Archimedes of Syracuse[1]

Archimedes of Syracuse (287 - 212 BCE), the most famous and probably the best mathematician of antiquity, made so many discoveries in mathematics and physics that it is difficult to point to any of them as his greatest.

He was born in Syracuse, the principal city-state of Sicily, the son of the astronomer Phidias. He spent considerable time in Alexandria, where he studied with Euclid's successors. It is there he met Conon of Samos (fl. 245 BCE) and Eratosthenes of Cyrene (c. 276 - 195 BCE), both leading mathematicians of their day. However, he resided most of his whole life in Syracuse, an intimate friend of the court of King Hieron II.

He was an accomplished engineer, indeed he is said to have disdained mechanical invention, who loved pure mathematics. With one exception, his only extant works are on pure mathematics. His methods of proof and discovery, though, were based substantially upon mechanical principles as revealed in his treatise *Method Concerning Mechanical Theorems*.

In fact, he seems to have disdained the source of his fame during his day, ingenious mechanical inventions, on which he left no written description. Said Plutarch, ”he possessed so high a spirit, so profound a soul, and such treasures of scientific knowledge that, thought these inventions had obtained for him the renown of more than human sagacity, he yet would not deign to leave behind him any written work on such subjects, ...”

Stories from Plutarch, Livy, and others describe machines invented by Archimedes for the defense of Syracuse. These include the catapult and the compound pulley. Also described is his instruments applying “burning-mirrors.” His fascination with geometry is beautifully described by Plutarch.[2]

> Often times Archimedes’ servants got him against his will to the baths, to wash and anoint him, and yet being there, he would ever be drawing out of the geometrical figures, even in the very embers of the chimney. And while they were anointing of him with oils and sweet savors, with his figures he drew lines upon his naked body, so far was he taken from himself, and brought into ecstasy or trance, with the delight he had in the study of geometry.

During the siege of Syracuse in the Second Punic War, inventions by Archimedes such as a catapult equally serviceable at a variety of ranges, caused great fear to the Roman attackers. Another invention, the compound pulley, was so powerfully built as to lift Roman ships from the sea and drop them back into it. However, the story that he used an array of mirrors, burning-mirrors, to destroy Roman ships is probably apocryphal. So much fear did these machines instill in the Romans that general Marcus Claudius Marcellus, the Roman commander, gave up on frontal assault and placed his hopes in a long siege. When at last Syracuse did fall in about 212 BCE, Archimedes was killed during the capture of Syracuse by the Romans Plutarch recounts this story of his killing: As fate would have it, Archimedes was intent on working out some problem by a diagram, and having fixed both his mind and eyes upon the subject of his speculation, he did not notice the entry of the Romans nor that the city was taken. In this transport of study a soldier unexpectedly came up to him and commanded that he accompany him. When he declined to do this before he had finished his problem, the enraged soldier drew his sword and ran him through. Marcellus was

[2]Plutarch(c. 46 - 119 CE), was a Greek biographer and author whose works influenced the evolution of the essay, the biography, even into our own times.

greatly saddened by this and arranged for Archimedes' burial.

1 Archimedes' Works

It was to Conon that Archimedes frequently communicated his results before they were published. There were no journals, as such, in that time. Major works were developed into books. There are nine extant books of Archimedes, that have come to us. Substantially in the form of advanced monographs, they are not works for students nor for the dilettante, as each requires serious study. Almost certainly, they were not as widely copied or studied as other works such as *The Elements* .

But how do we know about the works? Where did they come from? For most of the second millennium, the earliest sources of Archimedes works date from the Latin translations of Greek works made by William of Moerbeke (1215 - 1286). He used two Greek manuscripts. Both have disappeared, the first before 1311 and the second disappears about the 16^{th} century. No earlier versions were known until about 1899, when an Archimedes palimpsest was listed among hundreds of other volumes in a library in Istanbul. In 1906, the great Greek mathematical scholar was able to begin his examination of it.

A palimpsest is a document which has been copied over by another text. Two reasons are offered for doing this. First, parchment was expensive and reusing it was an economical measure. Second, at the time it was considered virtuous to copy over pagan texts. In the case at hand, the Archimedes palimpsest was covered over by a religious text. Moreover, the original sheets were folded in half, the resulting book of 174 pages having a sown binding.

What Heiberg found were four books already known but which had been copied in the 10^{th} century by a monk living in a Constantinople monastery. This version was independent of the two manuscripts used by William of Moerbeke. However, a new book was found. It was the *Method concerning Mechanical Theorems*. This book, though known to have been written, had not been found to that time. Its importance lies in that in this volume, Archimedes described his method of discovery of many of his other theorems.

The story of the Archimedes palimpsest over that past century is interesting with suggestions of theft and manuscript alteration. Having

disappeared in 1922, it reappeared in 1998 as an auction item displayed by Christie's in New York. It sold at auction for two million dollars in October of 1998 to an anonymous buyer. This buyer has agreed to make the manuscript available for scholarly research. For further details, the interested reader should consult
http://www-history.mcs.st-and.ac.uk/history/HistTopics/Greek_sources_1.html

The works themselves are

- *On Plane Equilibria*, Volume I
- *Quadrature of a Parabola*
- *On Plane Equilibria*, Volume II
- *On the Sphere and Cylinder, Volumes I and II*
- *On Spirals*
- *On Conoids and Spheroids*
- *The Sand-Reckoner*
- *On Floating Bodies*, Volumes I and II
- *On Measurement of the Circle*
- *Method Concerning Mechanical Theorems*

Another volume *Stomachion*, is known in fragments only. Yet another volume, a collection of Lemmas *Liber Assumptorum* comes down to us from the Arabic. In its present form, it could not been written by Archimedes as his name is referenced in it, though the results are likely due to Archimedes. Overall, we may say that he worked in the *Geometry of Measurement* in distinction to the *Geometry of Form* advanced by his younger colleague/competitor Apollonius (260 - 185 BCE). His methods anticipated the integral calculus 2,000 years before Newton and Leibniz. In the following subsections, we describe some of the results, recognizing the impossibility of rendering anything near an adequate description of the overwhelming depth and volume of his works.

1.1 Measurement of the Circle

Among Archimedes' most famous works is *Measurement of the Circle*, in which he determined the exact value of π to be between the values $3\frac{10}{71}$ and $3\frac{1}{7}$. This result is still used today, and most certainly every reader of these notes has used $\frac{22}{7} = 3\frac{1}{7}$ to approximate π. He obtained this result by circumscribing and inscribing a circle with regular polygons having up to 96 sides. However, the proof requires two fundamental relations about the perimeters and areas of these inscribed and circumscribed regular polygons.

The computation. With respect to a circle of radius r, let

$$\begin{aligned} b_1 &= \text{an inscribed hexagon with perimeter } p_1 \text{ and area } a_1 \\ B_1 &= \text{an circumscribed hexagon with perimeter } P_1 \text{ and area } A_1 \end{aligned}$$

Further, let $b_2, \ldots, b_n$ denote the regular inscribed $6{\cdot}2 \ldots 6{\cdot}2^n$ polygons, similarly, $B_2 \ldots B_n$ for the circumscribed polygons. The following formulae give the relations between the perimeters and areas of these $6 \cdot 2^n$ polygons.

$$P_{n+1} = \frac{2p_n P_n}{p_n + P_n} \qquad p_{n+1} = \sqrt{p_n P_{n+1}}$$

$$a_{n+1} = \sqrt{a_n A_n} \qquad A_{n+1} = \frac{2a_{n+1} A_n}{a_{n+1} + A_n}$$

Using *n*-gons up to 96 sides he derives the following
Proposition 3. The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$.

1.2 On the Sphere and Cylinder

In Volume I of *On the Sphere and Cylinder* Archimedes proved, among many other geometrical results, that the volume of a sphere is two-thirds the volume of a circumscribed cylinder. In modern notation, we have the familiar formula.

$$V_{\text{sphere}} = \frac{2}{3} V_{\text{circumscribed cylinder}}$$

This he considered his most significant accomplishments, requesting that a representation of a cylinder circumscribing a sphere be inscribed on his tomb. He established other fundamental results including

Proposition 33. The surface of any sphere is equal to four times the greatest circle on it.

Similarly, but for cones, we have

Proposition 34. Any sphere is four times the cone which has as its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere.

From this of course follows Archimedes relation above. In Volume II, Archimedes proves a number of results such as

Proposition 1. Given a cone or a cylinder, to find a sphere equal to the cone or to the cylinder.

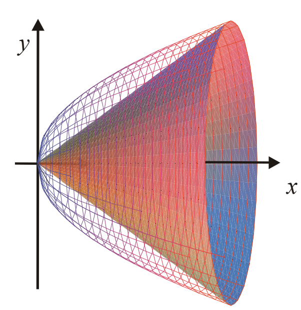
Proposition 3. To cut a given sphere by a plane so that the surfaces of the segments may have to one another a given ratio.

Proposition 9. Of all segments of spheres which have equal surfaces the hemisphere is the greatest in volume.

1.3 On Conoids and Spheroids

In *On Conoids and Spheroids*, he determined volumes of segments of solids formed by the revolutions of a conic, such as a parabola, about an axis. In modern terms these are problem of integration. For example, we have

Proposition 21. Any segment of a paraboloid of revolution is half as large again as the cone or segment of a cone which has the same base and the same axis.



Though easy to verify using calculus, this result requires a careful and lengthly proof using only the standard method of the day, i.e. double *reductio ad absurdum*.

1.4 On Floating Bodies

In *On Floating Bodies* Archimedes literally invented the whole study of hydrostatics. In one particular result he was able to compute the maximum angle that a (paraboloid) ship could list before it capsized — and he did it without calculus! This result, a *tour de force* of computation, is not nearly as well known as the story which describes Archimedes crying "Eureka" after discovering whether a newly made crown was truly pure gold.

The case of the fraudulent gold crown. King Hieron II commissioned the manufacture of a gold crown. Suspecting the goldsmith may have substituted silver for gold, he asked Archimedes to determine its authenticity. He was not allowed to disturb the crown in any way. What follows is a quote from Vitruvius.[3]

The solution which occurred when he stepped into his bath and caused it to overflow was to put a weight of gold equal to the crown and know to be pure, into a bowl which was filled with water to the brim. Then the gold would be removed and the king's crown put in, in its place. An alloy of lighter silver would increase the bulk of the crown and cause the bowl to overflow.

There are some technical exceptions to this method. A better solution applies Archimedes' Law of Buoyancy and his Law of the Lever:

Suspend the wreath from one end of a scale and balance it with an equal mass of gold suspended from the other end. Immerse the balanced apparatus into a container of water. If the scale remains in balance then the wreath and the gold have the same volume, and so the wreath has the same density as pure gold. But if the scale tilts in the direction of the gold, then the wreath has a greater volume

[3] Vitruvius's comments can be found in his work *De architectura* (about 27 BCE) a comprehensive treatise on architecture consisting of 10 books.

than the gold. For more details, consult the Archimedes home page, http://www.mcs.drexel.edu./~crorres/Archimedes/index.html.

1.5 Sand-Reckoner

The *Sand-Reckoner* is a small treatise that is addressed to Gelon, son of Hieron. Written for the layman, it nevertheless contains some highly original mathematics. One object of the book was to repair the inadequacies of the Greek numerical notation system by showing how to express a huge number, in particular the number of grains of sand that it would take to fill the whole of the universe. Apparently independent of the Babylonian base 60 system, Archimedes devised a place-value system of notation, with a base of 100,000,000. He constructed numbers up to 8×10^{17}. The work also gives the most detailed surviving description of the heliocentric system of Aristarchus of Samos, *the Ancient Copernican*.

1.6 On the Equilibrium of Planes

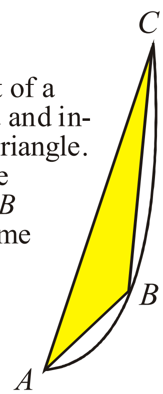
In a treatise of two volumes Archimedes discovered fundamental theorems concerning the center of gravity of plane figures and solids. His most famous theorem gives the weight of a body immersed in a liquid, called Archimedes' principal.

1.7 Quadrature of a Parabola

In the *Quadrature of a Parabola*, Archimedes proved using the *Method of Exhaustion* that

$$\text{area segment } ABC = \frac{4}{3} \cdot \triangle ABC$$

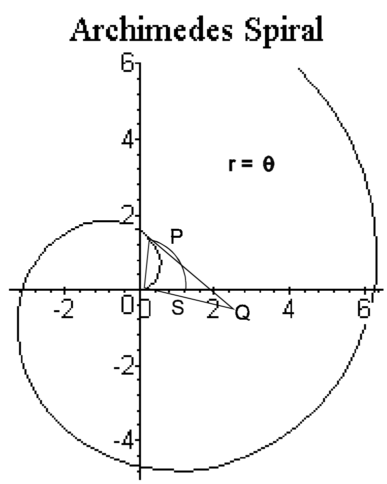
where the triangle and parabolic segment have the same base and height. The standard technique of proof, the method of exhaustion, was used.



Segment of a parabola and inscribed triangle. Note: the slope at *B* is the same as the line *AC*.

1.8 Spirals

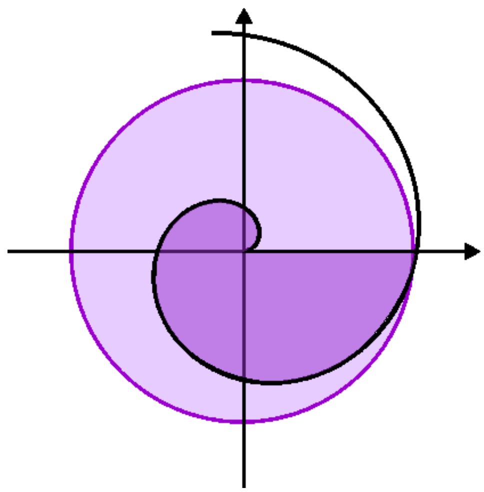
In *The Spiral* Archimedes squared the circle using the spiral.



He does this by proving that, in length, $OQ = arc\, PS$. Note, PQ is tangent to the spiral at P and $\angle POQ$ is a right angle.

- He also determined the area of one revolution ($0 \leq \theta \leq 2\pi$) of $r = a\theta$ to be[4]

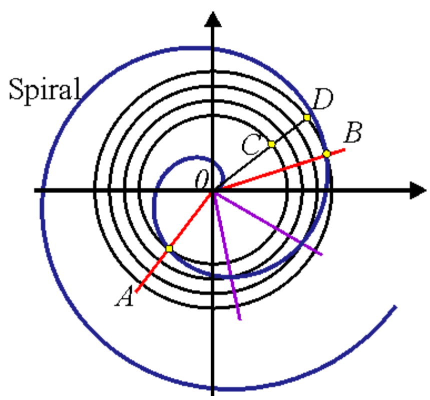
$$\text{Area } = \frac{1}{3}[\pi(a2\pi)^2]$$



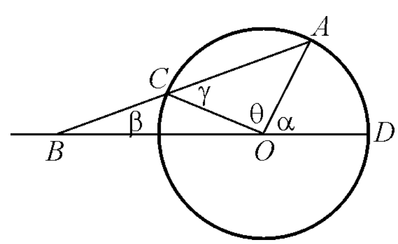
That is, the area enclosed by the spiral arc of one revolution is one third of the area of the circle with center at the origin and of radius at the terminus of the spiral arc.

[4] In polar coordinates the area of the curve $r = f(\theta)$ is given by $\frac{1}{2}\int_0^{2\pi} f^2(\theta)\,d\theta$. For the function $f(\theta) = a\theta$, we have $\frac{1}{2}\int_0^{2\pi} (a\theta)^2\,d\theta = \frac{4}{3}a^2\pi^3$.

He also showed how to trisect angles using the spiral. Suppose the particular angle to be trisected is $\angle AOB$. Construct circles with center O that intersect the spiral. Construct the line segment OD and mark the point C. Trisect the segment CD and construct circles with center O with the respective radii at the trisection points. Since the spiral sweeps out the radius in exact proportion to the respective angle, the new circles will intersect the spiral at equal angles from the lines OA and OB. The angle between them will be the same, as well. Thus the angles $\angle AOB$ is trisected.



In another argument using a compass and *ruler*, he trisected an angle. Using the diagram to the right, we trisect the angle $\angle DOA$. First, extend the diameter to B in such a way that $|BC| = |0C|$.



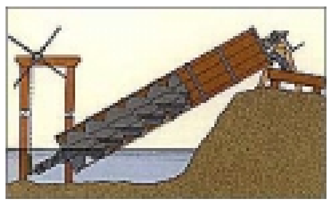
This is the part that requires the ruler. Now measure the angles. Note that $\gamma = 2\beta$. From this it follows that $\theta = 4\beta - \pi$. Finally observe that $\alpha + \theta + \beta = \pi$. Substitute $\theta = 4\beta - \pi$ and solve for β to get $\beta = \frac{1}{3}\alpha$.

1.9 The Method

In *Method Concerning Mechanical Theorems* Archimedes reveals how he discovered some of his theorems. The method is basically a "geometric method of the lever." He balances lines as one might balance weights. This work was found relatively recently, having being rediscovered only in 1906.

2 Inventions

Archimedes' mechanical skill together with his theoretical knowledge enabled him to construct many ingenious machines. Archimedes spent some time in Egypt, where he invented a device now known as Archimedes' screw. This is a pump, originally used for irrigation and for draining mines. It is still used in many parts of the world. The image below is but one example.[5]



From Pappus we have learned that in connection with his discovery of the solution to the problem of *moving a given weight by a given force*, that Archimedes upon applying the law of the lever[6] is to have said, "Give me a place to stand on, and I can move the earth." Another story related to this was the challenge to Archimedes by King Hieron to give a practical demonstration of this law. Thereupon, Archimedes, using only a compound pulley, steadily and smoothly pulled a ship from the sea onto dry dock. According to Proclus, Hieron was so impressed by Archimedes that he declared, "from that day forth Archimedes was to be believed in everything that he might say."

He is also said to have invented a sphere to imitate the motions of the sun, moon, and five planets known at that time. Cicero, who may have actually seen it, reported that it described details of the periodic nature of the rotations and even showed eclipses of the sun. How it operated is conjectural, but water power is often attributed.

[5] An interesting website maintained by Drexel University mathematics Professor Chris Rorres, located at http://www.mcs.drexel.edu/~crorres/Archimedes/Screw/Applications.html, shows many illustrations of Archimedian screws from the past and present. The illustration above can be found at this site.

[6] Archimedes was a master at the law of the lever and related mechanical principles, this is true. However, it is certain that this law must have been known much earlier in antiquity. In particular, the Egyptians must have applied some form of it in the construction of the pyramids.

3 Influence

The magnitude and originality of Archimedes' achievement is monumental. However, his influence on ancient mathematics was limited. Many reasons could be attributed, one being that mathematics in the Greek world was in something of an eclipse of his mathematics. Another is the hegemony of the Romans, who had little interest in theoretical works, particularly mathematics. Though some of his results, such as approximations to π by $\frac{22}{7}$, became commonplace, his deeper results of hydrostatics and quadrature were never continued in any important way — as far as is known. This seems true, despite his publication of *The Method*, in which he hoped to show others the basis of his techniques.

Nearly a millennium was to follow, when in the 8^{th} and 9^{th} centuries there were some substantial Arabic contributions that seem to be inspired by Arabic translations of Archimedes works.

The greatest influence of his work came much later in 16^{th} and 17^{th} centuries with the printing of texts derived from the Greek. Knowledge of these works was reflected in the work of the greatest mathematicians and physicists of the day, Galileo (1564 - 1642) and Kepler (1571 - 1630). Later, more mathematically sound editions such as David Rivault's edition and Latin translation (1615) of the complete works was profoundly influential on mathematicians of a stature no less than René Descartes (1595 - 1650) and Pierre de Fermat (1601 - 1665). The ancient works including Archimedes cast a pale across these times challenging mathematicians of the day to understand and advance the ancient results. It is widely regarded that the greatest advances of the 16^{th} century would have be delayed without them. Had *The Method* been discovered earlier than the late 19^{th} century, modern mathematics may have taken an entirely different course concluding, of course, the same essential results but with mechanical underpinnings instead of geometrical ones.