

The Hellenistic Period¹

In this brief section, we show what type of problems mathematicians solved, and the extent of Greek mathematics before the collapse of the mathematical world in about 400 CE.

1 Aristarchus of Samos (CA. 310-230 BCE)

Aristarchus of Samos was considered by the ancients, particularly Vitruvius², who considered him to be one of the few great men knowledgeable in all sciences, especially astronomy and mathematics. Note that his dates are only surmised. For example, he lived before the *Sand Reckoner* of Archimedes, a work published before 216 BCE. He is also known to have observed the summer solstice in about 280 BCE.

Perhaps, foremost among them is that he formulated the Copernican hypothesis. He was the first to formulate the Copernican hypotheses and is sometimes called the *Ancient Copernican*. His heliocentric theory was never accepted in his day and remained dormant until it was proposed again by Copernicus. His contributions to science and mathematics are many. He discovered an improved sundial, with a concave hemispherical circle. In supporting his heliocentric theory, he countered the *non parallax* objection by asserting that the stars to be so far distant that parallax was not measurable. For this he needed the following measure of distance to the stars.

$$\text{distance}(\text{earth, stars}) \sim \text{distance}(\text{radius of the sun})$$

He also determined a more accurate estimate of the solar year. He wrote as well on light and colors.

He wrote *On the Sizes and Distances of the Sun and Moon*, in which by the way the heliocentric ideas are not mentioned or in any event are not present in the copied version extant. He assumes the

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²Vitruvius's comments can be found in his work *De architectura* (about 27 BCE) which was a comprehensive treatise on architecture. Divided into 10 books, it deals with city planning, building materials, temple construction, the use of the Greek orders, public buildings such as theaters, private buildings, flooring, hydraulics, clocks, mensuration and astronomy, and engines. Though a Roman living probably during the time of Augustus, his outlook is essentially Hellenistic.

following two propositions that are equivalents of the following modern statements.

1. If $0 < \alpha < \frac{\pi}{2}$, then the ratio $\frac{\sin \alpha}{\alpha}$ decreases, and the ratio $\frac{\tan \alpha}{\alpha}$ increases as α increases from 0 to $\frac{\pi}{2}$.
2. If $0 < \beta < \alpha < \frac{\pi}{2}$ then

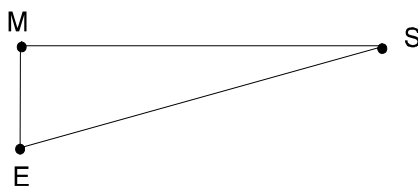
$$\frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$

(now known as Aristarchus' theorem)

He sets out to prove a number of results including, (1) When the moon is half full, the angle between the lines of sight to the sun and the moon is less than a right angle by $1/30$ of a quadrant. From this he concluded that the distance from the earth to the sun is more than 18 but less than 20 times the distance from the earth to the moon. (Actual ~ 400). Without trigonometry he was aware of and used the fact that

$$\frac{1}{20} < \sin 3^\circ < \frac{1}{18}.$$

This result gave a great improvement over all previous estimates.



(2) The angular diameters of the sun and the moon at the center of the earth are equal. (3) The diameter of the sun has to the diameter of the earth a ratio greater than 19:3 but less than 43:6. (Actual: $\sim 109:1$)

The proofs of all are geometrical in nature — as we would expect. He also made other trigonometric estimates — without trigonometry.

2 Archimedes, (287 - 212 BCE)

Looming far above almost all mathematicians of this period is Archimedes. His contributions are many and his mathematics clearly the deepest of all. A separate chapter is devoted to his life and works.

3 Apollonius of Perga (CA 262 BCE – 190 BCE)

★ Apollonius was born in Perga in Pamphilia (now Turkey), but was possibly educated in Alexandria where he spent some time teaching. Very little is known of his life. He seems to have felt himself a rival of Archimedes. In any event he worked on similar problems. However, in contrast to the older master we may say that Apollonius worked on the *Geometry of Form* as distinguished from the *Geometry of Measurement*. He was known as the “great geometer” mainly because of his work on conics.

Apollonius was 25 years younger than Archimedes, and they together with Euclid stood well above all other mathematicians of the first century of this period. Because of them, this period is sometimes called the “golden age” of Greek mathematics.

Apollonius wrote many books. All but one are lost. Among those we know he wrote are:

1. *Quick Delivery*
2. *Cutting-off of a Ratio*
3. *Cutting-off of an Area*
4. *Tangencies*
5. *Vergings (Inclinations)*
6. *Plane Loci*

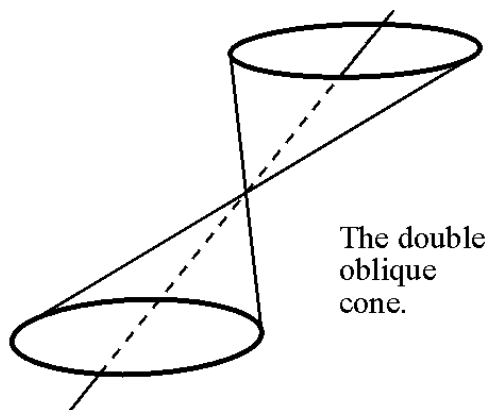
In his book *Quick Delivery* (lost), he gives the approximation to π as 3.1416. We do not know his method.

His only known work is *On Conics* in 8 Books of which only 4 survive in Greek. The 8th book is lost completely and the books V -

VII exist in Arabic. The fourth and later books were dedicated to King Attalus I (247 - 197 BCE), we have an approximate date of publication. The first two books were dedicated to Eudemus of Pergamum. It became immediately the authoritative treatise on conics, being referenced by many mathematicians. In about 500 CE, Eutocius wrote a commentary on them. It is this edition, altered somewhat in the ninth century in Constantinople, that our best manuscripts are derived. The *editio princeps* of the Greek text is due substantially to Edmund Halley³ in 1710. Originally, Gregory works on the Books I - IV, but died when the work was in progress. Features: Using the double oblique cone he constructs the conics

parabola, ellipse, hyperbola,

whose names he fixed for all time.

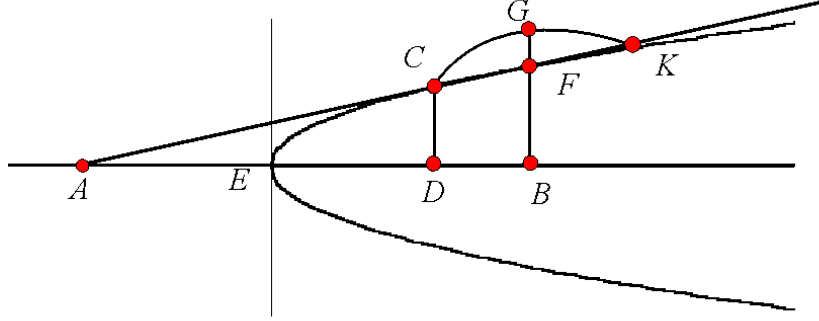


He made use of the idea of **Symptoms** which were similar to equations. There results an analytic-like geometry – but without coordinates! These come from what can be called a “symptoma” (from the Greek *συν'μπτωμα*), a constant relation between certain magnitudes which vary according to the position of an arbitrary point on the curve. The symptoma are very much like a modern equation. Even Archimedes used these relations for parabolas, ellipses and hyperbolas in his works.

³We will see in a later chapter that it was Edmund Halley (1656 - 1743) that was instrumental in persuading Isaac Newton to publish his monumental *Principia Mathematica*.

We consider two typical results found in *On Conics*.

Proposition I-33. If AC is constructed, where $|AE| = |ED|$, then AC is tangent to the parabola.



Proof. Assume AC cuts the parabola at K . Then CK lies within the parabola. Pick F on that segment and construct a \perp to AD at B and G on the curve. Then

$$\frac{|GB|^2}{|CD|^2} > \frac{|BF|^2}{|CD|^2} = \frac{|AB|^2}{|AD|^2}. \quad (*)$$

Since G and C lie on the curve, the symptom shows that $|BG|^2 = p|EB|$ and $|CD|^2 = p|ED|$ implies $\frac{|BG|^2}{|CD|^2} = \frac{|BE|}{|DE|}$. This implies from (*) that

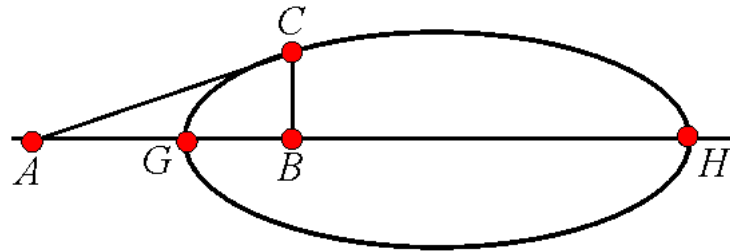
$$\frac{|BE|}{|DE|} > \frac{|AB|^2}{|AD|^2}, \text{ and so } \frac{4|BE| \cdot |EA|}{4|DE| \cdot |EA|} > \frac{|AB|^2}{|AD|^2}.$$

Thus $\frac{4|BE| \cdot |EA|}{|AB|^2} > \frac{4|DE| \cdot |EA|}{|AD|^2}$. Since $|AE| = |DE|$ it follows that $|AD|^2 = 4|DE| \cdot |EA|$ and since $|BE| > |EA|$ it follows that $4|BE| \cdot |EA| > |AB|^2$. (Why?) Since $AB = EA + EB$, this is a contradiction.

Proposition I-34. (ellipse) Choose A so that

$$\frac{|AH|}{|AG|} = \frac{|BH|}{|BG|}$$

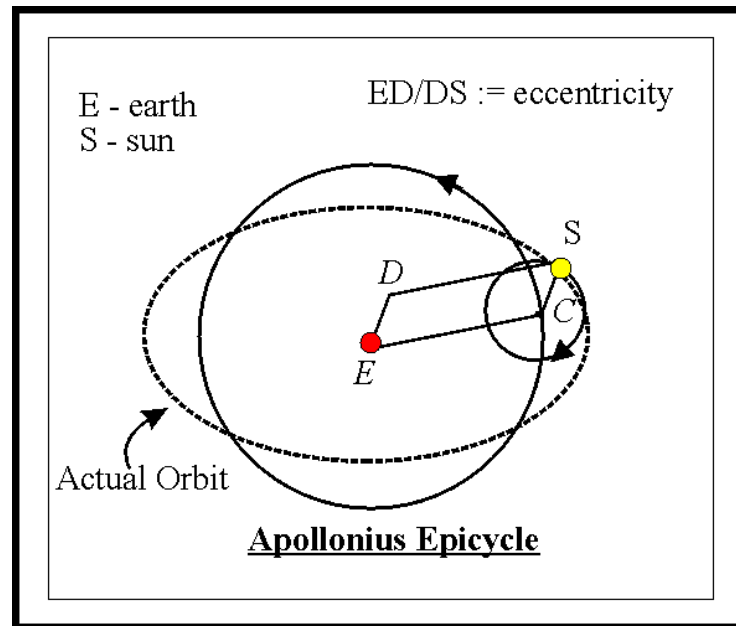
Then AC is tangent to the ellipse at C .



It is apparent that these *symptoma* used by Apollonius and by others before him were a definite precursor to analytic geometry. About two millennia would pass before coordinate geometry became common place.

The Apollonius model for the sun used epicycles. The sun rotates about a circle whose center revolves about the earth. The directions of rotation are opposed. The ratio

$$\frac{ED}{DS} := \text{eccentricity}$$



There was also the *eccenter* model, designed to explain the seasons better.

He improved upon the numbering system of Archimedes by using the base 10^4 .

In the works of Apollonius and Archimedes, Greek mathematics reached its zenith. Without his predecessors, the foremost being Euclid, Apollonius never would have reached this height. Together, they dominated geometry for two thousand years. With the works of Apollonius, mathematics was now well beyond the reaches of the dedicated amateur. Only a professional would be able to advance this theory further.

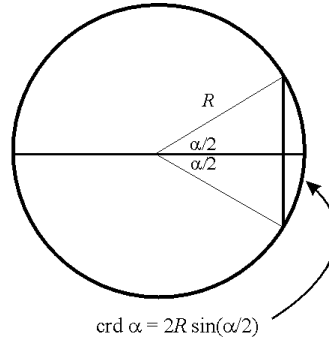
4 Hipparchus (fl. 140 BCE)

Hipparchus of Nicaea was a scientist of the first rank. So carefully accurate were his observations and calculations that he is known in antiquity as the “lover of truth.” He worked in nearly every field of astronomy and his reckonings were canon for 17 centuries.

Only one work of his remains, a commentary on *Phainomena* of Eudoxus and Aratus of Soli. We know him, however, from Ptolemy’s *The Almagest*. Indeed the “Ptolemaic Theory” should be called Hipparchian. His mathematical studies of astronomical models required a computation of a table of sines. He constructed a table of chords for astronomical use. Here a chord was given by

$$\text{Crd } \alpha = 2R \sin \alpha/2$$

in modern notation.



Hipparchus knew the half angle formula as well. He could compute the chord of every angle from $7\frac{1}{2}^\circ$ to 180° . Using Babylonian observations, he improved the lunar, solar and sidereal years. He reckoned the solar year at $365\frac{1}{4}$ days, less 4 minutes, 48 seconds — an error of 6 minutes from current calculations. He computed the lunar month at 29 days, 12 hours, 44 minutes, $2\frac{1}{2}$ seconds — less than one second off. He also computed the synodic periods of the planets with astonishing accuracy. He estimated the earth-moon distance at 250,000 miles, less than 5 percent off.

Hipparchus almost concluded the orbit of the earth about the sun

to be elliptic through his theory of “eccentrics” to account for orbital irregularities.

In about 129 BCE, he made a catalog of 1080 known fixed stars in terms of celestial longitude and latitude. Comparison of his chart with that of Timochares from 166 years earlier, he made his most brilliant observation. Noting a 2^0 shift in the apparent position of the stars he predicted the precession of the equinoxes⁴ — the advance, day by day, of the moment when the equinoctial points come to the meridian. He calculated the precession to be 36 seconds/year — 14 seconds slower than the current estimate of 50 seconds.

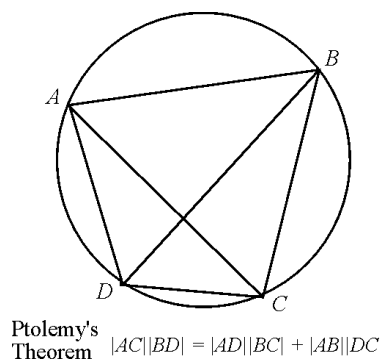
5 Claudius Ptolemy(100-178 AD)

Ptolemy was an astronomer/mathematician He wrote *The Geography*— a compilation of places in the known world along with their geographic coordinates. His most enduring work is the *Mathematical Collection*, later called The Almagest, from the Arabic ‘al-magisti’ meaning “the greatest.” It consists of 13 books, which contains

1. table of chords $1/2^\circ - 180^\circ$ $1/2^\circ$ intervals — $R = 60$, in sexagesimal.
2. He solve triangles, planes and spherical triangles.
3. The book was very algorithmic – this was a text/tutorial not a research monograph.
4. We also find the well known Ptolemy’s Theorem.

⁴equal nights

Theorem. $|AC||BD| = |AD||BC| + |AB||DC|$



The Almagest was the standard reference work for astronomers until the time of Copernicus.

6 Heron of Alexandria 1st century A.D.

Very little is known of Heron's life. However we do have his book *Metrica* which was more of a handbook for mathematics than what would now be called a research monograph. In it we find the famous **Heron's formula**. For any triangle of sides a , b , and c , and with perimeter $s = a + b + c$, the area is given by

$$A = (s(s - a)(s - b)(s - c))^{1/2}.$$

He also gives formulas for the area of regular polygons of n sides, each of length a :

$$\begin{aligned} A_3 &\approx \frac{13}{30}a^2 \\ A_5 &\approx \frac{5}{3}a^2 \\ A_7 &\approx \frac{43}{12}a^2 \end{aligned}$$

Heron gives an example of finding the cube of a non-cube number. The number is $A = 100$. Here are his instructions,

Take the nearest cube numbers to 100 both above and below. These are 125 and 64. Now compute the differences

$$125 - 100 = 25$$

$$100 - 64 = 36$$

Multiply 5 by 36 to get 180 and then add 100 to get 280. Take the fraction $\frac{180}{280} = \frac{9}{14}$ and add to the smaller cube obtaining $4\frac{9}{14}$. This is as nearly as possible the cube root (“cubic side”) of 100 units.

Just how Heron arrived at this method, and indeed, just what the method is we can only conjecture. However, in Heath⁵ we find the conjecture of Wertheim. Allowing the (positive) differences to be d_1 and d_2 , and the cube roots of the higher and lower cube to be a_1 and a_2 , an approximation can be taken as

$$a_1 + \frac{a_2 d}{a_2 d_1 + a_1 d_2}$$

Such was the state of the art for determining cube roots in the first century AD. In fact, with just elementary methods, it can be shown that this approximation is quite good, with accuracy exceeding that achieved by application of the differential method.⁶

7 Nicomachus of Gerasa(fl. 100 CE)

Nicomachus was probably a neo-Pythagorean as he wrote on numbers and music. The period from 30BCE to 641 CE is sometimes called the **Second Alexandrian School**.

He studied in Alexandria. Only two of his books are extant:

1. *Introduction to Arithmetic*
2. *Introduction to Harmonics*

A third book on geometry is lost.

⁵Thomas Heath, *A History of Greek Mathematics, II*, Dover, New York, page 341.

⁶The differential method: Approximate $f(x)$ by $f(x) \approx f'(a)(x - a) + f(a)$ where both $f(a)$ and $f'(a)$ are easily computed.

In the first of the works *Introduction to Arithmetic* given in two books we have in Book I a classification of integers **no without proofs** For example we have

| | | |
|----------|--|------------------|
| even: | even \times even | (powers of two) |
| | even \times odd | (doubles of odd) |
| | odd \times even | (all others) |
| odd: | primes | |
| | composites | |
| perfect: | calculates 6, 28, 496, 8128 ala Euclid | |

Also, Nicomachus provides a classification of ratios of numbers. Assume a/b is completely reduced form of A/B . (i.e. a and b are relative prime)

| relation | ratio of A to B is a |
|------------------------------|----------------------------------|
| $a = nb$ | <i>multiple</i> |
| $b = na$ | <i>submultiple</i> |
| $a = b + 1$ | <i>super particular</i> |
| $a + 1 = b$ | <i>sub super particular</i> |
| $a = nb + 1$ | <i>multiple super particular</i> |
| $a = nb + k \quad 1 < k < b$ | <i>multiple super particular</i> |
| etc. | |

In Book II Nicomachus discusses plane and solid numbers but again with no proofs. (Were the proofs removed in the many translations?) He considers the very Pythagorean:

triangular numbers
square numbers
pentagonal numbers
hexagonal numbers
heptagonal numbers

He notes an interesting result about cubes:

$$\begin{aligned}
 1 &= 1 \\
 3 + 5 &= 8 \\
 7 + 9 + 11 &= 27 \\
 13 + 15 + 17 + 19 &= 64 \\
 \dots &=
 \end{aligned}$$

This should be compared with the summation of odd numbers to achieve squares. (Recall, the square numbers of Pythagoras.)

The other known work, *Introduction to Arithmetic* was a “hand-book” designed for students, primarily. It was written at a much lower level than Euclid’s *Elements* but was studied intensively in Europe and the Arabic World throughout the early Middle Ages.

8 Other Great Geometers

8.1 Hypsicles of Alexandria

(fl. 175 BCE) added a fourteenth book to the *Elements* on regular solids. In short, it concerns the comparison of the volumes of the icosahedron and the dodecahedron inscribed in the same sphere.

★ In another work, *Risings*, we find for the first time in Greek mathematics the right angle divided in Babylonian manner into 90 degrees. He does not use exact trigonometry calculations, but only a rough approximation. For example, he uses as data the times of rising for the signs from Aries to Virgo as an arithmetic progression.

★ He also studied the **polygonal numbers**, the n^{th} one of which is given by

$$\frac{1}{2}n[2 + (n - 1)(a - 2)].$$

8.2 Diocles of Carystus, fl. 180 BCE

Diocles of Carystus invented the **cisoid**, or ivy shaped curve. It was used for the duplication problem. In modern terms it’s equation can be

written as

$$y^2(a - x) = x^3.$$

In fact, what we know of Diocles comes from references by Eutocius (6th cent. A.D.) to Apollonius' *Conics* in his version of Diocles, *Burning Mirrors*. Since the Cissoid was referenced by Geminus, who succeeded Apollonius, we have dates on Diocles of 190-180 B.C, the references to Apollonius an addition by Eutocius.

★ Diocles also solved an *open problem* posed by Archimedes, that of dividing a sphere into two parts whose volumes have a prescribed ratio. (Compare this with the result of Archimedes: *To cut a given sphere by a plane so that the surfaces of the segments may have to one another a given ratio*.)

He was able to solve certain cubics by intersecting an ellipse and a hyperbola. He studied refraction and reflection in the book *On Burning Mirrors*. However, this book may actually be a combination of three short works, one on burning mirrors, one on doubling the cube by intersecting parabolas, and the third on the problem of Archimedes.

8.3 Nicomedes,(fl. 260 BCE)

Nicomedes wrote *Introduction Arithmetica*. He discovered the **conchoid** and used it for angle trisection and finding two mean proportionals. We say the x and y are two mean proportionals between a and b if $a : x = x : y = y : b$. Of course this amounts to solving for the intersection of the parabola $ay = x^2$ and $y^2 = bx$. The conchoid is given by

$$r = a \sec \theta \pm d.$$

The little we know of these results come from Pappus (c. 350 CE) who described them and Eutocius (6th century CE) who indicated Nicomedes' great, even boastful, pride in his discovery. Pappus further associates Nicomedes with Dinostratus and others as having applied the *quadratrix*, invented by Hippias (c. 460 BCE), for squaring the circle.

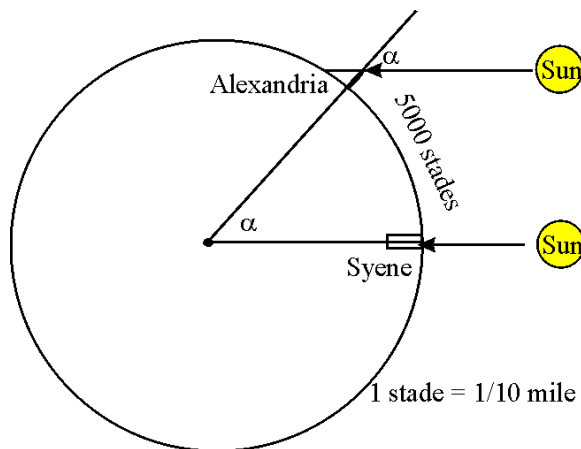
8.4 Eratosthenes of Cyrene, (c. 276 - c. 195 BCE)

Eratosthenes achieved distinction in many fields and ranked second only to the best in each. His admirers call him the second Plato and some

called him *beta*, indicating that he was the second of the wise men of antiquity. By the age of 40, his distinction was so great that Ptolemy III made him head of the Alexandrian Library. He wrote a volume of verse and a history of comedy, as well. He wrote mathematical monographs and devised mechanical means of finding mean proportions in continued proportion between two straight lines. He also invented the **sieve** for determining primes. This sieve is based on a simple concept:

Lay off all the numbers, then mark off all the multiples of 2, then 3, then 5, and so on. A prime is determined when a number is not marked out. So, 3 remains uncovered after the multiples of two are marked out; 5 remains uncovered after the multiples of two and three are marked out. Although it is not possible to determine truly large primes in this fashion, the sieve was used to determine early tables of primes.⁷

In a remarkable achievement he attempted the measurement of the earth's circumference, and hence diameter. Using a deep well in Syene (nowadays Aswan) and an Obelisk in Alexandria, he measured the angle cast by the sun at noonday in midsummer at both places. He measured the sun to be vertical in Syene and making an angle equal to $1/50$ of a circle at Alexandria he measured the circumference of the earth to be 25,000 miles. Remember, this measurement of the radius of the earth was made in ~ 250 B.C. Here's the diagram:



In modern terms, we have distance = radius \times angle. Using the angle to be $1/50$ of a circle, we obtain

⁷An interesting Java applet demonstrating this may be found at Peter Alfeld's web site: <http://www.math.utah.edu/~alfeld/Eratosthenes.html>

$$\begin{aligned}
 50 \times 5000 &= 250,000 \text{ stades} \\
 &= 25,000 \text{ miles}
 \end{aligned}$$

He also calculated the distance between the tropics as $\frac{11}{83}$ of the circumference, which make the obliquity of the ecliptic $23^0 51' 20''$, an error of $1/2$ of one percent.

Having measured the earth he set out to describe it, compiling accounts of all voyagers. He advocated regarding each person not as a Greek or Babylonian, but as an individual with individual merits — certainly a modern-sounding tenet.

It is said that in his old age he became blind and committed suicide by starvation.

8.5 Perseus (fl. 75 BCE)

Perseus discovered spiric sections, curves obtained by cutting solids, not cones. In modern terms a **spire** is given by

$$(x^2 + y^2 + z^2 + c^2 - a^2)^2 = 4c^2(z^2 + x^2)$$

8.6 Zenodorus (fl. 180 BCE)

About Zenodorus we know only by his reference to a result of Archimedes and his style of writing that he must have succeeded Archimedes but not by very much. In his book *On Isoperimetric Problems*, Zenodorus studied isoperimetric figures (same perimeter-different shape), a subject with a long history in Greek mathematics. This subject is extremely difficult and during the 18th century evolved into the Calculus of Variations. Indeed, even the great Isaac Newton was challenged to solve a difficult problem in this area, the so-called Brachistochone problem.⁸ This result, of course, was well beyond the technical level of Greek mathematics. Other modern problems include the “soap bubble” problem and others. However, Zenodorus (and later Pappus) offered some credibly difficult theorems. Here are a few typical results.

Proposition. Of all regular polygons of equal perimeter, that is greatest in areas has the most angles.

⁸Given two points in space, one higher than the other. If a wire is attached to the points what shape should it have for a frictionless bead to slide from the highest point to the lowest in least time.

Proposition. A circle is greater than any regular polygon of equal contour.

Proposition. Of all polygons of the same number of sides and equal perimeter the equilateral and equiangular polygon is the greatest in area.