) = A_0^{-1}A_0\lambda^m + A_0^{-1}A_1\lambda^{m-1} + \cdots + A_0^{-1}A_m \\ &= I\lambda^m + \tilde{A}_1\lambda^{m-1} + \cdots + \tilde{A}_m.\end{aligned}$$

The algebra of matrix polynomials mimics the normal polynomial algebra. Let

$$A(\lambda) = \sum_0^m A_i\lambda^{m-k} \qquad B(\lambda) = \sum_0^m B_i\lambda^{m-k}.$$

(1) **Addition:**

$$A(\lambda) \pm B(\lambda) = \sum_0^m (A_k \pm B_k)\lambda^{m-k}.$$

(2) **Multiplication:**

$$A(\lambda)B(\lambda) = \sum_{i=0}^m \lambda^m \left(\sum_{k=0}^m A_i B_{m-i} \right)$$

The term $\sum_{k=0}^m A_i B_{m-i}$ is called the *Cauchy product* of the sequences. Note that the matrices A_j *always* multiply on the left of the B_k.

(3) **Division:** Let $A(\lambda)$ and $B(\lambda)$ be two matrix polynomials of degree m (as above) and suppose $B(\lambda)$ is regular, i.e. $\det B_0 \neq 0$. We say that $Q_r(\lambda)$ and $R_r(\lambda)$ are **right** *quotient* and *remainder* of $A(\lambda)$ upon division by $B(\lambda)$ if

$$A(\lambda) = Q_r(\lambda)B(\lambda) + R_r(\lambda) \qquad (1)$$

[2]Recall that a monic polynomial is a polynomial where coefficient of the highest power is one. For matrix polynomials the corresponding coefficient is I, the identity matrix.

if the degree of $R_r(\lambda)$ is *less* than that of $B(\lambda)$. Similarly $Q_\ell(\lambda)$ and $R_\ell(\lambda)$ are respectively the **left** *quotient* and *remainder* of $A(\lambda)$ upon division by $B(\lambda)$ if

$$A(\lambda) = B(\lambda)Q_\ell(\lambda) + R_\ell(\lambda) \tag{2}$$

if the degree of $R_\ell(\lambda)$ is *less* than that of $B(\lambda)$.

In the case (1) we see

$$A(\lambda)B^{-1}(\lambda) = Q_r(\lambda) + R_r(\lambda)B^{-1}(\lambda), \tag{3}$$

which looks much like $\frac{a}{b} = q + \frac{r}{b}$, a way to write the quotient and remainder of a divided by b when a and b are numbers. Also, the form (3) *may not* properly exist for all λ.

Lemma 3.4.1. *Let $B_i \in M_n(\mathbb{C})$, $i = 0, \ldots, n$ with B_0 nonsingular. Then the polynomial*

$$B(\lambda) = B_0\lambda^n + B_1\lambda^{n-1} + \cdots + B_n$$

is invertible for sufficiently large $|\lambda|$.

Proof. We factor $B(\lambda)$ as

$$\begin{aligned} B(\lambda) &= B_0\lambda^n \left(I + B_0^{-1}B_1\lambda^{-1} + \cdots + B_0^{-1}B_n\lambda^{-n}\right) \\ &= B_0\lambda^n \left[I + \lambda^{-1}\left(B_0^{-1}B_1 + \cdots + B_0^{-1}B_n\lambda^{1-n}\right)\right] \end{aligned}$$

For $\lambda > 1$, the norm of the term $B_0^{-1}B_1 + \cdots + B_0^{-1}B_n\lambda^{1-n}$ is bounded by

$$\begin{aligned} \left\|B_0^{-1}B_1 + \cdots + B_0^{-1}B_n\lambda^{1-n}\right\| &\leq \left\|B_0^{-1}B_1\right\| + \left\|B_0^{-1}B_2\right\| |\lambda|^{-1} + \cdots \left\|B_0^{-1}B_n\right\| |\lambda|^{1-n} \\ &\leq \left(1 + |\lambda|^{-1} + \cdots + |\lambda|^{1-n}\right) \max_{1\leq i\leq n} \left\|B_0^{-1}B_i\right\| \\ &\leq \frac{1}{1-|\lambda|^{-1}} \max_{1\leq i\leq n} \left\|B_0^{-1}B_i\right\| \end{aligned}$$

Thus the conditions of our perturbation theorem hold and for sufficiently large $|\lambda|$, it follows $\left\|\lambda^{-1}\left(B_0^{-1}B_1 + \cdots + B_0^{-1}B_n\lambda^{1-n}\right)\right\| < 1$. Hence

$$I + \lambda^{-1}\left(B_0^{-1}B_1 + \cdots + B_0^{-1}B_n\lambda^{1-n}\right)$$

is invertible and therefore $B(\lambda)$ is also invertible. □

Theorem 3.4.1. *Let $A(\lambda)$ and $B(\lambda)$ be matrix polynomials in $M_n(\mathbb{C})$ or $(M_n(R))$. Then both left and right division of $A(\lambda)$ by $B(\lambda)$ is possible and the respective quotients and remainders are unique.*

Proof. We proceed by induction on $\deg B$, and clearly if $\deg B = 0$ the result holds. If $\deg B = p > \deg A(\lambda) = m$ then the result follows simply. For, take $Q_r(\lambda) = 0$ and $R_r(\lambda) = A(\lambda)$. The conditions of right division are met. Now suppose that $p \leq m$. It is easy to see that

$$\begin{aligned} A(\lambda) &= A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_{m-1}\lambda + A_m \\ &= A_0B_0^{-1}\lambda^{m-p}B(\lambda) - \sum_{j=1}^{p} A_0B_0^{-1}B_j\lambda^{m-j} + \sum_{j=1}^{m} A_j\lambda^{m-j} \\ &= Q_1(\lambda)B(\lambda) + A_1(\lambda) \end{aligned}$$

where $\deg A_1(\lambda) < \deg A(\lambda)$. Our inductive hypothesis assumed the division was possible for matrix polynomials $A(\lambda)$ of degree $< p$. Therefore, $A_1(\lambda) = Q_2(\lambda)B() + R(\lambda)$, where the degree of $B(\lambda) < p$. Finally, with $Q(\lambda) = Q_(\lambda) + Q_2(\lambda)$, there results $A(\lambda) = Q(\lambda)B() + R(\lambda)$.

To establish uniqueness we assume two right divisors and quotients have been determined. Thus

$$\begin{aligned} A &= Q_{r_1}(\lambda)B(\lambda) + R_{r_1}(\lambda) \\ A &= Q_{r_2}(\lambda)B(\lambda) + R_{r_2}(\lambda) \end{aligned}$$

Subtract to get

$$0 = (Q_{r_1}(\lambda) - Q_{r_2}(\lambda))B(\lambda) + R_{r_1}(\lambda) - R_{r_2}(\lambda).$$

If $Q_{r_1}(\lambda) - Q_{r_2}(\lambda) \neq 0$, we know the degree of $(Q_{r_1}(\lambda) - Q_{r_2}(\lambda))B(\lambda)$ is greater than the degree of $R_1(\lambda) - R_2(\lambda)$. This contradiction implies that $Q_{r_1}(\lambda) - Q_{r_2}(\lambda) = 0$, which in turn implies that $R_{r_1}(\lambda) - R_{r_2}(\lambda) = 0$. Hence the decomposition is unique. □

Hamilton-Cayley Theorem

Let

$$B(\lambda) = B_0\lambda^m + B_1\lambda^{m-1} + \cdots + B_{m-1}\lambda + B_m$$

with $B_0 \neq 0$. We can also, write $B(\lambda) = \sum_{i=0}^{m} \lambda^{m-i}B_i$. Both versions are the same. However, when $A \in M_n(F)$, there are two possible evaluations of $B(A)$.

Definition 3.4.2. Let $B(\lambda), A \in M_n(\mathbb{C})$ (or $M_n(R)$) and $B(\lambda) = B_0\lambda^m + B_1\lambda^{m-1} + \cdots + B_{m-1}\lambda + B_m$. Define

$$B(A) = B_0 A^m + B_1 A^{m-1} + \cdots + B_m \qquad \text{“right value”}$$
$$\widehat{B}(A) = A^m B_0 + A^{m-1} B_1 + \cdots + B_m \qquad \text{“left value”}$$

The generalized Bézout theorem gives the remainder of $B(\lambda)$ divided by $\lambda I - A$. In fact, we have

Theorem 3.4.2 (Generalized Bézout Theorem). *The right division of $B(\lambda)$ by $\lambda I - A$ has remainder*

$$R_r(\lambda) = B(A)$$

Similarly, the left division of $B(\lambda)$ by $(\lambda I - A)$ has remainder

$$R_\ell(\lambda) = \widehat{B}(A).$$

Proof. In the case $\deg B(\lambda) = 1$, we have

$$B_0\lambda + B_1 = B_0(\lambda I - A) + B_0 A + B_1.$$

The remainder $R_r(\lambda) = B_0 A + B_1 = B(A)$. Assume the result holds for all polynomials up to degree $p - 1$. We have

$$\begin{aligned} B(\lambda) &= B_0\lambda^p + B_1\lambda^{p-1} + \cdots + B_p \\ &= B_0\lambda^{p-1}(\lambda I - A) + B_0 A\lambda^{p-1} + B_1\lambda^{p-1} + \cdots \\ &= B_0\lambda^{p-1}(\lambda I - A) + B_1(\lambda) \end{aligned}$$

where $\deg B_1(\lambda) \leq p - 1$. By induction

$$B(\lambda) = B_0\lambda^{p-1}(\lambda I - A) + Q_r(\lambda)(\lambda I - A) + B_1(A)$$

$B_1(A) = (B_0 A + B_1)A^{p-1} + B_2 A^{p-2} + \cdots + B_{p-1}A + B_p = B(A)$. This proves the result. □

Corollary 3.4.1. *$(\lambda I - A)$ divides $B(\lambda)$ if and only if $B(A) = 0$ (resp $\widehat{B}(A) = 0$).*

Combining the Bézout result and the adjoint formulation of the matrix inverse, we can establish the important Hamilton-Cayley theorem.

Theorem 3.4.3 (Hamilton-Cayley). *Let $A \in M_n(\mathbb{C})$ (or $M_n(R)$) with characteristic polynomial $p_A(\lambda)$. Then $p_A(A) = 0$.*

Proof. Recall the adjoint formulation of the inverse as $\hat{C} = \frac{1}{\det \cdot C} C_{ij}^T = C^{-1}$. Now let $B = adj(A - \lambda I)$. Then

$$B(\lambda I - A) = \det(\lambda I - A)I$$
$$(\lambda I - A)B = \det(\lambda I - A)I.$$

These equations show that $p(\lambda)I = \det(\lambda I - A)I$ is divisible on the right *and* the left by $(\lambda I - A)$ without remainder. It follows from the generalized Bézout theorem that this is possible only if $p_A(A) = 0$. □

Let $A \in M_n(C)$. Now that we know any A satisfies its characteristic polynomial, we might also ask if there are polynomials of lower degree that it also satisfies. In particular, we will study the so-called *minimal* polynomial that a matrix satisfies. The nature of this polynomial will shed considerable light on the fundamental structure of A. For example, both matrices below have the same characteristic polynomial $P(\lambda) = (\lambda - 2)^3$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence $\lambda = 2$ is an eigenvalue of multiplicity three. However, A satifies the much simpler first degree polynomial $(\lambda - 2)$ while there is no polynomial of degree less that three that B satisfies. By this time you recognize that A has three linearly independent eigenvectors, while B has only one eigenvector. We will take this subject up in a later chapter.

Biographies

Arthur Cayley (1821-1895), one of the most prolific mathematicians of his era and of all time, born in Richmond, Surrey, and studied mathematics at Cambridge. For four years he taught at Cambridge having won a Fellowship and, during this period, he published 28 papers in the Cambridge Mathematical Journal. A Cambridge fellowship had a limited tenure so Cayley had to find a profession. He chose law and was admitted to the bar in 1849. He spent 14 years as a lawyer, but Cayley always considered it as a means to make money so that he could pursue mathematics. During these 14 years as a lawyer Cayley published about 250 mathematical papers! Part of that time he worked in collaboration with **James Joseph Sylvester**[3] (1814-

[3]In 1841 he went to the United States to become professor at the University of Virginia, but just four years later resigned and returned to England. He took to teaching private

1897), another lawyer. Together, but not in collaboration, they founded the algebraic theory of invariants 1843.

In 1863 Cayley was appointed Sadleirian professor of Pure Mathematics at Cambridge. This involved a very large decrease in income. However Cayley was very happy to devote himself entirely to mathematics. He published over 900 papers and notes covering nearly every aspect of modern mathematics.

The most important of his work is in developing the algebra of matrices, work in non-Euclidean geometry and n-dimensional geometry. Importantly, he also clarified many of the theorems of algebraic geometry that had previously been only hinted at, and he was among the first to realize how many different areas of mathematics were linked together by group theory.

As early as 1849 Cayley wrote a paper linking his ideas on permutations with Cauchy's. In 1854 Cayley wrote two papers which are remarkable for the insight they have of abstract groups. At that time the only known groups were groups of permutations and even this was a radically new area, yet Cayley defines an abstract group and gives a table to display the group multiplication.

Cayley developed the theory of algebraic invariance, and his development of n-dimensional geometry has been applied in physics to the study of the space-time continuum. His work on matrices served as a foundation for quantum mechanics, which was developed by Werner Heisenberg in 1925. Cayley also suggested that Euclidean and non-Euclidean geometry are special types of geometry. He united projective geometry and metrical geometry which is dependent on sizes of angles and lengths of lines.

Heinrich Weber (1842-1913) was born and educated in Heidelberg, where he became professor 1869. He then taught at a number of institutions in Germany and Switzerland. His main work was in algebra and number theory. He is best known for his outstanding text *Lehrbuch der Algebra* published in 1895.

Weber worked hard to connect the various theories even fundamental concepts such as a field and a group, which were seen as tools and not properly developed as theories in his *Die partiellen Differentialgleichungen der mathematischen Physik* 1900-01, which was essentially a reworking of a book of the same title based on lectures given by Bernhard Riemann and

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pupils and had among them Florence Nightingale. By 1850 he became a barrister, and by 1855 returned to an academic life at the Royal Military Academy in Woolwich, London. He returned to the US again in 1877 to become professor at the new Johns Hopkins University, but returned to England once again in 1877. Sylvester coined the term 'matrix' in 1850.

written by Karl Hattendorff.

Etienne Bézout (1730-1783) was a mathematician who represents a characteristic aspect of the subject at that time. One of the many successful textbook projects produced in the 18^{th} century was Bézout's *Cours de mathematique*, a six volume work that first appeared in 1764-1769, which was almost immediately issued in a new edition of 1770-1772, and which boasted many versions in French and other languages. (The first American textbook in analytic geometry, incidentally, was derived in 1826 from Bézout's *Cours*.)

It was through such compilations, rather than through the original works of the authors themselves, that the mathematical advances of Euler and d'Alembert became widely known. Bézout's name is familiar today in connection with the use of determinants in algebraic elimination. In a memoir of the Paris Academy for 1764, and more extensively in a treatise of 1779 entitled *Theorie generale des equations algebriques*, Bézout gave artificial rules, similar to Cramer's, for solving n simultaneous linear equations in n unknowns. He is best known for an extension of these to a system of equations in one or more unknowns in which it is required to find the condition on the coefficients necessary for the equations to have a common solution. To take a very simple case, one might ask for the condition that the equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_ay + c_3 = 0$ have a common solution. The necessary condition is that the eliminant a special case of the "Bezoutiant," should be 0.

Somewhat more complicated eliminants arise when conditions are sought for two polynomial equations of unequal degree to have a common solution. Bézout also was the first one to give a satisfactory proof of the theorem, known to Maclaurin and Cramer, that two algebraic curves of degrees m and n respectively intersect in general in $m \cdot n$ points; hence, this is often called Bézout's theorem. Euler also had contributed to the theory of elimination, but less extensively than did Bézout.

Taken from *A History of Mathematics* by Carl Boyer

William Rowen Hamilton
Born Aug. 3/4, 1805, Dublin, Ire. and died Sept. 2, 1865, Dublin

Irish mathematician and astronomer who developed the theory of quaternions, a landmark in the development of algebra, and discovered the phenomenon of conical refraction. His unification of dynamics and optics, moreover, has had lasting influence on mathematical physics, even though the full significance of his work was not fully appreciated until after the rise of quantum mechanics.

Like his English contemporaries Thomas Babington Macaulay and John Stuart Mill, **Hamilton** showed unusual intellect as a child. Before the age of three his parents sent him to live with his father's brother, James, a learned clergyman and schoolmaster at an Anglican school at Trim, a small town near Dublin, where he remained until 1823, when he entered Trinity College, Dublin. Within a few months of his arrival at his uncle's he could read English easily and was advanced in arithmetic; at five he could translate Latin, Greek, and Hebrew and recite Homer, Milton and Dryden. Before his 12th birthday he had compiled a grammar of Syriac, and by the age of 14 he had sufficient mastery of the Persian language to compose a welcome to the Persian ambassador on his visit to Dublin.

Hamilton became interested in mathematics after a meeting in 1820 with Zerah Colburn, an American who could calculate mentally with astonishing speed. Having read the *Eléments d'algèbre* of Alexis–Claude Clairaut and Isaac Newton's *Principia*, **Hamilton** had immersed himself in the five volumes of Pierre–Simon Laplace's *Traité de mécanique céleste* (1798-1827; *Celestial Mechanics*) by the time he was 16. His detection of a flaw in Laplace's reasoning brought him to the attention of John Brinkley, professor of astronomy at Trinity College. When **Hamilton** was 17, he sent Brinkley, then president of the Royal Irish Academy, an original memoir about geometrical optics. Brinkley, in forwarding the memoir to the Academy, is said to have remarked: "This young man, I do not say *will be*, but *is*, the first mathematician of his age."

In 1823 **Hamilton** entered Trinity College, from which he obtained the highest honours in both classics and mathematics. Meanwhile, he continued his research in optics and in April 1827 submitted this "theory of Systems of Rays" to the Academy. The paper transformed geometrical optics into a new mathematical science by establishing one uniform method for the solution of all problems in that field. **Hamilton** started from the principle, originated by the 17th-century French mathematician Pierre de Fermat, that light takes the shortest possible time in going from one point to another, whether the path is straight or is bent by refraction. **Hamilton**'s key idea was to consider the time (or a related quantity called the "action") as a function of the end points between which the light passes and to show that this quantity varied when the coordinates of the end points varied, according to a law that he called the law of varying action. He showed that the entire theory of systems of rays is reducible to the study of this characteristic function.

Shortly after -**Hamilton** submitted his paper and while still an undergraduate, Trinity College elected him to the post of Andrews professor of astronomy and royal astronomer of Ireland, to succeed Brinkley, who had been made bishop. Thus an undergraduate (not quite 22 years old) became ex officio an examiner of graduates who were candidates for the Bishop Law Prize in mathematics. The electors' object was to provide **Hamilton** with a research post free from heavy teaching duties. Accordingly, in October 1827 **Hamilton** took up residence next to Dunsink Observatory, 5 miles (8 km) from Dublin, where he lived for the rest of his life. He proved to be an unsuccessful observer, but large audiences were attracted by the distinctly literary flavour of his lectures on astronomy. Throughout his life **Hamilton** was attracted to literature and considered the poet **William** Wordsworth among his fiends, although Wordsworth advised him to write mathematics rather than poetry.

With eigenvalues we are able to begin spectral analysis. That part is the derivation of the various normal forms for matrices. We begin with a relatively weak form of the Jordan form, which is coming up.

First of all, as you have seen diagonal matrices furnish the easiest form for matrix analysis. Also, linear systems are very simple to solve for diagonal matrices. The next simplest class of matrices are the triangular matrices. We begin with the following result based on the idea of similarity.

3.5 Similarity

Definition 3.5.1. A matrix $B \in M_n$ is said to be *similar* to $A \in M_n$ if there exists a nonsingular matrix $S \in M_n$ such that

$$B = S^{-1}AS.$$

The transformation $A \to S^1AS$ is called a *similarity transformation*. Sometimes we write $A \sim B$. Note that similarity is an *equivalence* relation:

(i)	$A \sim A$	reflexivity
(ii)	$B \sim A \Rightarrow A \sim B$	symmetry
(iii)	$B \sim A$ and $A \sim C \Rightarrow B \sim C$	transitivity

Theorem 3.5.1. *Similar matrices have the same characteristic polynomial.*

Proof. We suppose A, B, and $S \in M_n$ with S invertible and $B = S^{-1}AS$. Then

$$\begin{aligned}\lambda I - B &= \lambda I - S^{-1}AS \\ &= \lambda S^{-1}IS - S^{-1}AS \\ &= S^{-1}(\lambda I - A)S.\end{aligned}$$

Hence

$$\begin{aligned}\det(\lambda I - B) &= \det(S^{-1}(\lambda I - A)S) \\ &= \det(S^{-1})\det(\lambda I - A)\det S \\ &= \det(\lambda I - A)\end{aligned}$$

because $1 = \det I = \det(SS^{-1}) = \det S \det S^{-1}$. □

A simple consequence of Theorem 3.5.1 and Corollary 2.3.1 follows.

Corollary 3.5.1. *If A and B are in M_n and if A and B are similar, then they have the same eigenvalues counted according to multiplicity, and therefore the same rank.*

We remark that even though $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ have the same eigenvalues, 0 and 0, they are not similar. Hence the converse is not true.

Another immediate corollary of Theorem 3.5.1 can be expressed in terms of the invariance of the trace and determinant of *similarity transformations*, that is functions on $M_n(F)$ defined for any invertible matrix S by $T_S(A) = S^{-1}AS$. Note that such transformations are linear mappings from $M_n(F) \to M_n(F)$. We shall see how important are those properties of matrices that are invariant (i.e. do not change) under similarity transformations.

Corollary 3.5.2. *If A, $B \in M_n$ are similar, then they have the same trace and same determinant. That is,* $\mathrm{tr(A)} = \mathrm{tr(B)}$ *and* $\det A = \det B$.

Theorem 3.5.2. *If $A \in M_n$, then A is similar to a triangular matrix.*

Proof. The following sketch shows the first two steps of the proof. A formal induction can be applied to achieve the full result.

Let λ_1 be an eigenvalue of A with eigenvector u_1. Select a basis , say S_1, of $\mathbb{C}_n$ and arrange these vectors into columns of the matrix P_1, with u_1

in the first column. Define $B_1 = P_1^{-1} A P_1$. Then B_1 is the representation of A in the new basis and so

$$B_1 = \begin{bmatrix} \lambda & \alpha_1 & \dots & \alpha_{n-1} \\ 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{bmatrix} \quad \text{where } \mathrm{A}_2 \text{ is } (\mathrm{n}-1)\times(\mathrm{n}-1)$$

because $Au_1 = \lambda_1 u_1$. Remembering that B_1 is the representation of A in the basis S_1, then $[u_1]_{S_1} = e_1$ and hence $B_1 e_1 = \lambda e_1 = \lambda_1 [1, 0, \dots, 0]^T$. By similarity, the characteristic polynomial of B_1 is the same as that of A, but more importantly (using expansion by minors down the first column)

$$\det(\lambda I - B_1) = (\lambda - \lambda_1) \det(\lambda I_{n-1} - A_2).$$

Now select an eigenvalue λ_2 of A_2 and pertaining eigenvector $v_2 \in \mathbb{C}_n$; so $A_2 v_2 = \lambda_2 v_2$. Note that λ_2 is also an eigenvalue of A. With this vector we define

$$\hat{u}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \hat{u}_2 = \begin{pmatrix} 0 \\ \\ v_2 \end{pmatrix}.$$

Select a basis of $\mathbb{C}_{n-1}$, with v_2 selected first and create the matrix $\widehat{P}_2$ with this basis as columns

$$P_2 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \widehat{P}_2 & \\ 0 & & & \end{bmatrix}.$$

It is an easy matter to see that P_2 is invertible and

$$\begin{aligned} B_2 &= P_2^{-1} B_1 P_2 \\ &= \begin{bmatrix} \lambda_1 & * & \dots & \dots & * \\ 0 & \lambda_2 & * & \dots & * \\ 0 & 0 & & A_3 & \\ \vdots & \vdots & & & \\ 0 & 0 & & & \end{bmatrix}. \end{aligned}$$

Of course, $B_2 \sim B_1$, and by the transitivity of similarity $B_2 \sim A$. Continue this process, deriving ultimately the triangular matrix $B_n \sim B_{n-1}$ and hence $B_n \sim A$. This completes the proof. □

Definition 3.5.2. We say that $A \in M_n$ is *diagonalizable* if A is similar to a diagonal matrix.

Suppose $P \in M_n$ is nonsingular and $D \in M_n$ is a diagonal matrix. Let $A = PDP^{-1}$. Suppose the columns of P are the vectors $v_1, v_2, \dots, v_n$. Then

$$\begin{aligned} Av_j &= PDP^{-1}v_j \\ &= PDe_j = \lambda_j Pe_j = \lambda_j v_j \end{aligned}$$

where λ_j is the j^{th} diagonal element of D, and e_j is the j^{th} standard vector.

Similarly, if $u_1, \dots, u_n$ are n linearly independent vectors of A pertaining to eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then with Q, the matrix with columns $u_1 \dots u_n$, we have

$$AQ = QD$$

where

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \bigcirc \\ & & \ddots & \\ & \bigcirc & & \lambda_n \end{bmatrix}.$$

Therefore

$$Q^{-1}AQ = D.$$

We have thus established the

Theorem 3.5.3. *Let $A \in M_n$. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.*

As a practical measure, the conditions of this theorem are remarkably difficult to verify.

Example 3.5.1. The matrix $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is *not* diagonalizable.

Solution. First we note that the spectrum $\sigma(A) = \{0\}$. Solving

$$Ax = 0$$

we see that $x = c(1,0)^T$ is the only solution. That is to say, there are *not two* linearly independent eigenvectors. hence the result.

Corollary 3.5.3. *If $A \in M_n$ is diagonalizable and B is similar to A, then B is diagonalizable.*

Proof. The proof follows directly from transitivity of similarity. However, more directly, suppose that $B \sim A$ and S is the invertible matrix such that $B = SAS^{-1}$ Then $BS = SA$. If u is an eigenvector of A with eigenvalue λ, then $SAu = \lambda Su$ and therefore $BSu = \lambda Su$. This is valid for each eigenvector. We see that if $u_1, \ldots, u_n$ are the eigenvalues of A, then $Su_1, \ldots, Su_n$ are the eigenvalues of B. The similarity matrix converts the eigenvectors of one matrix to the eigenvectors of the transformed matrix. □

In light of these remarks, we see that similarity transforms preserve completely the dimension of eigenspaces. It is just as significant to note that if a similarity transformation diagonalizes a given matrix A, the similarity matrix must consist of the eigenvectors of A.

Corollary 3.5.4. *Let $A \in M_n$ be nonzero and nilpotent. Then A is not diagonalizable.*

Proof. Suppose $A \sim T$, where T is triangular. Since A is nilpotent (i.e.($A^m = 0$), T is nilpotent, as well. Therefore the diagonal entries of T are zero. The spectrum of T and hence A is zero, it's null space has dimension n. Therefore, by Theorem 2.3.3(f), its rank is zero. Therefore $T = 0$, and hence $A = 0$. The result is proved. □

Alternatively, if the nilpotent matrix A is similar to a diagonal matrix with zero diagonal entries, then A is similar to the zero matrix. Thus A is itself the zero matrix. From the obvious fact that the power of a similarity transformations is the similarity transformation of the power of a matrix, that is $(SAS^{-1})^m = SA^mS^{-1}$ (see Exercise 26), we have

Corollary 3.5.5. *Suppose $A \in M_n(\mathbb{C})$ is diagonalizable. (i) Then A^m is diagonalizable for every positive integer m. (ii) If $p(\cdot)$ is any polynomial, then $p(A)$ is diagonalizable.*

Corollary 3.5.6. *If all the eigenvalues of $A \in M_n(\mathbb{C})$ are distinct, then A is diagonalizable.*

Proposition 3.5.1. *Let $A \in M_n(\mathbb{C})$ and $\varepsilon > 0$. Then for any matrix norm $\|\cdot\|$ there is a matrix B with norm $\|B\| < \varepsilon$ for which $A+B$ is diagonalizable and for each $\lambda \in \sigma(A)$ there is a $\mu \in \sigma(B)$ for which $|\lambda - \mu| < \varepsilon$.*

Proof. First triangularize A to T by a similarity transformation, $T = SAS^{-1}$. Now add to T any diagonal matrix D so that the resulting triangular matrix $T + D$ has all distinct values. Moreover this can be accomplished by a diagonal matrix of arbitrarily small norm for any matrix norm. Then we have

$$\begin{aligned} S^{-1}(T+D)S &= S^{-1}TS + S^{-1}DS \\ &= A + B \end{aligned}$$

where $B := S^{-1}DS$. Now by the submultiplicative property of matrix norms $\|B\| \le \left\|S^{-1}DS\right\| \le \left\|S^{-1}\right\| \|D\| \|S\|$. Thus to obtain the estimate $\|B\| < \varepsilon$, it is sufficient to take $\|D\| \le \frac{\epsilon}{\|S^{-1}\|\|S\|}$, which is possible as established above. Since the spectrum $\sigma(A + B)$ of $A + B$ has n distinct values, it follows that $A + B$ is diagonalizable. □

There are many, many results on diagonalizable and non-diagonalizable matrices. Here is an interesting class of nondiagonalizable matrices we will encounter later.

Proposition 3.5.2. *pro Every matrix of the form*

$$A = \lambda I + N$$

where N is nilpotent and is <u>not</u> zero is not diagonalizable.

An important subclass has the form:

$$A = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & \bigcirc & \\ & & \lambda & 1 & \\ & \bigcirc & & \ddots & 1 \\ & & & & \lambda \end{bmatrix} \qquad \text{"Jordan block".}$$

Eigenvectors

Once an eigenvalue is determined, it is a relatively simple matter to find the pertaining eigenvectors. Just solve the homogeneous system $(\lambda I - A)x = 0$. Eigenvectors have a more complex structure and their study merits our attention.

Facts

(1) $\sigma(A) = \sigma(A^T)$ including mutliplicities.

(2) $\overline{\sigma(A)} = \sigma(A^*)$, including multiplicities.

Proof. $\det(\lambda I - A) = \det((\lambda I - A)^T) = \det(\lambda I - A^T)$. Similarly for A^*. □

Definition 3.5.3. The linear space spanned by all the eigenvectors pertaining to an eigenvalue λ is called the *eigenspace* corresponding to the eigenvalue λ.

For any $A \in M_n$ any subspace $V \in \mathbb{C}_n$ for which

$$AV \subset V$$

is called an *invariant subspace* of A. The determination of invariant subspaces for linear transformations has been an important question for decades.

Example 3.5.2. For any upper triangular matrix T the spaces $V_j = \mathfrak{S}(e_1, e_2, \ldots, e_j)$, $j = 1, \ldots, n$ are invariant.

Example 3.5.3. Given $A \in M_n$, with eigenvalue λ. The eigenspace corresponding to λ and all of its subspaces are invariant subspaces. Corollary to this, the null space $N(A) = \{x | Ax = 0\}$ is an invariant subspace of A corresponding to the eigenvalue $\lambda = 0$.

Definition 3.5.4. Let $A \in M_n$ with eigenvalue λ. The dimension of the eigenspace corresponding to λ is called the *geometric multiplicity* of λ. The multiplicity of λ as a zero of the characteristic polynomial $p_A(\lambda)$ is called the *algebraic multiplicity*.

Theorem 3.5.4. *If $A \in M_n$ and λ_0 is an eigenvalue of A with geometric and algebraic multiplicities m_g and m_a, respectively. Then*

$$m_g \leq m_a.$$

Proof. Let $u_1 \ldots u_{m_g}$ be linearly independent eigenvectors pertaining to λ_0. Let S be a matrix consisting of a basis of $\mathbb{C}_n$ with $u_1 \ldots u_{m_g}$ selected among them and placed in the first m_g columns. Then

$$B = S^{-1}AS = \left[\begin{array}{c:c} \lambda_0 I & * \\ \hdashline 0 & \widehat{B} \end{array}\right]$$

where $I = I_{m_g}$. It is easy to see that

$$p_B(\lambda) = p_A(\lambda) = (\lambda - \lambda_0)^{m_g} p_{\widehat{B}}(\lambda),$$

whence the algebraic multiplicity $m_a \geq m_g$. □

Example 3.5.4. For $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, the algebraic multiplicity of a is 2, while the geometric multiplicity of 0 is 1. Hence, the equality $m_g = m_a$ is not possible, in general.

Theorem 3.5.5. *If $A \in M_n$ and for each eigenvalue $\mu \in \sigma(A)$, $m_g(\mu) = m_a(\mu)$, then A is diagonalizable. The converse is also true.*

Proof. Extend the argument given in the theorem just above. Alternatively, we can see that A must have n linearly independent eigenvalues, which follows from the □

Lemma 3.5.1. *If $A \in M_n$ and $\mu \neq \lambda$ are eigenvalues with eigenspaces E_μ and E_λ respectively. Then E_μ and E_λ are linearly independent.*

Proof. Suppose $u \in E_\mu$ can be expressed as

$$u = \sum^k c_j v_j$$

where, of course $u \neq 0$ and $v_j \neq 0, j = 1, \dots, k$ where $v_1 \dots v_k \in E_\lambda$. Then

$$\begin{aligned} Au &= A\Sigma c_j v_j \\ \Rightarrow \quad \mu u &= \lambda \Sigma c_j v_j \\ \Rightarrow \quad u &= \frac{\lambda}{\mu} \Sigma c_j v_j, \text{ if } \mu \neq 0. \end{aligned}$$

Since $\lambda/\mu \neq 1$, we have a contradiction. If $\mu = 0$, then $\Sigma c_j v_j = 0$ and this implies $u = 0$. In either case we have a contradiction, the result is therefore proved.

While both A and A^T have the same eigenvalues, counted even with multiplicities, the eigenspaces can be very much different. Consider the example where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

The eigenvalue $\lambda = 1$ has eigenvector $u = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ for A and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ for A^T. □

A new concept of left eigenvector yield some interesting results.

Definition 3.5.5. We say λ is a *left eigenvector* of $A \in M_n$ if there is a vector $y \in \mathbb{C}_n$ for which

$$y^* A = \lambda y^*$$

or in $\mathbb{R}_n$ if $y^T A = \lambda y^T$. Taking adjoints the two sides of this equality become

$$(y^* A)^* = A^* y$$
$$(\lambda y^*)^* = \bar{\lambda} y.$$

Putting these lines together $A^* y = \bar{\lambda} y$ and hence $\bar{\lambda}$ is an eigenvalue of A^*. Here's the big result.

Theorem 3.5.6. *Let $A \in M_n(\mathbb{C})$ with eigenvalues λ and eigenvector u. Let u, v be left eigenvectors pertaining to μ, λ, respectively. If $\mu \neq \lambda$, $v^* u = \langle u, v \rangle = 0$.*

Proof. We have

$$v^* u = \frac{1}{\mu} v^* A \mu = \frac{\lambda}{\mu} v^* u.$$

Assuming $\mu \neq 0$ we have a contradiction. If $\mu = 0$, argue as $v^* u = \frac{1}{\lambda} v^* A u = \frac{\mu}{\lambda} c^* u = 0$, which was to be proved. The result is proved. □

Note that left eigenvectors of A are (right) eigenvectors of A^* (A^T in the real case).

3.6 Equivalent norms and convergent matrices

Equivalent norms

So far we have defined a number of different norms. Just what "different" means is subject to different interpretations. For example, we might agree that different means that the two norms have a different value for some matrix A. On the other hand if we have two matrix norms $\|\cdot\|_a$ and $\|\cdot\|_b$, we might be prepared to say that if for two positive constants $m\, M > 0$

$$0 < m \leq \frac{\|A\|_a}{\|A\|_b} \leq M \quad \text{for all } A \in M_n(F)$$

then these norms are not really so different but rather are equivalent because "small" in one norm implies "small" in the other and the same for "large." Indeed, when two norms satisfy the condition above we will call them equivalent. The remarkable fact is that all subordinate norms on a finite dimensional space are equivalent. This is a direct consequence of the similar result for vector norms. We state it below but leave the details of the proof to the reader.

Theorem 3.6.1. *Any two matrix norms, $\|\cdot\|_a$ and $\|\cdot\|_b$, on $M_n(\mathbb{C})$ are equivalent in the sense that there exist two positive constants $m\, M > 0$*

$$0 < m \leq \frac{\|A\|_a}{\|A\|_b} \leq M \quad \textit{for all } A \in M_n(F)$$

We have already proved one convergence result about the invertibility of $I - A$ when $\|A\| < 1$. A deeper version of this result can be proved based on a new matrix norm. This result has important consequences in general matrix theory and particularly in computational matrix theory. It is most important in applications, where having $\rho(A) < 1$ can yield the same results as having $\|A\| < 1$.

Lemma 3.6.1. *Let $\|\cdot\|$ be a matrix norm that is subordinate to the vector norm (on C_n) $\|\cdot\|$. Then for each $A \in M_n$*

$$\rho(A) \leq \|A\|.$$

Note: We use the same notation for both vector and matrix norms.

Proof. Let $\lambda \in \sigma(A)$. Then with corresponding eigenvector x_λ we have $Ax_\lambda = \lambda x_\lambda$. Normalizing x_λ so that $\|x_\lambda\| = 1$, we have $\|A\| = \max\limits_{\|x\|=1} \|Ax\| \geq \|Ax_\lambda\| = |\lambda|\ \|x_\lambda\| = |\lambda|$. Hence

$$\|A\| \geq \max_{\lambda \in \sigma(A)} |\lambda| = \rho(A).$$

□

Theorem 3.6.2. *For each $A \in M_n$ and $\varepsilon > 0$ there is a vector norm $\|\cdot\|$ on $\mathbb{C}_n$ for which the subordinate matrix norm $\|\cdot\|$ satisfies*

$$\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon.$$

Proof. The proof follows in a series of simple steps, the first of which is interesting in its own right. First we know that A is similar to a triangular matrix B—from a previous theorem

$$B = SAS^{-1} = \Lambda + U$$

where Λ is the diagonal part of B and U is strictly upper triangular part of B, with zeros filled in elsewhere.

Note that the diagonal of B *is* the spectrum of B and hence that of A. This is important. Now select a $\delta > 0$ and form the diagonal matrix

$$D = \begin{bmatrix} 1 & & & \\ & \delta^{-1} & & \bigcirc \\ & & \ddots & \\ & \bigcirc & & \delta^{1-n} \end{bmatrix}.$$

A brief computation reveals that

$$\begin{aligned} C = DBD^{-1} &= DSAS^{-1}D^{-1} \\ &= D(\Lambda + U)D^{-1} \\ &= \Lambda + DUD^{-1} \qquad\qquad = \Lambda + V \end{aligned}$$

where $V = DUD^{-1}$ and more specifically $v_{ij} = \delta^{j-i}u_{ij}$ for $j > i$ and of course Λ is a diagonal matrix with the spectrum of A for the diagonal elements. In this way we see that for δ small enough we have arranged that A is similar to the diagonal matrix of its spectral elements plus a small triangular perturbation. We now define the new vector norm on $\mathbb{C}_n$ by

$$\|x\|_A = \langle (DS)^*(DS)x, x\rangle^{1/2}.$$

We recognise this to be a norm from a previous result. Now compute the matrix norm

$$\|A\|_A = \max_{\|x\|_A=1} \|Ax\|_A.$$

Thus

$$\begin{aligned} \|Ax\|_A^2 &= \langle DSAx, DSAx\rangle \\ &= \langle CDSx, CDSx\rangle \\ &\le \|C\|_2^2 \|DSx\|^2 \\ &= \|C^*C\|_2 \|x\|_A^2. \end{aligned}$$

From $C = \Lambda + V$, it follows that

$$C^*C = \Lambda^*\Lambda + \Lambda^*V + V^*\Lambda + V^*V.$$

Because the last three terms have a δ^p multiplier for various **positive** powers p, we can make the terms $\Lambda^*V + V^*\Lambda + V^*V$ small in *any* norm by taking δ sufficiently small. Also the diagonal elements of $\Lambda^*\Lambda$ have the form

$$|\lambda_i|^2 \qquad \lambda_i \in \mathfrak{S}(A).$$

Talking δ sufficiently small so that each of $\|\Lambda^* V\|_2$, $\|V^*\Lambda\|_2$ and $\|V^*V\|_2$ is less than $\epsilon/3$. With $\|x\|_A^2 = 1$ as required, we have

$$\|Ax\|_A \leq (\rho(A) + \varepsilon)\|x\|_A$$

and we know already that $\|A\|_A \geq \rho(A)$. □

An important corollary places a lower bound on vector norms is given below.

Corollary 3.6.1. *For any* $A \in M_n$

$$\rho(A) = \inf_{\| \ \|} (\max_{\|x\|=1} \|Ax\|)$$

where the infimum is taken over all vector norms.

Convergent matrices

Definition 3.6.1. We say that a matrix $A \in M_n(F)$ for $F = R$ or $\mathbb{C}$ is *convergent* if

$$\lim_{m\to\infty} A^m = 0 \qquad \leftarrow \text{ the zero matrix.}$$

That is, for each $1 \leq i, j \leq n$ the $lim_{m\to\infty}(A^m)_{ij} = 0$. This is sometimes called *pointwise* convergence.

Theorem 3.6.3. *The following three statements are equivalent:*

(a) *A is convergent.*

(b) $\lim_{n\to\infty} \|A^m\| = 0$*, for some matrix norm.*

(c) $\rho(A) < 1$.

Proof. These results follow substantially from previously proven results. However, for completeness, assume (a) holds. Then

$$\lim_{n\to\infty} \max_{ij} |(A^n)_{ij}| = 0$$

or what is the same we have

$$\lim_{n\to\infty} \|A^n\|_\infty = 0$$

which is (b). Now suppose that (b) holds. If $\rho(A) \geq 1$ there is a vector x for which $\|A^n x\| = \|\rho(A)\|^n \|x\|$. Therefore $\|A^n\| \geq 1$, which contradicts (b). Thus (c) holds. Next if (c) holds we can apply the above theorem to establish that there is a norm for which $\|A\| < 1$. Hence (b) follows. Finally, we know that

$$\|A^m\|_\infty \leq M\|A^m\| \tag{$\star$}$$

whence $\lim_{n\to\infty} A^m = 0$. (That is (a) $\equiv$ (b)). In sum we have shown that (a) $\equiv$ (b) and (b) $\equiv$ (c). □

To establish ($\star$) we need the following.

Theorem 3.6.4. *If* $\| \quad \|$ *and* $\|\cdot\|'$ *are two vector norms on* $\mathbb{C}_n$ *then there are constants* m *and* M *so that*

$$m\|x\|' \leq \|x\| < M\|x\|'$$

for all $x \in \mathbb{C}_n$.

This result carries over to matrix norms subordinate to vector norms by simple inheritance. To prove this result we use compactness ideas. Suppose there is a sequence of vectors x_j for which

$$1 \leq \|x_j\| < \frac{1}{j}\|x_j\|'$$

and for which the components of x_j are bounded in modulus. By compactness there is a convergent subsequence of the x_j, for which $\lim x_j = x \neq 0$. But $\|x\|' < \infty$. Hence $\|x\| = 0$, a contradiction.

Here is the new and improved version of our previous result.

Theorem 3.6.5. *If* $\rho(A) < 1$ *then* $(I - A)^{-1}$ *exists and*

$$(I - A)^{-1} = I + A + A^2 + \cdots \ .$$

Proof. Select a matrix norm $\|\cdot\|$ for which $\|A\| < 1$. Apply previous calculations. We know that every matrix can be triangularized and "almost" diagonalized we may ask if we can eliminate altogether the off-diagonal terms. The answer is unfortunately **no**. But we can resolve the diagonal question completely. □

3.7 Exercises

1. Suppose that $U \in M_n$ is orthogonal and let $\|\cdot\|$ be a matrix norm. Show that $\|U\| \geq 1$.

2. In two dimensions the counter-clockwise rotations through the angle θ are given by
$$B_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
Find the eigenvalues and eigenvectors for all θ. (Note the two special cases, θ is not equal to an even multiple of π and $\theta = 0$.

3. Given two matrices A, $B \in M_n(\mathbb{C})$. Define the *commutant* of A and B by $[A, B] - AB - BA$. Prove that $\operatorname{tr}[A, B] = 0$.

4. Given two finite sequences $\{c_k\}_{k=1}^n$ and $\{d_k\}_{k=1}^n$. Prove that $\sum_k |c_k d_k| \leq \max_k |d_k| \sum_k |c_k|$.

5. Verify that a matrix norm which is subordinate to a vector norm satisfies norm conditions (i) and (ii).

6. Let $A \in M_n(\mathbb{C})$. Show that the matrix norm subordinate to the vector norm $\|\cdot\|_\infty$ is given by
$$\|A\|_\infty = \max_i \|r_i(A)\|_1$$
where as usual $r_i(A)$ denotes the i^{th} row of the matrix A and $\|\cdot\|_1$ is the ℓ_1 norm.

7. The Hilbert matrix, H_n of order n is defined by
$$h_{ij} = \frac{1}{i+j-1} \quad 1 \leq i,\, j \leq n$$

8. Show that $\|H_n\|_1 < \ln n$.

9. Show that $\|H_n\|_\infty = 1$.

10. Show that $\|H_n\|_2 \sim n^{\frac{1}{2}}$.

11. Show that the spectral radius of H_n is bounded by 1.

12. Show that for each $\varepsilon > 0$ there exists an integer N such that if $n > N$ there is a vector $x \in R_n$ with $\|x\|_2=1$ such that $\|H_n x\|_2 < \varepsilon$.

13. Same as the previous question except that you need to show that $N = \left\lfloor \frac{1}{\varepsilon^{1/2}} \right\rfloor + 1$ will work.

14. Show that the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & -1 & 2 \end{bmatrix}$ is not diagonalizable.Let

    $A \in M_n(\mathbb{C})$.

15. We know that the characteristic polynomial of a matrix $A \in M_{12}(\mathbb{C})$ is equal to $p_A(\lambda) = (\lambda - 1)^{12} - 1$. Show that A is not similar to the identity matrix.

16. We know that the spectrum of $A \in M_3(\mathbb{C})$ is $\sigma(A) = \{1, 1, -2\}$ and the corresponding eigenvectors are $\{[1, 2, 1]^T, [2, 1, -1]^T, [1, 1, 2]^T\}$.

    (i) Is it possible to determine A ? Why or why not? If so, prove it. If not show two different matrices with the given spectrum and eigenvectors.

    (ii) Is A diagonalizable?

17. Prove that if $A \in M_n(\mathbb{C})$ is diagonalizable then for each $\lambda \in \sigma(A)$, the algebraic and geometric multiplicities are equal. That is, $m_a(\lambda) = m_g(\lambda)$.

18. Show that $B^{-1}(\lambda)$ exists for $|\lambda|$ sufficiently large.

19. We say that a matrix $A \in M_n(R)$ is row stochastic if all its entries are non negative and the sum of the entries of each row is one.

    (i) Prove that $1 \in \sigma(A)$.

    (ii) Prove that $\rho(A) = 1$.

20. We say that $A, B \in M_n(\mathbb{C})$ are *quasi*-commutative if the spectrum of $AB - BA$ is just zero, i.e. $\sigma(AB - BA) = \{0\}$.

21. Prove that if AB is nilpotent then so is BA.

22. A matrix $A \in M_n(\mathbb{C})$ has a square root if there is a matrix $B \in M_n(\mathbb{C})$ such that $B^2 = A$. Show that if A is diagonalizable then it has a square root.

23. Prove that every matrix that commutes with every diagonal matrix is itself diagonal.

24. Suppose that if $A \in M_n(\mathbb{C})$ is diagonalizable and for each $\lambda \in \sigma(A)$, $|\lambda| < 1$. Prove directly from similarity ideas that $\lim_{n\to\infty} A^n = 0$.(That is, do not apply the more general theorem from the lecture notes.)

25. We say that A is *right quasi*-invertible if there exists a matrix $B \in M_n(\mathbb{C})$ such that $AB \sim D$, where D is a diagonal matrix with diagonal entries nonzero. Similarly, we say that A is *left quasi*-invertible if there exists a matrix $B \in M_n(\mathbb{C})$ such that $BA \sim D$, where D is a diagonal matrix with diagonal entries nonzero.

    (i) Show that if A is *right quasi*-invertible then it is invertible.

    (ii) Show that if A is *right quasi*-invertible then it is *left quasi*-invertible.

    (iii) Prove that *quasi*-invertibility is not an equivalence relation. (Hint. How do we usually show that an assertion is false?)

26. Suppose that $A, S \in M_n$, with S invertible, and m is a positive integer. Show that $(SAS^{-1})^m = SA^mS^{-1}$.

27. Consider the rotation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

    Show that the eigenvalues are the same as for the rotation

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

    See Example 2 of section 3.2.

28. Let $A \in M_n(\mathbb{C})$. Define $e^A = \sum_{n=0}^{\infty} \dfrac{A^n}{n!}$. (i) Prove that exists. (ii) Suppose that $A, B \in M_n(\mathbb{C})$. Is $e^{A+B} = e^A e^B$? If not, when is it true?

29. Suppose that $A, B \in M_n(\mathbb{C})$. We say $A \smile B$ if $[A, B] = 0$, where $[\cdot, \cdot]$ is the commutant. Prove or disprove that "$\smile$" is an equivalence relation. Answer the same question in the case of quasi-commutivity. (See Exercise 20

30. Let $u, v \in \mathbb{C}_n$. Find $(I + uv^*)^m$.

31. Define the function f on $M_{m\times n}(\mathbb{C})$ by $f(A) = \operatorname{rank} A$, and suppose that $\|\cdot\|$ is any matrix norm. Prove the following. (1) If for any matrix A for which $f(A) = n$ show that f is continuous in $\|\cdot\|$. That is, for every $\varepsilon > 0$ then $f(B) = n$ for every matrix B with $\|B_A\| < \varepsilon$. is continuous in an ε-neighborhood of A. That is, for any B with $\|A - B\| < \varepsilon$, then $f(B) = f(A)$. (This deviates slightly from the usual definition of continuity because the function f is integer valued.) (2) If $f(A) < n$, then f is not continuous in every ε-neighborhood of A.

3.8 Appendix A

It is desired to solve the equation $p(\lambda) = 0$ for coefficients in $p_0, \ldots, p_n \in \mathbb{C}$ or R. The basic theorem on this subject is called the Fundamental Theorem of Algebra (FTA), whose importance is manifest by its hundreds of applications. Concommitant with the FTA is the notion of reducible and irreducible factors.

Theorem 3.8.1 (Theorem Fundamental Theorem of Algebra). *Given any polynomial $p(\lambda) = p_0\lambda^n + p_1\lambda^{n-1} + \cdots + p_n$ with coefficients in $p_0, \ldots, p_n \in \mathbb{C}$. There is at least one solution $\lambda \in \mathbb{C}$ to the equation $p(\lambda) = 0$.*

Though proving this result would take us too far afield of our subject, we remarks that the simplest proof of this result no doubt comes as a direct application of Liouville's Theorem, a result in complex analysis. As a corollary to the FTA, we have that there are exactly n solutions to $p(\lambda) = 0$ when counted with multiplicitiy. Proved originally by Gauss, the proof of this theorem eluded mathematicians for many years. Let us assume that $p_0 = 1$ to make the factorization simpler to write. Thus

$$p(\lambda) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{m_i} \tag{4}$$

.

As we know, in the case that the coefficients $p_0, \ldots, p_n$ are real, there may be complex solutions. As is easy to prove, complex solutions must come in complex conjugate pairs. For if $\lambda = r + is$ is a solution

$$p(r + is) = (r + is)^n + p_1 (r + is)^{n-1} + \cdots p_{n-1} (r + is) + p_n = 0$$

Because the coefficients are real, the real and imaginary parts of the powers $(r + is)^j$ remain respectively real or imaginary upon multiplication by p_j.

Thus

$$\begin{aligned} p(r+is) &= \operatorname{Re}(r+is)^n + p_1 \operatorname{Re}(r+is)^{n-1} + \cdots p_{n-1}\operatorname{Re}(r+is) + p_n \\ &\quad + i\left(\operatorname{Im}(r+is)^n + p_1 \operatorname{Im}(r+is)^{n-1} + \cdots p_{n-1}\operatorname{Im}(r+is) + p_n\right) \\ &= 0 \end{aligned}$$

Hence the real and imaginary parts are each zero. Since $\operatorname{Re}(r-is)^j = \operatorname{Re}(r+is)^j$ and $\operatorname{Im}(r-is)^j = -\operatorname{Im}(r+is)^j$ it follows that $p(r-is) = 0$.

In the case that the coefficients are real it may be of interest to note what statement of factorization analogous to (4) above. To this end we need the definition

Definition 3.8.1. The real polynomial $x^2 + bx + c$ is called **irreducible** if it has no real zeros.

In general, any polynomial that cannot be factored over the underlying field is called irreducible. Irreducible polynomials have played a very important role in fields such as abstract algebra and number theory. Indeed, they were central in early attempts to prove Fermat's last theorem and also so but more indirectly to the acutal proof. Our attention here is restricted to the fields $\mathbb{C}$ and R.

Theorem 3.8.2. *Let $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_n$ with coefficients in $p_0, \ldots, p_n \in R$. Then $p(\lambda)$ can be factored as a product of linear factors pertaining to real zeros of $p(\lambda) = 0$ and irreducible quadratic factors.*

Proof. The proof is an easy application of the FTA and the observation above. If $\lambda_k = r + is$ is a zero of $p(\lambda)$, then so also is $\bar{\lambda}_k = r - is$. Therefore the product $(\lambda - \lambda_k)\left(\lambda - \bar{\lambda}_k\right) = \lambda^2 - 2\lambda r + r^2 + s^2$ is an irreducible quadratic. Combine such terms with the linear factors generated from the real zeros, and use (4). □

Remark 3.8.1. It is worth noting that there are **no** higher order irreducible factors with real coefficients. Even though there are certainly higher order polynomials with complex roots, they can always factored as products of either real linear or real quadratic factors. Proving this without the FTA may prove challenging.

3.9 Appendix B

3.9.1 Infinite Series

Definition 3.9.1. An infinite series, denoted by

$$a_0 + a_1 + a_2 + \cdots$$

is a sequence $\{u_n\}$, where u_n is defined by

$$u_n = a_0 + a_1 + \cdots + a_n$$

If the sequence $\{u_n\}$ converges to some limit A, we say that the infinite series converges to A and use the notation

$$a_0 + a_1 + a_2 + \cdots = A$$

We also say that the sum of the infinite series is A. If $\{u_n\}$ diverges, the infinite series $a_0 + a_1 + a_2 + \cdots$ is said to be divergent.

The sequence $\{u_n\}$ is called the sequence of partial sums, and the sequence $\{a_n\}$ is called the sequence of terms of the infinite series $a_0 + a_1 + a_2 + \cdots$.

Let us now return to the formula

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

where $r \neq 1$. Since the sequence $\{r^{n+1}\}$ converges (and, in fact, to 0) if and only if $-1 < r < 1$ (or $|r| < 1$), r being different from 1, the infinite series

$$1 + r + r^2 + \cdots$$

converges to $(1 - r)^{-1}$ if and only if $|r| < 1$. This infinite series is called the geometric series with ratio r.

Definition 3.9.2. Geometric Series with Ratio r

$$1 + r + r^2 + \cdots = \frac{1}{1 - r}, \quad \text{if } |r| < 1$$

and diverges if $|r| \geq 1$.

Multiplying both sides by r yields the following.

Definition 3.9.3. If $|r| < 1$, then

$$r + r^2 + r^3 + \cdots = \frac{r}{1-r}$$

Example 3.9.1. Investigate the convergence of each of the following infinite series. If it converges, determine its limit (sum).
a. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ b. $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$ c. $\frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots$

Solution A careful study reveals that each infinite series is a geometric series. In fact, the ratios are, respectively, (a) $-\frac{1}{2}$, (b) $\frac{2}{3}$, and (c) $\frac{3}{4}$. Since they are all of absolute value less than 1, they all converge. In fact, we have:
a. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$
b. $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots = \frac{1}{1-\frac{2}{3}} = \frac{3}{1} = 3$
c. $\frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \cdots = \frac{\frac{3}{4}}{1-\frac{3}{4}} = \frac{3}{4} \cdot \frac{4}{1} = 3$

Since an infinite series is defined as a sequence of partial sums, the properties of sequences carry over to infinite series. The following two properties are especially important.

1. Uniqueness of Limit If $a_0 + a_1 + a_2 + \cdots$ converges, then it converges to a unique limit.

This property explains the notation

$$a_0 + a_1 + a_2 + \cdots = A$$

where A is the limit of the infinite series. Because of this notation, we often say that the infinite series sums to A, or the sum of the infinitely many terms $a_0, a_1, a_2, \ldots$ is A. We remark, however, that the order of the terms cannot be changed arbitrarily in general.

Definition 3.9.4. 2. Sum of Infinite Series If
$a_0 + a_1 + a_2 + \cdots = A$ and $b_0 + b_1 + b_2 + \cdots = B$, then

$$\begin{aligned}(a_0+b_0)+(a_1+b_1)+(a_2+b_2)+\cdots &= (a_0+a_1+a_2+\cdots)\\ &\quad +(b_0+b_1+b_2+\cdots)\\ &= A+B\end{aligned}$$

This property follows from the observation that

$$\begin{aligned}(a_0+b_0)+(a_1+b_1)+\cdots+(a_n+b_n) &= (a_0+a_1+\cdots+a_n)\\ &\quad +(b_0+b_1+\cdots+b_n)\end{aligned}$$

converges to $A + B$. Another property is

Definition 3.9.5. 3. Constant Multiple of Infinite Series If $a_0+a_1+a_2+\cdots=A$ and c is a constant, then

$$ca_0+ca_1+ca_2+\cdots=cA$$

Example 3.9.2. Determine the sums of the following convergent infinite series.
a. $\frac{5}{3\cdot 2}+\frac{13}{9\cdot 4}+\frac{35}{27\cdot 8}+\cdots$ b. $\frac{1}{3\cdot 2}+\frac{1}{9\cdot 2}+\frac{1}{27\cdot 2}+\cdot$

Solution a. By the Sum Property, the sum of the first infinite series is

$$\begin{aligned}&\left(\frac{1}{3}+\frac{1}{2}\right)+\left(\frac{1}{9}+\frac{1}{4}\right)+\left(\frac{1}{27}+\frac{1}{8}\right)+\cdots\\=&\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)+\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)\\=&\frac{\frac{1}{3}}{1-\frac{1}{3}}+\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{1}{2}+1=\frac{3}{2}\end{aligned}$$

b. By the Constant-Multiple Property, the sum of the second infinite series is

$$\frac{1}{2}\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots\right)=\frac{1}{2}\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

Another type of infinite series can be illustrated by using Taylor polynomial extrapolation as follows.

Example 3.9.3. In extrapolating the value of $\ln 2$, the nth-degree Taylor polynomial $P_n(x)$ of $\ln(1+x)$ at $x=0$ was evaluated at $x=1$ in Example 4 of Section 12.3. Interpret $\ln 2$ as the sum of an infinite series whose nth partial sum is $P_n(1)$.

Solution We have seen in Example 3 of Section 12.3 that

$$P_n(x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots+(-1)^{n-1}\frac{1}{n}x^n$$

so that $P_n(1)=1-\frac{1}{2}+\frac{1}{3}-\cdots+(-1)^{n-1}\frac{1}{n}$, and it is the nth partial sum of the infinite series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

Since $\{P_n(1)\}$ converges to $\ln 2$ (see Example 4in Section 12.3), we have

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\cdots$$

It should be noted that the terms of the infinite series in the preceding example alternate in sign. In general, an infinite series of this type always converges.

> **Alternating Series Test** Let $a_0 \geq a_1 \geq \cdots \geq 0$. If a_n approaches 0, then the infinite series
>
> $$a_0 - a_1 + a_2 - a_3 + \cdots$$
>
> converges to some limit A, where $0 < A < a_0$.

The condition that the term a_n tends to zero is essential in the above Alternating Series Test. In fact, if the terms do not tend to zero, the corresponding infinite series, alternating or not, must diverge.

> **Divergence Test** If a_n does not approach zero as $n \to +\infty$, then the infinite series
>
> $$a_0 + a_1 + a_2 + \cdots$$
>
> diverges.

Example 3.9.4. Determine the convergence or divergence of the following infinite series.
a. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ b. $-\frac{2}{3} + \frac{4}{5} - \frac{6}{7} + \cdots$

Solution a. This series is an alternating series with

$$1 > \frac{1}{3} > \frac{1}{5} > \cdots > 0$$

and with the terms approaching 0. In fact, the general (nth) term is

$$(-1)^n \frac{1}{2n+1}$$

Hence, by the Alternating Series Test, the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ converges.
b. Let $a_1 = -\frac{2}{3}, a_2 = \frac{4}{5}, a_3 = -\frac{6}{7}, \ldots$. It is clear that a_n does not approach 0, and it follows from the Divergence Test that the infinite series is divergent. [The nth term is $(-1)^n 2n/(2n+1)$.] Note, however, that the terms do alternate in sign.

Another useful tool for testing convergence or divergence of an infinite series is the Ratio Test, to be discussed below. Recall that the geometric series

$$1 + r + r^2 + \cdots$$

where the nth term is $a_n = r^n$, converges if $|r| < 1$ and diverges otherwise. Note also that the ratio of two consecutive terms is

$$\frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{r^n} = r$$

Hence, the geometric series converges if and only if this ratio is of absolute value less than 1. In general, it is possible to draw a similar conclusion if the sequence of the absolute values of the ratios of consecutive terms converges.

> **Ratio Test** Suppose that the sequence
>
> $$\{|a_{n+1}|/|a_n|\}$$
>
> converges to some limit R. Then, the infinite series $a_0 + a_1 + a_2 + \cdots$ converges if $R < 1$ and diverges if $R > 1$.

Example 3.9.5. In each of the following, determine all values of r for which the infinite series converges.
a. $1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \cdots$ b. $1 + 2r + 3r^3 + 4r^3 + \cdots$

Solution a. The nth term is $a_n = r^n/n!$, so that

$$\frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{(n+1)!} \cdot \frac{n!}{r^n} = \frac{r}{n+1}$$

For each value of r, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1}{n+1}|r| \to 0 < 1$$

Hence, by the Ratio Test, the infinite series converges for all values of r.
b. Let $a_n = (n+1)r^n$. Then,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+2)|r|^{n+1}}{(n+1)|r|^n} = \frac{n+2}{n+1}|r| \to |r|$$

Hence, the Ratio Test says that the infinite series converges if $|r| < 1$ and diverges if $|r| > 1$. For $|r| = 1$, we have $|a_n| = n+1$, which does not converge to 0, so that the corresponding infinite series must be divergent as a result of applying the Divergence Test.

Sometimes it is possible to compare the partial sums of an infinite series with certain integrals, as illustrated in the following example.

Example 3.9.6. Show that $1+\frac{1}{2}+\cdots+1/n$ is larger than the define integral $\int_1^{n+1} dx/x$, and conclude that the infinite series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ diverges.

Solution Consider the function $f(x)=1/x$. Then, the definite integral

$$\int_1^{n+1} \frac{dx}{x} = \ln(n+1) - \ln 1 = \ln(n+1)$$

is the area under the graph of the function $y=f(x)$ between $x=1$ and $x=n+1$. On the other hand, the sum

$$1+\frac{1}{2}+\cdots+\frac{1}{n} = [f(1)+f(2)+\cdots+f(n)]\Delta x$$

where $\Delta x = 1$, is the sum of the areas of n rectangles with base $\Delta x = 1$ and heights $f(1),\ldots,f(n)$, consecutively, as shown in Figure 12.5. Since the union of these rectangles covers the region bounded by the curve $y=f(x)$, the x-axis, and the vertical lines $x=1$ and $x=n+1$, we have:

$$1+\frac{1}{2}+\cdots+\frac{1}{n} > \int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$

Now recall that $\ln(n+1)$ approaches ∞ as n approaches ∞. Hence, the sequence of partial sums of the infinite series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots$$

must be divergent.

The above infinite series is called the harmonic series. It diverges "to infinity" in the sense that its sequence of partial sums becomes arbitrarily large for all large values of n. If an infinite series diverges to infinity, we also say that it sums to infinity and use the notation "$=\infty$" accordingly. We have
The **Harmonic Series** is defined by

$$1+\frac{1}{2}+\frac{1}{3}+\cdots = \infty$$