# The Three Ancient Problems[1]

Three problems left unsolved by the ancient Greek school challenged later mathematicians, amateur and professional, for two millennia before their resolution. In this brief section we prove two of them, the Delian problem and the trisection problem, using the absolute bare minimum of mathematical machinery. What we will see is that the (negative) resolution of these problems while conceptually not difficult was nonetheless across an unbridgable chasm between the Greek paradigms of proper mathematical technique and the symbolic requirements of the proofs.

## 1 Constructible Numbers and Number Fields

Begin with the ring of integers, positive and negative. From them we construct using just a compass and straight edge a value not rational. This, as all subsequent constructions is accomplished by taking two lengths, $r$ and $s$, placing them at right angles, and constructing the diagonal of the implied rectangle It has length squared given by $w = r^2 + s^2$. Applying basic constructions methods we generate all products and sums with $\sqrt{w}$ and the rationals. This generates the set $\{p+q\sqrt{w}\}$. It is obvious that additions and subtractions leaves this set invariant. However, the set is also closed under products and quotients, as the computations below demonstrate.

$$\begin{aligned} \left(p+q\sqrt{w}\right)\left(a+b\sqrt{w}\right) &= pa+qwb+(pb+qa)\sqrt{w} \\ \frac{(p+q\sqrt{w})}{(a+b\sqrt{w})} &= \frac{pa-qwb+(-pb+qa)\sqrt{w}}{a^2-b^2w} \\ &= r+s\sqrt{w} \end{aligned}$$

This set is called the *extension field* of the rationals by $\sqrt{w}$. Application of the Pythagorean theorem (i.e. constructing with a compass and straight edge) on two of these values yields the root for the next extension field. The only implied condition is that two values selected yield a "diagonal" not in the originating field. Is this possible? Suppose the new extension equals the previous one. That is,

$$\sqrt{\left(a+b\sqrt{w}\right)^2+\left(c+d\sqrt{w}\right)^2} = p+q\sqrt{w}$$

for every selection of coefficients. Compute

$$\begin{aligned}(a+b\sqrt{w})^2+(c+d\sqrt{w})^2 &= a^2+d^2w+b^2w+c^2+(2cd+2ab)\sqrt{w}\\(p+q\sqrt{w})^2 &= p^2+q^2w+2pq\sqrt{w}\end{aligned}$$

If $a^2+d^2w+b^2w+c^2=p^2+q^2w$ then $a^2+c^2=p^2$ and $(d^2+b^2)w=q^2w$. From this it is a simple matter to demonstrate that it not necessary that $cd+ab=pq$. Thus each extension field can be chosen to be larger than the previous one. It is also a fact that each extension field contains only constructible numbers. In this way we can construct a hierarchy of extension fields. For example, we can construct the value $\sqrt{p+q\sqrt{w}}$; the resulting extension field has elements of the form

$$a+b\sqrt{p+q\sqrt{w}}$$

Note that extension fields can be constructed in infinitely many ways.

Example. What does the extension field structure look like for the constructible number

$$9\sqrt{5}+2\sqrt{3+\sqrt{\sqrt{4-\sqrt{5}}}}$$

Construct the first field extension by adding $\sqrt{5}$ to the rationals. Then adjoin $\sqrt{4-\sqrt{5}}$ in the next extension. A third extension is determined by adjoining $\sqrt{\sqrt{4-\sqrt{5}}}$. Finally, a fourth and last extension is created by adjoining $\sqrt{3+\sqrt{\sqrt{4-\sqrt{5}}}}$.

Notice that the new ideas here amount to a heavy use of symbolism and the abstraction through which these inductive constructions can be realized. We have systematically arranged all possible numbers that can be constructed with only a compass and straight edge into a hierarchy of extension fields by adding one new number at a time. In Euclid's *Elements* we see only the first and second levels of construction with the numbers $\sqrt{p+q\sqrt{w}}$ with $p$, $q$, and $w$ all integers. The proofs below require arbitrary but finite levels.

## 2 The Delian Problem

"Can the cube be duplicated?" was one of the three great problems of antiquity. In order to resolve it, mathematicians needed to invent the extension fields of numbers as above. Once done, and this took a mere two millennia, the Delian problem is resolved rather quickly.

**Theorem** There is no constructible solution of $x^3 = 2$.

Proof. To show that $x^3 = 2$ can have no solution among the Constructible numbers, we assume it does have the solution $x = p + q\sqrt{w}$, where $\sqrt{w}$ is in the smallest extension field based on the rationals. Then

$$\begin{aligned}\left(p + q\sqrt{w}\right)^3 - 2 &= p^3 + 3p^2q\sqrt{w} + 3pq^2w + q^3\left(\sqrt{w}\right)^3 - 2 \\ &= p^3 + 3pq^2w - 2 + \left(3p^2q + q^3w\right)\sqrt{w}\end{aligned}$$

We must therefore have

$$\begin{aligned}p^3 + 3pq^2w - 2 &= 0 \\ 3p^2q + q^3w &= 0\end{aligned}$$

Similarly, the "conjugate" of $p - q\sqrt{w}$ satisfies

$$\left(p - q\sqrt{w}\right)^3 - 2 = p^3 + 3pq^2w - \left(3p^2q + q^3w\right)\sqrt{w}$$

From the system of equations above this is zero. It follows that there are two *real* cube roots of 2. This furnishes the desired contraction as their clearly cannot be two *cube* roots of any number. Thus, there is no constructible value equal to $\sqrt[3]{2}$.

## 3 The Trisection Problem

Almost as old as geometry itself is the problem of constructing the "trisector" of a given angle. Throughout antiquity and even into modern times constructions have been offered that claim to work. One of the most clever, given by Archimedes, reveals just how delicate the problem is. For the goal is to trisect any given angle using only a compass and *straight edge*. Archimedes' solution (which appears in the chapter, *Archimedes*) is correct using a compass and *ruler*. That slight, but added on power of the straight edge, makes the difference in this case.

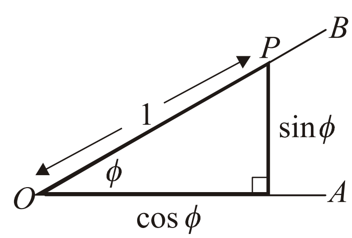
Our resolution of the trisection problem — in the negative — will depend on the extension field ideas. construction field Begin with any angle $\theta$. To explore the trisection problem, we examine what it means trigonometrically. Let $g = \cos\theta$. Then

$$\begin{aligned}
\cos\theta &= \cos\left(\frac{2\theta}{3} + \frac{\theta}{3}\right) \\
&= \cos\frac{2\theta}{3}\cos\frac{\theta}{3} - \sin\frac{2\theta}{3}\sin\frac{\theta}{3} \\
&= \left(\cos^2\frac{\theta}{3} - \sin^2\frac{\theta}{3}\right)\cos\frac{\theta}{3} - 2\sin^2\frac{\theta}{3}\cos\frac{\theta}{3} \\
&= \left(2\cos^2\frac{\theta}{3} - 1\right)\cos\frac{\theta}{3} - 2\left(1 - \cos^2\frac{\theta}{3}\right)\cos\frac{\theta}{3} \\
&= 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}
\end{aligned}$$

Thus, trigonometrically the trisection problem reduces to the solution of the polynomial

$$4x^3 - 3x = g$$

Now, from the diagram below that it is an elementary construction to construct any angle knowing its cosine and conversely to construct the cosine of any given angle. For example, given the angle $\angle AOB$, denoted by $\phi$, construct the a




length of one to $P$ as shown. Drop the perpendicular to $Q$. The resulting sides must be the values $\cos\phi$ and $\sin\phi$.

Now we select an angle for which the equation $4x^3 - 3x = g$ has no solution. Select $\theta = 60^o$. This gives the equation $4x^3 - 3x = \cos 60^o = \frac{1}{2}$. Multiply through and make the change of variable $y = 2x$ to get

$$y^3 - 3y = 1$$

First we must show there is no rational solution. Suppose $y = \frac{p}{q}$ is a solution and assume $(p, q) = 1$; that is $p$ and $q$ are relatively prime. Then

$$\left(\frac{p}{q}\right)^3 - 3\left(\frac{p}{q}\right) = 1$$

Multiplying through by $q^3$, we have

$$\begin{aligned} p^3 - 3pq^2 &= q^3 \\ p\left(p^2 - 3q^2\right) &= q^3 \end{aligned}$$

Thus $p$ divides $q^3$, which implies $(p, q) > 1$, unless $p = \pm 1$. Similarly from $p^3 - 3pq^2 = q^3$, we obtain

$$p^3 = q^2\,(3p + q)$$

This means $q^2$ divides $p^3$ unless $q = \pm 1$. Finally, if $p = \pm 1$ it follows that $q^2\,(3p + q) = \pm 1$. This implies both factors are $\pm 1$, which is impossible. On the other hand if $q = \pm 1$ then $p^3 - 3p = p\,(p^2 - 3) = \pm 1$ which is of course impossible as this would imply that both factors $p$ and $(p^2 - 3)$ are both $\pm 1$, which again is impossible. We thus have that rationals as solutions are impossible.

Next, let the three roots of a cubic be given as $y_1$, $y_2$, and $y_3$, then

$$\begin{aligned} (y - y_1)\,(y - y_2)\,(y - y_3) &= y^3 - (y_1 + y_2 + y_3)y^2 \\ &\quad + (y_2y_3 + y_1y_3 + y_1y_2)\,y + y_1y_2y_3 \end{aligned}$$

Thus if $y_1$, $y_2$, and $y_3$ are the three roots of $y^3 - 3y = 1$ it follows that

$$y_1 + y_2 + y_3 = 0$$

We are ready to begin the main part of the proof which is very much like that of the Delian problem. We assume there is a constructible solution given by $y = p + q\sqrt{w}$, where $\sqrt{w}$ is in the smallest extension field based on the construction procedures given in Section 1. Substitute to get

$$\begin{aligned} \left(p + q\sqrt{w}\right)^3 - 3\left(p + q\sqrt{w}\right) - 1 &= p^3 + 3p^2q\sqrt{w} + 3pq^2w + q^3\left(\sqrt{w}\right)^3 \\ &\quad - 3\left(p + q\sqrt{w}\right) - 1 \\ &= p^3 + 3pq^2w - 3p - 1 \\ &\quad + \left(3p^2q + q^3w - 3q\right)\sqrt{w} = 0 \end{aligned}$$

We must therefore have

$$\begin{aligned} p^3 + 3pq^2w - 3p - 1 &= 0 \\ 3p^2q + q^3w - 3q &= 0 \end{aligned}$$

Similarly, the "conjugate" $p - q\sqrt{w}$ satisfies

$$\begin{aligned} \left(p + q\sqrt{w}\right)^3 - 3\left(p + q\sqrt{w}\right) - 1 &= p^3 - 3p^2q\sqrt{w} + 3pq^2w \\ &\quad -q^3\left(\sqrt{w}\right)^3 - 3\left(p - q\sqrt{w}\right) - 1 \\ &= p^3 + 3pq^2w - 3p - 1 \\ &\quad -\left(3p^2q + q^3w - q\right)\sqrt{w} \end{aligned}$$

It is clear that $p - q\sqrt{w}$ is also a root of our equation. Letting $y_1 = p + q\sqrt{w}$ and $y_2 = p - q\sqrt{w}$, it follows that

$$\begin{aligned} y_1 + y_2 + y_3 &= p + q\sqrt{w} + \left(p - q\sqrt{w}\right) + y_3 \\ &= 2p + y_3 = 0 \end{aligned}$$

The implication is that the third solution is $y_3 = -2p$. But $p$ is an element of a field one lower in the hierarchy of extension fields, and we have assumed that $p + q\sqrt{w}$ is in the smallest extension field for which there is a solution. Thus, this contradiction completes the proof. The general theorem is

**Theorem.** Not all angles can be trisected using a compass and straightedge.

## 4 Circle Squaring

We postpone discussion of the circle squaring problem to a later section. But for the record the problem is this: For a given circle, can a square be constructed with the same area? It is not possible, as well. However, there is no known proof using just the extension field techniques applied above. A new kind of number far beyond the reach of constructible numbers, called *transcendental* numbers, are needed to resolve the issues here.

## 5 Exercises

1. Adapt the proof of the Delian problem to answer the question as to what numbers *do* have constructable solutions.
2. Adapt the proof of the Delian problem to show that $x^5 = 2$ can have no constructible solution.
3. Show where the proof that the angle cannot be trisected breaks down for a $90^{\text{o}}$ angle. Show how the resulting cubic equation can be solved in this case.
4. Show where the proof that the angle cannot be trisected breaks down for a $45^{\text{o}}$ angle. Can it be constructed by any other means? Can a $15^{\text{o}}$ be constructed by trisecting a $45^{\text{o}}$ angle?

## References

1. Geometrical Constructions, Courant, R, and Robbins, H., in *Mathematics: People, Problems, Results*, Volume II, Campbell, Douglas M, and Higginds, John C., eds., Wadsworth International, Belmont, 1984.